APPLICATIONES MATHEMATICAE
24,2 (1996), pp. 203–221

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GLOBAL EXISTENCE FOR A ONE-DIMENSIONAL MODEL IN GAS DYNAMICS

Abstract. We prove the existence of a global solution for a one-dimensional Navier–Stokes system for a gas with internal capillarity.

0. Introduction. Several works on Navier–Stokes equations for viscous compressible fluids give global existence and regularity results for small initial data (for an example see Kazhikhov and Shelukin [K-S], Matsumura and Nishida [M-N] and their bibliography). One difficulty lies in the fact that the system is neither parabolic nor hyperbolic, and the regularization (by viscous effects) acts only on the velocity field. Works by Serre [Se1], [Se2] avoid smallness hypotheses on initial data, for Lagrange's variables in one space dimension.

In the present paper we consider a system of partial differential equations describing the adiabatic flow of a one-dimensional viscous gas, with the theory of second gradient taking in account the internal capillarity (Germain [G], Gatignol and Seppecher [Ga-S], Seppecher [S], and Serre [Se3]) and a fourth order (positive) viscosity. We denote by ϱ, u, f the mass density, the velocity field, and a volume force. $\gamma(\in\,]1,2[)$ is the polytropic index of the gas, λ is the (positive) capillarity coefficient, Re, M, ε the Reynolds and Mach numbers, and a fourth order viscosity coefficient (depending on λ).

The system is

$$\varrho_t + (\varrho u)_x = 0, \tag{0.1}$$

$$\varrho(u_t + uu_x) + \frac{1}{\gamma M^2}(\varrho^\gamma)_x - \lambda\varrho\varrho_{xxx} - \frac{1}{Re}u_{xx} + \varepsilon u_{xxxx} = \varrho f \tag{0.2}$$

for $(x,t) \in\,]0,1[\,\times\,]0,T[\ (T \in \mathbb{R}^+_*)$.

1991 *Mathematics Subject Classification*: 35Q30, 76N10, 76N15.
Key words and phrases: Navier–Stokes equations, gas dynamics, capillarity.

We have the following periodic boundary conditions:

$$\varrho(0,t)=\varrho(1,t), \tag{0.3}$$

$$u(0,t)=u(1,t),\quad u_x(0,t)=u_x(1,t),\quad u_{xx}(0,t)=u_{xx}(1,t),\quad u_{xxx}(0,t)=u_{xxx}(1,t), \tag{0.4}$$

and an initial data.

Remark. Introducing λ and ε does not allow us to define a global solution of the classical Navier–Stokes system by passing to the limit (in λ and ε). ■

This paper is divided in three parts. In the first part we define the function spaces and a system derived from (0.1)–(0.2) for which we build a solution by an iterative method. In the second one, we show a local existence result. The capillarity coefficient gives regularity to the mass density, and allows us to avoid smallness hypotheses on initial data. In the third part a uniform Gronwall lemma gives the global existence when the volume force is equal to zero.

I. An iterative method. Let $I=]0,1[$, $J=]0,T[$, and let $f_{\underbrace{x\ldots x}_{k\text{ times}}}$ be the spatial derivative of order k of f. We consider the function spaces (see for example [A] or [B])

$$L^p(I)=\Big\{f:\int_0^1|f(x)|^p\,dx<\infty\Big\},$$

$$W^{m,p}(I)=\{f:\forall k\in[0,m]\cap\mathbb{N},\ f_{\underbrace{x\ldots x}_{k}}\in L^p(I)\},$$

$$H^m(I)=W^{m,2}(I),$$

$$H^m_{\rm per}(I)=\{f\in H^m(I):\forall k\in[0,m-1]\cap\mathbb{N},\ f_{\underbrace{x\ldots x}_{k}}(0)=f_{\underbrace{x\ldots x}_{k}}(1)\},$$

