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OPTIMAL STOPPING OF A RISK PROCESS

Abstract. Optimal stopping time problems for a risk process $U_t = u + ct - \sum_{n=0}^{N(t)} X_n$where the number $N(t)$ of losses up to time t is a general renewal process and the sequence of X_i's represents successive losses are studied. $N(t)$ and X_i's are independent. Our goal is to maximize the expected return before the ruin time. The main results are closely related to those obtained by Boshuizen and Gouweleew [2].

1. Introduction. Let $\{N(t), t \geq 0\}$be a renewal process representing the stream of losses of an insurance company, so $N(t)$ is the number of losses up to the time t. If T_i denotes the time of occurrence of the ith loss, then random variables (r.v.'s) $S_i = T_i - T_{i-1}$ are independent identically distributed (i.i.d.) with a cumulative distribution function (c.d.f.) F, $T_0 = 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.'s with c.d.f. H, representing the successive losses. As a capital assets model for the insurance company we take the risk process

$$U_t = u + ct - \sum_{n=0}^{N(t)} X_n, \tag{1}$$

where $u > 0$ represents the initial capital and $c > 0$ is a constant rate of income from the insurance premium, $X_0 = 0$. The return at time t will be defined by the process $\{Z(t), t \geq 0\}$ where

$$Z(t) = \begin{cases} g_1(U_t)\mathbf{I}\{U_s > 0,\ s \leq t\} & \text{if } t \leq t_0, \\ 0 & \text{if } t > t_0, \end{cases} \tag{2}$$

where g_1 is a utility function. For simplicity define $g(u,t) = g_1(u)\mathbf{I}\{t \geq 0\}$.

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Then

$$Z(t) = g(U_t, t_0 - t) \prod_{j=1}^{N(t)} \mathbf{I}\{U_{T_j} > 0\}. \tag{3}$$

Let

$$\mathcal{F}(t) = \sigma(U_s, s \le t) = \sigma(X_1, T_1, \ldots, X_{N(t)}, T_{N(t)}) \tag{4}$$

be the σ-field generated by all events up to time t, $t \ge 0$, and $\mathcal{T}$ be the set of stopping times with respect to the family $\{\mathcal{F}(t),\ t \ge 0\}$. Moreover, for $n = 0, 1, 2, \ldots, n < K$, denote by $\mathcal{T}_{n,K}$ the subset of $\mathcal{T}$ such that

$$\tau \in \mathcal{T}_{n,K} \quad \text{if and only if} \quad T_n \le \tau \le T_K \text{ a.s.} \tag{5}$$

Set $\mathcal{F}_n \triangleq \mathcal{F}(T_n)$. We will be interested in finding optimal stopping times τ^*, $\tau^*_{n,K}$, τ^*_K such that

$$EZ(\tau^*) = \sup\{EZ(\tau) : \tau \in \mathcal{T}\}, \tag{6}$$

$$EZ(\tau^*_K) = \sup\{EZ(\tau) : \tau \in \mathcal{T}_{0,K}\}, \tag{7}$$

$$E\{Z(\tau^*_{n,K}) \mid \mathcal{F}_n\} = \operatorname{ess\,sup}\{E(Z(\tau) \mid \mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}. \tag{8}$$

The crucial role in the subsequent considerations is played by the following representation theorem for stopping times (see for example Davis [4]):

LEMMA 1. *If $\tau \in \mathcal{T}_{n,K}$, then there exists a positive $\mathcal{F}_n$-measurable r.v. R_n such that*

$$\tau \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1} \quad \textit{a.s.} \tag{9}$$

2. Finite horizon case. In this section we will find the form of optimal stopping rules in the finite horizon case, i.e. optimal in the class $\mathcal{T}_{0,K}$, where K is finite and fixed. First, in Theorem 1, we will derive dynamic programming equations satisfied by

$$\Gamma_{n,K} = \operatorname{ess\,sup}\{E(Z(\tau) \mid \mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}, \quad n = K, K-1, \ldots, 1. \tag{10}$$

