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A NOTE ON THE FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR LÉVY'S AREA PROCESS

Abstract. By using large deviation techniques, we prove a Strassen type law of the iterated logarithm, in Hölder norm, for Lévy's area process.

1. Introduction. In the last years there have been several attempts to study the Brownian motion and diffusion processes by endowing the path-space with stronger topologies than the uniform one. For example, Baldi, Ben Arous and Kerkyacharian [2] showed that the large deviations principle for the Brownian motion still holds under the topology induced by any Hölder norm with exponent $\alpha < 1/2$. As a consequence of this result they deduced Strassen's law, in Hölder norm, for the Brownian motion.

The aim of this short note is to prove an analogue of this law for Lévy's area process which is the stochastic analogue of the area contained in a lens-shaped domain. More precisely, let $\xi = \{(\xi_1(t), \xi_2(t)) : t \geq 0\}$ be a 2-dimensional Gaussian process with independent components. Lévy's stochastic area process $L = \{L(t) : t \geq 0\}$ associated with ξ is defined by

$$L(t) = \frac{1}{2}\Big(\int_0^t \xi_1(u)\,\xi_2(du) - \int_0^t \xi_2(u)\,\xi_1(du)\Big), \quad t \geq 0.$$

This process has been thoroughly studied in recent years (see e.g. Ikeda, Kusuoka and Manabe [7] and Chan *et al.* [4]) and plays an important role in the study of various problems in analysis, geometry, mathematical physics and statistics. For example, let $B = \{(B_1(t), B_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion and let ξ be the stationary Gaussian process

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defined by the stochastic differential equation

$$\xi(dt) = A\xi(t)dt + B(dt)$$

and such that $\xi(0)$ is independent of B, where

$$A = \begin{pmatrix} -\theta_1 & -\theta_2 \\ \theta_2 & -\theta_1 \end{pmatrix},$$

$\theta_1 > 0$ and $-\infty < \theta_2 < \infty$ being some unknown parameters to be estimated from the observations $\{\xi(t) : 0 \le t \le T\}$ of ξ until time $T \ge 0$.

Liptser and Shiryaev [9, p. 212] have proved that ξ has independent components and that the maximum likelihood estimates $\widehat{\theta}_1(T,\xi)$ and $\widehat{\theta}_2(\xi,T)$ of θ_1 and θ_2 respectively are given by the equations

$$\frac{1}{\widehat{\theta}_1(T,\xi)} - \widehat{\theta}_1(T,\xi)\int_0^T(\xi_1^2(t)+\xi_2^2(t))\,dt = \int_0^T \xi_1(t)\,\xi_1(dt) + \int_0^T \xi_2\,(t)\,\xi_2(dt),$$

$$\widehat{\theta}_2(T,\xi) = \frac{\int_0^T \xi_1(t)\,\xi_2(dt) - \int_0^T \xi_2(t)\,\xi_1(dt)}{\int_0^T(\xi_1^2(t)+\xi_2^2(t))dt}.$$

Here, we study the asymptotic behaviour of Lévy's area associated with a Brownian motion B. We shall use a recent result of Ben Arous and Ledoux [3] on large deviations, in Hölder norm, for diffusion processes.

The paper is organized as follows: in Section 2 we state the results and in Section 3 we give the proofs. Before closing this section, let us note that the asymptotic behaviour of Lévy's area process via the law of the iterated logarithm can be found in Helmes, Rémillard and Theodorescu [5], N'zi, Rémillard and Theodorescu [10] and Rémillard [11].

2. Strassen's law in Hölder norm for the area process. Let us denote by $\mathcal{C}$ (resp. $\mathcal{C}^\alpha$) the space of all continuous functions $f : [0,1] \to \mathbb{R}^2$ (resp. $f : [0,1] \to \mathbb{R}$) such that $f(0) = 0$ endowed with the uniform (resp. α-Hölder) norm

$$\|f\| = \sup_{t\in[0,1]} |f(t)|, \qquad \Big(\text{resp. } \|f\|_\alpha = \sup_{0\le s,t\le 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha}\Big).$$

For every h in the Cameron–Martin space $\mathcal{H}$, i.e. the space of all absolutely continuous functions null at the origin with square integrable derivatives, we put

$$|h|_{\mathcal{H}}^2 = \int_0^1 |\dot{h}|^2\,ds,$$

where $\dot{h}$ denotes the derivative of h.