$$\dot L^2(I)=\Big\{f\in L^2(I):\int_0^1 f(x)\,dx=0\Big\},$$

$$\dot H^m_{\rm per}(I)=H^m_{\rm per}(I)\cap\dot L^2(I).$$

In the whole work, the scalar products will be L^2 scalar products and $C_{\rm e}$ will represent all the Sobolev embedding constants. We shall write L^p (resp. H^m) instead of $L^p(I)$ (resp. $H^m(I)$) and we shall denote by $\|\ \|_p$ (resp. $\|\ \|_{H^m}$) the norm in L^p (resp. H^m) and by $\|\ \|_{p,q}$ the norm in $L^p(0,t;L^q)$.

Let ϱ_0 be a positive constant. We make the following change of unknown function: $\varrho(x,t)\to\varrho(x,t)+\varrho_0$. Then (0.1)–(0.2) becomes

$$(1.1)\qquad \varrho_t + ((\varrho+\varrho_0)u)_x = 0,$$

$$(1.2)\qquad (\varrho+\varrho_0)(u_t+uu_x) + \frac{1}{\gamma M^2}((\varrho+\varrho_0)^\gamma)_x - \lambda(\varrho+\varrho_0)\varrho_{xxx} - \frac{1}{Re}u_{xx} + \varepsilon u_{xxxx} = (\varrho+\varrho_0)f.$$

We add to (1.1)–(1.2) the boundary conditions (0.3)–(0.4), and some initial data.

Let k be a positive integer. We consider the linear system

$$(1.3)\qquad (\varrho^k+\varrho_0)(u_t^{k+1}+u^k u_x^{k+1}) - \frac{1}{Re}u_{xx}^{k+1} + \varepsilon u_{xxxx}^{k+1} - (\varrho^k+\varrho_0)f = -l_x^k = -\left(\frac{1}{\gamma M^2}((\varrho^k+\varrho_0)^\gamma) + \frac{\lambda}{2}(\varrho_x^k)^2 - \lambda(\varrho^k+\varrho_0)\varrho_{xx}^k\right)_x,$$

$$(1.4)\qquad \varrho_t^{k+1} + ((\varrho^{k+1}+\varrho_0)(u^{k+1}))_x = 0,$$

with periodic boundary conditions.

We write $(\varrho^k,u^k)=(r,v)$ and $(\varrho^{k+1},u^{k+1})=(\varrho,u)$, with the initial data $(r,v)(x,0)=(\varrho,u)(x,0)=(\varrho_{\rm i},u_{\rm i})$ satisfying

$$(1.5)\qquad 0<\varrho_{\rm m}\le \varrho_{\rm i}(x)+\varrho_0\le\varrho_{\rm M}<\infty$$

for x in $\bar I$ and $(\varrho_{\rm i},u_{\rm i})$ in $\dot H^1_{\rm per}\times\dot L^2$. Let $t_k\in\mathbb{R}_*^+$ and $J_k=]0,t_k[$. We make the following hypothesis: $r(x,t)+\varrho_0>0$ for (x,t) in $\bar I\times\overline{J}_k$, $(r+\varrho_0,r_x)\in(L^\infty(0,t;L^4))^2$ for t in $\overline{J}_k$ and f in L^∞.

The existence result for (1.3)–(1.4) is given in

PROPOSITION 1.1. *Under the previous conditions and for t in $\overline{J}_k$, the problem* (1.3)–(1.4)–(0.3)–(0.4) *has a unique solution which satisfies*

$$(1.6)\qquad ((r+\varrho_0)^{1/2}u,u_x)\in L^\infty(0,t;L^2)\times L^2(0,t;\dot H^1_{\rm per}),$$

$$(1.7)\qquad \varrho(x,t)+\varrho_0\ge 0;$$

moreover, $\varrho+\varrho_0\in L^\infty(0,t;H^1_{\rm per})$.

Proof. The a priori estimates are classical.