Then, in Theorem 2, we will find optimal stopping times $\tau^*_{n,K}$ and τ^*_K and corresponding optimal conditional mean rewards and optimal mean rewards, respectively. Define

$$\mu_n = \prod_{j=1}^{n} \mathbf{I}(U_{T_j} > 0), \quad \mu_0 = 1. \tag{11}$$

Note that

$$\Gamma_{K,K} = Z(T_K) = g(U_{T_K}, t_0 - T_K)\mu_K. \tag{12}$$

THEOREM 1. (i) *For* $n = K-1, K-2, \ldots, 0$,

$$\Gamma_{n,K} = \operatorname{ess\,sup}\{\mu_n \overline{F}(R_n) g(U_{T_n} + cR_n, t_0 - T_n - R_n) + E(\mathbf{I}\{R_n \ge S_{n+1}\}\Gamma_{n+1,K} \mid \mathcal{F}_n) : R_n \ge 0, R_n \text{ is } \mathcal{F}_n\text{-measurable}\} \text{ a.s.},$$

where $\overline{F} = 1 - F$ *denotes the survival function.*

(ii) *For* $n = K, K-1, \ldots, 0$,

$$\Gamma_{n,K} = \mu_n \gamma_{K-n}(U_{T_n}, T_n) \quad \text{a.s.,} \tag{13}$$

where the sequence of functions $\{\gamma_j(u,t), u \in \mathbb{R},\ t \ge 0\}$ *is defined recursively as follows*:

$$\gamma_0(u,t) = g(u, t_0 - t), \tag{14}$$

$$\gamma_j(u,t) = \sup_{r \ge 0} \Big[\overline{F}(r) g(u + cr, t_0 - t - r) + \int_0^r dF(s) \int_0^{u+cs} \gamma_{j-1}(u + cs - x, t + s)\, dH(x)\Big],\ j = 1, 2, \ldots \tag{15}$$

Proof. (i) Let $\tau \in \mathcal{T}_{n,K}$, $0 \le n < K < \infty$. From Lemma 1 we get

$$A_n \stackrel{\triangle}{=} \{\tau < T_{n+1}\} = \{T_n + R_n < T_{n+1}\} = \{R_n < S_{n+1}\}$$

and

$$\overline{A}_n = \{\tau \ge T_{n+1}\} = \{R_n \ge S_{n+1}\}.$$

Then, using the properties of the conditional expectation, we can obtain the conditional expectation of the return at τ:

$$E(Z(\tau) \mid \mathcal{F}_n) = E(Z(\tau)\mathbf{I}_{A_n} \mid \mathcal{F}_n) + E(Z(\tau)\mathbf{I}_{\bar{A}_n} \mid \mathcal{F}_n) \stackrel{\triangle}{=} \alpha_n + \beta_n,$$

where

$$\begin{aligned} \alpha_n &= E(\mathbf{I}\{R_n < S_{n+1}\} g(U_\tau, t_0 - \tau)\mu_n \mid \mathcal{F}_n) \\ &= \mu_n E(\mathbf{I}\{R_n < S_{n+1}\} g(U_{T_n} + cR_n, t_0 - T_n - R_n) \mid \mathcal{F}_n) \\ &= \mu_n \overline{F}(R_n) g(U_{T_n} + cR_n, t_0 - T_n - R_n). \end{aligned}$$

Note that β_n can be expressed as follows:

$$\beta_n = E[\mathbf{I}\{S_{n+1} \le R_n\} E(Z(\tau') \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n],$$

where $\tau' = \tau \vee T_{n+1} \in \mathcal{T}_{n+1,K}$. Hence,

$$\begin{aligned} E(Z(\tau) \mid \mathcal{F}_n) = \mu_n \overline{F}(R_n) g(U_{T_n} + cR_n, t_0 - T_n - R_n) \\ + E[\mathbf{I}\{S_{n+1} \le R_n\} E(Z(\tau') \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n]. \end{aligned}$$

Now, following the standard reasoning of optimal stopping theory, we get the dynamic programming equation for $\Gamma_{n,K}$, $n = K, K-1, \ldots, 0$, given in (i), with $\Gamma_{K,K} = \mu_K g(U_{T_K}, t_0 - T_K)$.

(ii) We will prove (ii) using the backward induction method for $n = K-1,\dots,1$. First note that (ii) is satisfied for $n = K$ since $\Gamma_{K,K} = \mu_K\gamma_0(U_{T_K},T_K)$.