For every $A \in \mathcal{C}^\alpha$, we put

$$\Lambda(A) = \begin{cases} \inf\left\{\frac{1}{2}|h|^2_{\mathcal{H}} : h \in \mathcal{H},\ F(h) \in A\right\} & \text{if } F^{-1}(A) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$F(h)(t) = \frac{1}{2}\Big(\int_0^t h_1(u)\dot{h}_2(u)\,du - \int_0^t h_2(u)\dot{h}_1(u)\,du\Big), \qquad h = (h_1, h_2).$$

In particular, for every $g \in \mathcal{C}^\alpha$, we denote $\Lambda(\{g\})$ by $\lambda(g)$. We also set $K = \{g \in \mathcal{C}^\alpha : \lambda(g) \le 1\}$.

For every $u \ge 0$, let us put

$$\phi(u) = \begin{cases} \log\log u & \text{if } u \ge 3, \\ 1 & \text{if } 0 < u < 3, \end{cases}$$

and

$$Z_u = \frac{L(u\,\cdot)}{u\phi(u)}.$$

Now, we state the main result of this paper. From now on, we assume that $0 < \alpha < 1/2$.

THEOREM 2.1. *The process* $\{Z_u : u > 0\}$ *is* P*-a.s. relatively compact and has* K *as set of limit points in the Hölder topology.*

The proof of Theorem 2.1 follows the classical lines in Baldi [1], which consists in proving the two propositions below:

PROPOSITION 2.2. *For every* $\varepsilon > 0$, *there exists* $u^0 > 0$ P*-a.s. such that if* $u > u^0$ *then* $d(Z_u, K) < \varepsilon$, *where*

$$d(g, K) = \inf_{h \in K} \|g - h\|_\alpha.$$

PROPOSITION 2.3. *Let* $g \in K$. *Then, for every* $\varepsilon > 0$, *there exists* $c = c_\varepsilon \in (1, \infty)$ *such that*

$$P(\|Z_{c^j} - g\|_\alpha \le \varepsilon \ i.o.) = 1.$$

3. Proofs of Propositions 2.2 and 2.3. Let us first state a large deviations principle, in Hölder norm, for L, which is an immediate consequence of the main theorem in Ben Arous and Ledoux [3] and the scaling property of L in Helmes and Schwane [6].

THEOREM 3.1. *For every* $A \in \mathcal{C}^\alpha$, *we have*

$$-\Lambda(\mathring{A}) \le \liminf_{\varepsilon\to 0} \varepsilon^2 \log P(\varepsilon^2 L \in A) \le \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(\varepsilon^2 L \in A) \le -\Lambda(\overline{A})$$

where $\mathring{A}$ *and* $\overline{A}$ *are respectively the interior and the closure of* A *in the Hölder topology.*

Proof of Proposition 2.2. We divide the proof in three steps.

Step 1. We first prove that for every $c \in (1,\infty)$ and every $\varepsilon > 0$, there exists $j^0 \in \mathbb{N}$ such that if $j \ge j^0$ then $d(Z_{c^j}, K) < \varepsilon$.

Let $K_\varepsilon = \{g \in \mathcal{C}^\alpha : d(g,K) \ge \varepsilon\}$. In view of the Borel–Cantelli Lemma, we only have to check

$$\sum_j P(Z_{c^j} \in K_\varepsilon) < \infty.$$

By virtue of the scaling property of L, we have

$$P(Z_{c^j} \in K_\varepsilon) = P\left(\frac{L}{\phi(c^j)} \in K_\varepsilon\right). \tag{3.1}$$

Now, let us prove that $\Lambda(K_\varepsilon) > 1$. Since K is compact and λ is lower semicontinuous, there exists $g_0 \in K_\varepsilon$ such that $\lambda(g_0) = \inf_{g\in K_\varepsilon} \lambda(g)$. If $\Lambda(K_\varepsilon) \le 1$ then $\lambda(g_0) \le 1$. Therefore, $g_0 \in K$, which contradicts $g_0 \in K_\varepsilon$.