(i) We take the scalar product of (1.4) and $u^2/2$, and of (1.3) and u. By the Gagliardo–Nirenberg inequalities, we obtain

$$\frac{d}{dt}(\|(r+\varrho_0)^{1/2}u\|_2^2) + \frac{1}{2Re}\|u_x\|_2^2 + \varepsilon\|u_{xx}\|_2^2 \le \frac{2Re}{(\gamma M^2)^2}\|r+\varrho_0\|_{2\gamma}^{2\gamma} + \frac{9\lambda^2 Re}{2}\|r_x\|_4^4 + \frac{\lambda^2}{\varepsilon}\|r+\varrho_0\|_4^2\|r_x\|_4^2 + \frac{C_{\rm e}^2\varrho_0 Re}{2}\|f\|_\infty^2.$$

Then

$$\|((r+\varrho_0)^{1/2}u)(t)\|_2^2 + \frac{1}{Re}\|u_x\|_{2,2}^2 + \varepsilon\|u_{xx}\|_{2,2}^2 \tag{1.8}$$
$$\le \|(r_{\rm i}+\varrho_0)^{1/2}u_i\|_2^2$$
$$+ \left(\frac{\lambda^2}{2\varepsilon}\|r+\varrho_0\|_{\infty,4}^4 + \frac{\lambda^2}{2\varepsilon}(1+9\varepsilon Re)\|r_x\|_{\infty,4}^4 + \frac{C_{\rm e}^2\varrho_0 Re}{2}\|f\|_\infty^2\right)t$$
$$+ \left(\frac{2Re}{(\gamma M^2)^2}\|(r+\varrho_0)\|_{\infty,2\gamma}^{2\gamma}\right)t$$

and the existence for the velocity field results from the application of the Lions theorem [B].

(ii) For the mass density, we work along the characteristic lines. Let $x(t)$ be a regular solution of $dx/dt = u(x(t),t), x(0) = y$. Then

$$\varrho_0 + \varrho(x(t),t) = (\varrho_0 + \varrho(y,0))\exp\Big(\int_0^t -u_x(x(\tau),\tau)\,d\tau\Big). \tag{1.9}$$

The positivity of the initial data gives the positivity of $\varrho(x,t)+\varrho_0$.

Let $p>1$. We take the scalar product of (1.4) and $(\varrho+\varrho_0)^{p-1}$ (resp. of (1.4) and $(1/(\varrho+\varrho_0))^p$). Setting $g_n(t) = \exp(n\int_0^t \|u_x\|_\infty\,d\tau)$ we easily obtain the estimates

$$\|(\varrho+\varrho_0)(t)\|_p \le \varrho_{\rm M} g_2(t), \tag{1.10}$$
$$\Big\|\frac{1}{\varrho+\varrho_0}(t)\Big\|_p \le \frac{1}{\varrho_{\rm m}} g_2(t). \tag{1.11}$$

We take the spatial derivative of (1.4), and the scalar product of the result and ϱ_x. By the Gronwall lemma we have

$$\|\varrho_x(t)\|_2 \le \Big(\|(\varrho_{\rm i})_x\|_2 + \int_0^t (\|\varrho+\varrho_0\|_\infty \|u_{xx}\|_2)(\tau)\,d\tau\Big) g_{5/2}(t) \tag{1.12}$$

and the proposition follows. ∎

Let $(\varrho_{\rm i}, u_{\rm i}) \in \dot H^3_{\rm per} \times \dot H^2_{\rm per}$ and $v \in L^\infty(0,t;\dot H^1)$. We obtain more regularity in

Proposition 1.2. *The solution of* (1.2)–(1.3)–(0.3) *satisfies*

$$u_x \in L^\infty(0,t;H^1), \qquad \frac{u_{xxxx}}{(r+\varrho_0)^{1/2}} \in L^2(0,t;L^2), \tag{1.13}$$
$$\varrho_{xx} \in L^\infty(0,t;\dot H^1). \tag{1.14}$$