Let $n = K-1$. Then from (i) and the definition of the risk process (1) we get

$$\begin{aligned}\Gamma_{K-1,K} = \operatorname{ess\,sup}\{&\mu_{K-1}\overline{F}(R_{K-1})g(U_{T_{K-1}}+cR_{K-1},t_0-T_{K-1}-R_{K-1})\\ &+E(\mu_K\mathbf{I}\{R_{K-1}\ge S_K\})\gamma_0(U_{T_{K-1}}+cS_K-X_K,T_{K-1}+S_K)\mid\mathcal{F}_{K-1}):\\ &R_{K-1}\text{ is }\mathcal{F}_{K-1}\text{-measurable, }R_{K-1}\ge 0\}.\end{aligned}$$

Now, to get $\Gamma_{K-1,K} = \mu_{K-1}\gamma_1(U_{T_{K-1}},T_{K-1})$ it is sufficient to note that $\mu_K = \mu_{K-1}\mathbf{I}\{U_{T_{K-1}}+cS_K-X_K>0\}$. Moreover, the random variables S_K, X_K and the σ-field $\mathcal{F}_{K-1}$ are independent; S_K and X_K have c.d.f. F and H, respectively.

Let $1\le n\le K-1$ and suppose that $\Gamma_{n,K} = \mu_n\gamma_{K-n}(U_{T_n},T_n)$. From (i) we have

$$\begin{aligned}\Gamma_{n-1,K} = \operatorname{ess\,sup}\{&\mu_{n-1}\overline{F}(R_{n-1})g(U_{T_{n-1}}+cR_{n-1},t_0-T_{n-1}-R_{n-1})\\ &+\mu_nE(\mathbf{I}\{R_{n-1}\ge S_n\})\gamma_{K-n}(U_{T_n},T_n)\mid\mathcal{F}_{n-1}):\\ &R_{n-1}\ge 0, R_{n-1}\text{ is }\mathcal{F}_{n-1}\text{-measurable}\}.\end{aligned}$$

The second term under ess sup can be rewritten in the following way:

$$\begin{aligned}\mu_{n-1}E[&\mathbf{I}\{R_{n-1}\ge S_n\}\mathbf{I}\{U_{T_{n-1}}+cS_n-X_n>0\}\\ &\times\gamma_{K-n}(U_{T_{n-1}}+cS_n-X_n,T_{n-1}+S_n)\mid\mathcal{F}_{n-1}].\end{aligned}$$

Then we have

$$\begin{aligned}\Gamma_{n-1,K} &= \mu_{n-1}\operatorname{ess\,sup}\Big\{\overline{F}(R_{n-1})g(U_{T_{n-1}}+cR_{n-1},t_0-T_{n-1}-R_{n-1})\\ &\quad+\int_0^{R_{n-1}}dF(s)\int_0^{U_{T_{n-1}}+cs}\gamma_{K-n}(U_{T_{n-1}}+cs-x,T_n+s)\,dH(x):\\ &\quad R_{n-1}\ge 0,\ R_{n-1}\text{ is }\mathcal{F}_{n-1}\text{-measurable}\Big\}\\ &= \mu_{n-1}\cdot\gamma_{K-(n-1)}(U_{T_{n-1}},T_{n-1}).\ \blacksquare\end{aligned}$$

To find the form of optimal stopping times τ_K^* we need to analyze properties of the sequence of functions $\{\gamma_n,\,n\ge 0\}$ defined in the second part of Theorem 1.

Let $B = B[(-\infty,\infty)\times[0,\infty)]$ be the space of all bounded continuous functions, with the norm $\|\delta\| = \sup_{u,t}|\delta(u,t)|$, and

$$B^0 = \{\delta:\delta(u,t) = \delta_1(u,t)\mathbf{I}\{t\le t_0\}\text{ and }\delta_1\in B\}. \tag{16}$$

For any $\delta\in B^0$ and any $u\in\mathbb{R}$, $t,r\ge 0$ define

$$\phi_\delta(r,u,t) \stackrel{\triangle}{=} \overline{F}(r)g(u+cr,t_0-t-r) + \int_0^r dF(s)\Big[\int_0^{u+cs} \delta(u+cs-x,t+s)\,dH(x)\Big]. \tag{17}$$