Let $\delta > 0$ be such that $\Lambda(K_\varepsilon) > 1 + 2\delta$. In view of Theorem 3.1, we have for j large

$$P\left(\frac{L}{\phi(c^j)} \in K_\varepsilon\right) \le \exp(-(1+\delta)\phi(c^j)) = \frac{cte}{j^{1+\delta}},$$

which leads to the conclusion by virtue of (3.1).

Step 2. Now, we want to prove that for every $\varepsilon > 0$, there exists $c_\varepsilon > 1$ such that for every $1 < c < c_\varepsilon$ there exists $j_0 = j_0(\omega)$ such that $Y_j(\omega) \le \varepsilon$ for every $j \ge j_0$, where

$$Y_j = \sup_{c^j \le u \le c^{j+1}} \frac{1}{c^j \phi(c^j)} \|L(u\cdot) - L(c^j\cdot)\|_\alpha.$$

By virtue of the Borel–Cantelli Lemma, we only have to show that

$$\sum_j P(Y_j > \varepsilon) < \infty.$$

By using the scaling property of L we obtain

$$\begin{aligned} P(Y_j \ge \varepsilon) &= P\left(\sup_{c^j \le u \le c^{j+1}} \frac{u}{c^j\phi(c^j)} \|L - L(c^j/u\,\cdot)\|_\alpha \ge \varepsilon\right) \\ &\le P\left(\sup_{c^j \le u \le c^{j+1}} \frac{1}{\phi(c^j)} \|L - L(c^j/u\,\cdot)\|_\alpha \ge \varepsilon/c\right) \end{aligned}$$

$$= P\bigg(\sup_{1\le v\le c} \frac{1}{\phi(c^j)}\|L - L(\cdot/v)\|_\alpha \ge \varepsilon/c\bigg) = P\bigg(\frac{1}{\phi(c^j)}L \in A\bigg)$$

where

$$A = \{g \in \mathcal{C}^\alpha : \sup_{1\le v\le c} \|g - g(\cdot/v)\|_\alpha \ge \varepsilon/c\}.$$

By virtue of Theorem 3.1 and since A is closed, for every $\delta > 0$ and j sufficiently large, we have

$$P\bigg(\frac{1}{\phi(c^j)}L \in A\bigg) \le \exp(-(\Lambda(A) - \delta)\phi(c^j)).$$

It remains to show that we can choose δ such that for c small, $\Lambda(A) > 1 + \delta$. Let $g \in A$ be such that $\lambda(g) < \infty$. There exist $1 \le v \le c$ and $(s,t) \in [0,1]^2$ such that

$$\begin{aligned}(3.2)\qquad \frac{\varepsilon}{c}|t-s|^\alpha &\le |(g(t) - g(t/v)) - (g(s) - g(s/v))| \\ &= \Big|\int\limits_{s\vee(t/v)}^{t} \dot g(u)\,du - \int\limits_{s/v}^{s\wedge(t/v)} \dot g(u)\,du\Big| \\ &\le (|t - s\vee(t/v)|^{1/2} + |s\wedge(t/v) - s/v|^{1/2})\|\dot g\|_{L^2}\end{aligned}$$

where $\dot g$ is the derivative of g.

Now, let $f \in \mathcal{H}$ be such that $\lambda(g) = \frac12|f|^2_{\mathcal{H}}$ and $F(f) = g$. Since it is easy to prove that $\|\dot g\|_{L^2} \le |f|^2_{\mathcal{H}}$, we deduce from (3.2) that

$$\lambda(g) \ge \frac{\varepsilon}{2c}\bigg(\frac{|t-s|^\alpha}{(|t - s\vee(t/v)|^{1/2} + |s\wedge(t/v) - s/v|^{1/2})}\bigg).$$

By virtue of Lemma 3.4 in Baldi [1], we obtain

$$\lambda(g) \ge \frac{\varepsilon}{4c}(c-1)^{\alpha-1/2}.$$

Therefore, we have $\Lambda(A) \ge \frac{\varepsilon}{4c}(c-1)^{\alpha-1/2}$.