Proof. We take the scalar product of (1.3) and $\frac{1}{\varrho+\varrho_0}\left(\frac{-1}{Re}u_{xx}+\varepsilon u_{xxxx}\right)$ to obtain

$$\frac{d}{dt}\left(\frac{1}{Re}\|u_x\|_2^2+\varepsilon\|u_{xx}\|_2^2\right)+\frac{\varepsilon^2}{4}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_2^2$$
$$\le \frac{4}{\varepsilon^4 Re^4}\left\|\frac{1}{r+\varrho_0}\right\|_\infty^2\|(r+\varrho_0)^{1/2}u\|_2^2+3\|r+\varrho_0\|_\infty\|v\|_4^2\|u_x\|_4^2$$
$$+3\lambda^2\|r+\varrho_0\|_\infty\|r_{xxx}\|_2^2+\frac{3}{M^4}\int_0^1(r+\varrho_0)^{2\gamma-3}dx+C_{\mathrm{e}}^2\|f\|_\infty^2.$$

An integration with respect to time gives

$$\frac{1}{Re}\|u_x\|_2^2+\varepsilon\|u_{xx}\|_2^2+\frac{\varepsilon^2}{4}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_{2,2}^2 \tag{1.15}$$
$$\le\frac{1}{Re}\|(u_{\mathrm{i}})_x\|_2^2+\varepsilon\|(u_{\mathrm{i}})_{xx}\|_2^2+\frac{3}{M^4}\|(r+\varrho_0)^{2\gamma-3}\|_{1,1}$$
$$+\frac{4}{\varepsilon^4Re^4}\left\|\frac{1}{r+\varrho_0}\right\|_{\infty,\infty}^2\|(r+\varrho_0)^{1/2}u\|_{2,2}^2$$
$$+3C_{\mathrm{e}}\|(r+\varrho_0)\|_{\infty,\infty}\|v_x\|_{\infty,2}^2\|u_x\|_{2,2}^2+tC_{\mathrm{e}}^2\|f\|_\infty^2$$
$$+3\lambda^2\|(r+\varrho_0)\|_{\infty,\infty}\|r_{xxx}\|_{2,2}^2.$$

For the mass density, we proceed as for the estimate (1.12) to obtain

$$\|\varrho_{xx}(t)\|_2\le\|\varrho_{\mathrm{i}xx}\|_2 g_{5/2}(t) \tag{1.16}$$
$$+3C_{\mathrm{e}}\left(\int_0^t\|\varrho_x\|_2\|r+\varrho_0\|_\infty^{1/4}\|u_{xx}\|_2^{1/2}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_2^{1/2}d\tau\right)g_{5/2}(t)$$
$$+C_{\mathrm{e}}\left(\int_0^t\|\varrho+\varrho_0\|_\infty\|r+\varrho_0\|_\infty^{1/2}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_2d\tau\right)g_{5/2}(t),$$

$$\|\varrho_{xxx}(t)\|_2\le\|(\varrho_{\mathrm{i}})_{xxx}\|_2 g_{5/2}(t) \tag{1.17}$$
$$+6C_{\mathrm{e}}\left(\int_0^t\|\varrho_{xx}\|_2\|r+\varrho_0\|_\infty^{1/4}\|u_{xx}\|_2^{1/2}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_2^{1/2}d\tau\right)g_{5/2}(t)$$
$$+\left(\int_0^t\left(4\|\varrho_x\|_\infty\|u_{xx}\|_2+\|\varrho+\varrho_0\|_\infty\|r+\varrho_0\|_\infty^{1/2}\left\|\frac{u_{xxxx}}{(r+\varrho_0)^{1/2}}\right\|_2\right)d\tau\right)g_{5/2}(t)$$

and the proposition follows. ∎

The proposition has the following consequence:

Corollary 1.3.