Note that the properties of the c.d.f. F imply that $\phi_\delta(r,u,t)$ has an at most countable number of points of discontinuity with respect to r and is continuous with respect to (u,t) provided that $g_1(\cdot)$ is continuous and $t \neq t_0 - r$. In what follows we will use the following

ASSUMPTION 1. *The function $g_1(\cdot)$ is bounded and continuous.*

For all $\delta \in B^0$ define

$$(\Phi\delta)(u,t) = \sup_{r\geq 0}\{\phi_\delta(r,u,t)\}. \tag{18}$$

LEMMA 2. *For any $\delta \in B^0$ we have*

$$(\Phi\delta)(u,t) = \max_{0\leq r\leq t_0-t}\{\phi_\delta(r,u,t)\} \in B^0$$

and there exists a function r_δ such that $(\Phi\delta)(u,t) = \phi_\delta(r_\delta(u,t),u,t)$.

Proof. Observe that for all $\delta \in B^0$ and for any $r > t_0 - t$ we have

$$\phi_\delta(r,u,t) = \int_0^{t_0-t} dF(s)\Big[\int_0^{u+cs} \delta(u+cs-x,t+s)\,dH(x)\Big]. \tag{19}$$

Hence, Assumption 1 and the fact that F has an at most finite number of discontinuity points in the compact interval $[0,t_0]$ imply the form of Φ. ■

Observe that for $i = 1,2,\ldots$, $u \in \mathbb{R}$, $t \geq 0$, $\gamma_i(u,t)$ can be rewritten as follows:

$$\gamma_i(u,t) = \begin{cases} (\Phi\gamma_{i-1})(u,t) & \text{if } u \geq 0 \text{ and } t \leq t_0,\\ 0 & \text{otherwise,}\end{cases} \tag{20}$$

and from Lemma 2 there exist functions $r_i \stackrel{\triangle}{=} r_{\gamma_{i-1}}$ such that

$$\gamma_i(u,t) = \begin{cases} \phi_{\gamma_{i-1}}(r_i(u,t),u,t) & \text{if } u \geq 0 \text{ and } t \leq t_0,\\ 0 & \text{otherwise.}\end{cases} \tag{21}$$

To determine the form of optimal stopping times $\tau^*_{n,K}$ we need to define the following r.v.'s:

$$R^*_i \stackrel{\triangle}{=} r_{K-i}(U_{T_i},T_i) \tag{22}$$

and

$$\sigma_{n,K} = K \wedge \inf\{i \geq n : R^*_i < S_{i+1}\}. \tag{23}$$

THEOREM 2. *Let*

$$\tau^*_{n,K} = T_{\sigma_{n,K}} + R^*_{\sigma_{n,K}} \quad \textit{and} \quad \tau^*_K = \tau^*_{0,k}. \tag{24}$$

Then, for any $0 \le n \le K$, *we have*

$$\Gamma_{n,K} = E(Z(\tau^*_{n,K}) \mid \mathcal{F}_n) \quad a.s. \quad and \quad \Gamma_{0,K} = E(Z(\tau^*_K)) = \gamma_K(u,0). \tag{25}$$

Proof. This is a straightforward consequence of the formulas (22)–(24) and Theorem 1.

3. Infinite horizon case. In this section we will show that there exists an optimal stopping rule τ^* in the infinite horizon case, maximizing over $\mathcal{T}$ the mean return (2), i.e. (6) is fulfilled. Moreover, the optimal stopping time τ^* can be defined as a limit of the finite horizon optimal stopping times.

ASSUMPTION 2. $F(t_0) < 1$.