Since $\alpha < 1/2$, it follows that for c small we have $\Lambda(A) > 1$, which ends Step 2.

Step 3. For every $c^j \le u \le c^{j+1}$ we have

$$\begin{aligned}(3.3)\qquad d(Z_u, K) \le \bigg\|Z_u - \frac{c^j\phi(c^j)}{u\phi(u)}Z_{c^j}\bigg\|_\alpha \\ + \bigg|1 - \frac{c^j\phi(c^j)}{u\phi(u)}\bigg|\|Z_{c^j}\|_\alpha + d(Z_{c^j}, K).\end{aligned}$$

Let us deal with the right member of (3.3). In view of Step 2, the first term is $\le \frac{1}{3}\varepsilon$. Now, Step 1 implies that $\|Z_{c^j}\|_\alpha$ is bounded for j large. Since

$$\lim_{j\to\infty}\left|1-\frac{c^j\phi(c^j)}{u\phi(u)}\right| = \frac{c-1}{c},$$

for c close to 1 and j large we see that the second term is $\le \frac{1}{3}\varepsilon$. Step 1 also implies that the third term is $\le \frac{1}{3}\varepsilon$. The assertion of Proposition 2.2 follows immediately. ■

Proof of Proposition 2.3. Let $g \in K$ and $f \in \mathcal{H}$ be such that $\frac{1}{2}|f|^2_{\mathcal{H}} = \lambda(g)$ and $F(f) = g$. By virtue of the Proposition in Ben Arous and Ledoux [3] and the scaling property of L, for δ small and j large, we have

$$P\biggl(\|Z_{c^j}-g\|_\alpha > \varepsilon,\ \biggl\|\frac{B(c^j\cdot)}{\sqrt{c^j\phi(c^j)}}-f\biggr\| \le \delta\biggr) \le \exp(-2\phi(c^j)).$$

It follows that

$$\sum_j P\biggl(\|Z_{c^j}-g\|_\alpha > \varepsilon,\ \biggl\|\frac{B(c^j\cdot)}{\sqrt{c^j\phi(c^j)}}-f\biggr\| \le \delta\biggr) < \infty.$$

Now, since there exists $c = c_\varepsilon$ such that

$$P\biggl(\biggl\|\frac{B(c^j\cdot)}{\sqrt{c^j\phi(c^j)}}-f\biggr\| \le \delta \text{ i.o.}\biggr) = 1,$$

we deduce that $P(\|Z_{c^j}-g\|_\alpha \le \varepsilon \text{ i.o.}) = 1$ for $c = c_\varepsilon$. ■

Remarks. (i) Theorem 3.1 gives a stronger result than the law of the iterated logarithm obtained by Helmes, Rémillard and Theodorescu [5].

(ii) Theorem 3.1 can be easily generalized to Brownian functionals $F(B)$ satisfying the following conditions:

(H1) For every $a \ge 0$ the restriction of F to $K_a = \{f \in \mathcal{H} : \frac{1}{2}|f|^2_{\mathcal{H}} \le a\}$ is continuous;

(H2) For every $R > 0, \varrho > 0, a > 0$ there exist $\varepsilon_0 > 0, \beta > 0$ such that for every $f \in K_a$,

$$P(\|F(\varepsilon B)-F(f)\|_\alpha > \varrho,\ \|\varepsilon B - f\| \le \beta) \le \exp(-R/\varepsilon^2);$$

(H3) There exists $\delta > 0$ such that for every $\varepsilon > 0$ and every $(u,t) \in [0,\infty)^2$,

$$F(\varepsilon B(u\cdot))(t) = \varepsilon^\delta F(B)(ut);$$

(H4) For every $\varepsilon > 0$, there exists $c_\varepsilon > 1$ such that for every $1 < c \le c_\varepsilon$, we have $\Lambda(A) > 1$, where

$$A = \{g \in \mathcal{C}^\alpha : \sup_{1\le v\le c}\|g - g(\cdot/v)\|_\alpha \ge \varepsilon/c\}.$$

Let us note that the class of Brownian functionals satisfying (H1)–(H4) contains the iterated stochastic integrals considered by Baldi [1] and the stochastic integrals in Rémillard [11].

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