$$\varrho \in C([0,t_k];\dot H^2_{\mathrm{per}}) \cap L^\infty(0,t_k;\dot H^3_{\mathrm{per}}), \tag{1.18}$$

$$(u,u_x) \in C([0,t_k];\dot H^2_{\mathrm{per}}) \times C([0,t_k];\dot H^1_{\mathrm{per}}). \ \blacksquare \tag{1.19}$$

II. Local existence

II.I. *Uniform estimates for the sequence* (ϱ^k,u^k). Let N be an upper bound of

$$\frac{1}{Re}\|u_x^k(t)\|_2^2 + \varepsilon\|u_{xx}^k(t)\|_2^2 + \frac{\varepsilon^2}{4}\left\|\frac{u_{xxxx}^k}{(\varrho^{k-1}+\varrho_0)^{1/2}}(t)\right\|_{2,2}^2$$

and

$$\frac{1}{Re}\|u_x^{k-1}(t)\|_2^2 + \varepsilon\|u_{xx}^{k-1}(t)\|_2^2 + \frac{\varepsilon^2}{4}\left\|\frac{u_{xxxx}^{k-1}}{(\varrho^{k-2}+\varrho_0)^{1/2}}(t)\right\|_{2,2}^2$$

for t in $]0,\min(t_{k-1},t_k)[$. We define

$$b_n(t) = \exp\left(n\left(\frac{C_{\mathrm{e}}^2tNRe}{2\varepsilon}\right)^{1/2}\right).$$

By (1.8), we have the inequalities

$$\int_0^t \|u_x^k\|_\infty\,d\tau \le \left(\frac{C_{\mathrm{e}}^2tNRe}{2\varepsilon}\right)^{1/2},$$

$$\|(\varrho^k+\varrho_0)(t)\|_2 \le \varrho_{\mathrm{M}}b_2(t), \qquad \left\|\frac{1}{\varrho^k+\varrho_0}(t)\right\|_2 \le \frac{1}{\varrho_{\mathrm{m}}}b_2(t),$$

from which we deduce the estimates

$$\|\varrho_{xx}^k(t)\|_2 \le \|(\varrho_{\mathrm{i}})_{xx}\|_2 b_{5/2}(t) + \frac{C_1(tN)^{1/2}}{\varepsilon}(1+(\varepsilon^2\|(\varrho_{\mathrm{i}})_x\|_2)^{2/3})b_{11/2}(t) + \frac{C_1N^{1/2}t}{\varepsilon^{3/2}}(1+Nt^2)b_{25/2}(t) \tag{2.1}$$

and

$$\begin{aligned}\|\varrho_{xxx}^k(t)\|_2 &\le \left(\|(\varrho_{\mathrm{i}})_{xxx}\|_2 + \frac{C_1(tN)^{1/2}}{\varepsilon}(1+\varepsilon(t)^{1/2}\|(\varrho_{\mathrm{i}})_{xx}\|_2^2)\right)b_{7/2}(t)\\ &\quad + \frac{C_1(tN)^{1/2}}{\varepsilon}b_{13/2}(t)\\ &\quad + \frac{C_1t}{N^{1/2}\varepsilon^{7/2}}(1+\varepsilon^2\|(\varrho_{\mathrm{i}})_x\|_2)^{4/3}(1+(tN)^4)(Nt+N\varepsilon^2+\varepsilon)^2b_{30}(t).\end{aligned} \tag{2.2}$$