LEMMA 3. *The operator* $\Phi : B^0 \to B^0$ *defined by* (18) *is a contraction.*

Proof. Let $\delta_1, \delta_2 \in B^0$. By Lemma 2 there exist $\varrho_i \stackrel{\triangle}{=} r_{\delta_i}(u,t)$, $i = 1, 2$, such that $(\Phi\delta_i)(u,t) = \phi_{\delta_i}(\varrho_i, u, t)$, $i = 1, 2$. Since $\phi_{\delta_2}(\varrho_2, u, t) \ge \phi_{\delta_2}(\varrho_1, u, t)$ we obtain the inequalities

$$\begin{aligned}(\Phi\delta_1)(u,t) - (\Phi\delta_2)(u,t) &\le \int_0^{\varrho_1} dF(s) \int_0^{u+cs} [\delta_1 - \delta_2](u + cs - x, t + s)\, dH(x) \\ &\le \|\delta_1 - \delta_2\| \int_0^{\varrho_1} dF(s) \int_0^{u+cs} dH(x) \le \varrho\|\delta_1 - \delta_2\|,\end{aligned}$$

where

$$\varrho = \sup_{u>0} \int_0^{t_0} dF(s) \int_0^{u+cs} dH(x) \le F(t_0) < 1. \tag{26}$$

Similarly, we get $(\Phi\delta_2)(u,t) - (\Phi\delta_1)(u,t) \le \varrho\|\delta_1 - \delta_2\|$. Hence, $\|\Phi\delta_2 - \Phi\delta_1\| \le \varrho\|\delta_1 - \delta_2\|$. ■

Since $\gamma_0(u,t) = g(u, t_0 - t)$ it follows that $\gamma_i \in B^0$ for all i. Hence, from the Fixed Point Theorem we get the following lemma.

LEMMA 4. *There exists* $\gamma \in B^0$ *such that*

$$\gamma = \Phi\gamma \quad and \quad \lim_{K\to\infty} \|\gamma_K - \gamma\| = 0. \tag{27}$$

Remark 1. Note that all optimal stopping times are less than t_0 a.s., which is a consequence of the definition (2) of the return.

THEOREM 3. *Assume that the utility function* g_1 *is differentiable and nondecreasing, and* F *has the density function* f. *Then*

(i) *for* $n = 0, 1, \ldots,$ *the limit* $\widehat{\tau}_n \stackrel{\triangle}{=} \lim_{K\to\infty} \tau^*_{n,K}$ *exists and* $\widehat{\tau}_n$ *is an optimal stopping rule in* $\mathcal{T} \cap \{\tau \ge T_n\}$,

(ii) $E[Z(\widehat{\tau}_n) \mid \mathcal{F}_n] = \mu_n \gamma(U_{T_n}, T_n)$ *a.s.*

P r o o f. (i) Let $n \ge 0$. Note that $\tau^*_{n,K} \le \tau^*_{n,K+1}$ a.s. Hence, the stopping rule $\hat\tau_n = \lim_{K\to\infty} \tau^*_{n,K} \ge T_n$ exists.

To prove optimality of $\widehat{\tau}_n$ we will apply similar arguments to those used by Boshuizen and Goeweleew [2] in the proof of the existence of optimal stopping times for semi-Markov processes. Let $\xi_t = (t, U_t, Y_t, V_t)$, $Y_t = t - T_{N(t)}$, $V_t = \mu_{N(t)}$, $t \ge 0$. Then $\xi = \{\xi_t : t \ge 0\}$ is a Markov process with the state space $\mathbb{R}^1_+ \times \mathbb{R}^1 \times \mathbb{R}^1_+ \times \{0,1\}$. Note that the return $Z(t)$ is a function, say $\widetilde{g}$, of ξ_t. Let A be a strong generator of ξ. Then we get

$$(A\widetilde{g})(t,u,y,v) = \left\{ c g_1'(u) - \frac{f(y)}{\overline{F}(y)} \Big[g_1(u) - \int_0^u g_1(u-x)\,dH(x) \Big] \right\} v, \tag{28}$$

where $t < t_0$, $y \ge 0$ and $v \in \{0,1\}$.

Now, note that $\widetilde{g}(\xi_t) - \widetilde{g}(\xi_0) - \int_0^t (A\widetilde{g})(\xi_s)\,ds$, $t \ge 0$, is a martingale with respect to $\sigma(\xi_s, s \le t)$, which is the same as $\mathcal{F}(t)$ (see [3], p. 31). Applying the optional sampling theorem ([3], p. 22) we get

$$E[\widetilde{g}(\xi_{\tau^*_{n,K}}) \mid \xi_{T_n}] - \widetilde{g}(\xi_{T_n}) = E\Big[\int_{T_n}^{\tau^*_{n,K}} (A\widetilde{g})(\xi_s)\,ds \,\Big|\, \mathcal{F}_n \Big] \quad \text{a.s.} \tag{29}$$

Since

$$(A\widetilde{g})(\xi_s) = \left\{ c g_1'(U_s) + \frac{f(s - T_{N(s)})}{\overline{F}(s - T_{N(s)})} \times \Big[\int_0^{U_s} g_1(U_s - x)\,dH(x) - g_1(U_s) \Big] \right\} \mu_{N(s)}, \tag{30}$$

the right hand side of (29) can be expressed as the difference $E(I^1_{n,K} \mid \mathcal{F}_n) - E(I^2_{n,K} \mid \mathcal{F}_n)$, where

$$I^2_{n,K} = \int_{T_n}^{\tau^*_{n,K}} \frac{f(s - T_{N(s)})}{\overline{F}(s - T_{N(s)})} g_1(U_s) \mu_{N(s)}\,ds.$$

Now, $I^1_{n,K}$, $I^2_{n,K}$ are positive r.v.'s and $I^2_{n,K}$ is bounded by $g_1(u + ct_0) \times E(L)/\overline{F}(t_0)$, where $L = \inf\{n \in \mathbb{N} : T_n < t_0,\ T_{n+1} \ge t_0\}$. Note that

$$E(L) = \sum_{n=1}^{\infty} F^{*(n)}(t_0) \le \sum_{n=1}^{\infty} [F(t_0)]^n < \infty.$$

Hence, from the convergence of $\tau^*_{n,K}$ to $\widehat{\tau}_n$ as $K \to \infty$ and the Monotone Convergence Theorem we see that the right hand side of (29) converges to

$$E\Big[\int_{T_n}^{\hat\tau_n} (A\widetilde{g})(\xi_s)\,ds \,\Big|\, \mathcal{F}_n \Big].$$

Again, applying Dynkin's formula, since $\widehat{\tau}_n < \infty$ a.s. we get

$$E\Big[\int_{T_n}^{\hat{\tau}_n} (A\widetilde{g})(\xi_s)\,ds \,\Big|\, \mathcal{F}_n\Big] = E[\widetilde{g}(\xi_{\hat{\tau}_n}) \mid \mathcal{F}_n] - \widetilde{g}(\xi_{T_n}) \quad \text{a.s.} \tag{31}$$

Hence, we have

$$E[\widetilde{g}(\xi_{\tau^*_{n,K}}) \mid \mathcal{F}_n] \xrightarrow[K\to\infty]{} E[\widetilde{g}(\xi_{\hat{\tau}_n}) \mid \mathcal{F}_n] \quad \text{a.s.} \tag{32}$$

Now, we will prove that $\widehat{\tau}_n$ is optimal in the class $\mathcal{T} \cap \{\tau : \tau \ge T_n\}$. Let τ be any stopping rule from $\mathcal{T} \cap \{\tau : \tau \ge T_n\}$. Then, as $\tau^*_{n,K}$ is optimal in $\mathcal{T}_{n,K}$, we have for any K,

$$E[\widetilde{g}(\xi_{\tau^*_{n,K}})\mathcal{F}_n] \ge E[\widetilde{g}(\xi_{\tau\wedge T_K}) \mid \mathcal{F}_n] \quad \text{a.s.} \tag{33}$$

Hence, a reasoning similar to that which led to (32) gives

$$E[\widetilde{g}(\xi_{\hat{\tau}_n}) \mid \mathcal{F}_n] \ge E[\widetilde{g}(\xi_\tau) \mid \mathcal{F}_n] \quad \text{a.s.,} \tag{34}$$

which completes the proof of (i).

(ii) $E[\widetilde{g}(\xi_{\tau^*_{n,K}}) \mid \mathcal{F}_n] = \mu_n\gamma_{K-n}(U_{T_n}, T_n)$ from Theorem 1(ii). Now, Lemma 4 and (34) give

$$E[\widetilde{g}(\xi_{\tau^*_{n,K}}) \mid \mathcal{F}_n] \xrightarrow[K\to\infty]{} E[\widetilde{g}(\xi_{\hat{\tau}_n}) \mid \mathcal{F}_n] = \mu_n\gamma(U_{T_n}, T_n) \quad \text{a.s.} \ \blacksquare \tag{35}$$

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