Here $C_1 > 0$ depends only on $\varrho_{\rm m}$, $\varrho_{\rm M}$ and $C_{\rm e}$. We denote by P_{12} a polynomial of degree 12 of variable tN, and set

$$\varepsilon_1 = \frac{\varepsilon^{4(\gamma-1)}}{\lambda^{8\gamma}M^5Re^2}, \qquad \varepsilon_2 = \frac{1}{M^4}\max\left(\left(\frac{1}{\varrho_{\rm m}}\right)^{2\gamma-3}, (\varrho_{\rm M})^{2\gamma-3}\right),$$
$$\varepsilon_3 = \max\left(\varepsilon_2; \frac{M^4}{Re^2}\left(\frac{\lambda^2}{\varepsilon}\right)^{4\gamma+1}; \frac{\lambda^2(1+Re\varepsilon)}{\varepsilon^4Re^5}; \frac{1}{\varepsilon^7}; \varepsilon^4Re^5\right).$$

Then we deduce the following estimate from (2.1)–(2.2)–(1.15):

$$\frac{1}{Re}\|u_x^{k+1}(t)\|_2^2 + \varepsilon\|u_{xx}^{k+1}(t)\|_2^2 + \frac{\varepsilon^2}{4}\left\|\frac{u_{xxxx}^{k+1}}{(\varrho^k+\varrho_0)^{1/2}}\right\|_{2,2}^2 \tag{2.3}$$
$$\leq \frac{1}{Re}\|(u_{\rm i})_x\|_2^2 + \varepsilon\|(u_{\rm i})_{xx}\|_2^2 + C_1\lambda^2\|(\varrho_{\rm i})_{xxx}\|_2^2\exp\left(7\left(\frac{C_1^2tNRe}{2\varepsilon}\right)^{1/2}\right)$$
$$+ t\varepsilon_1\varepsilon_2\left(1 + \|f\|_\infty^2 + P_{12}(tN)\exp\left(60\left(\frac{C_1^2tNRe}{2\varepsilon}\right)^{1/2}\right)\right).$$

For λ small enough, t (depending on N) small enough (denoted from now on by t_*), and for an initial data smaller than N, the right hand side of (2.3) has N as upper bound. By induction on k we obtain the required upper bound.

II.2. *Convergence of the sequence* (ϱ^k, u^k). Let $(\varrho_{\rm i}, u_{\rm i}) \in \dot H^3_{\rm per} \times \dot H^2_{\rm per}$. We have

PROPOSITION 2.1. *The sequence* (ϱ^k, u^k) *is a Cauchy sequence in* $L^\infty([0,t_*]; \dot H^2_{\rm per}) \times C([0,t_*]; \dot H^1_{\rm per})$.

Proof. We denote by (r^{k+1}, u^{k+1}) the difference $(\varrho^{k+1} - \varrho^k, u^{k+1} - u^k)$. Then (r^{k+1}, u^{k+1}) is a solution of the system

$$(\varrho^k + \varrho_0)(w_t^{k+1} + u^kw_x^{k+1}) - \frac{1}{Re}w_{xx}^{k+1} + \varepsilon w_{xxxx}^{k+1} \tag{2.4}$$
$$= r^kf - (l^k - l^{k-1})_x - r^ku_t^k - (r^ku^k + \varrho^{k-1}w^k)u_x^k$$
$$= h_1^{k+1,k} - h_{2,x}^{k+1,k} - h_3^{k+1,k} - h_4^{k+1,k},$$

$$r_t^{k+1} + (\varrho^{k+1}w^{k+1})_x + (r^{k+1}u^k)_x = 0, \tag{2.5}$$

with periodic boundary conditions and a null initial data. We recall that ϱ^k is a solution of

$$\varrho_t^k + ((\varrho^k + \varrho_0)u^k)_x = 0. \tag{2.6}$$

We proceed as in Proposition 1.1 and so the details will be omitted.

(i) We take the scalar product of (2.4) and w^{k+1}, and of (2.7) and $\frac{1}{2}(w^{k+1})^2$. Then we obtain a differential inequality with left-hand side

$$\frac{1}{2}\frac{d}{dt}(\|(\varrho^k+\varrho_0)^{1/2}w^{k+1}\|_2^2)+\frac{1}{Re}\|w_x^{k+1}\|_2^2+\varepsilon\|w_{xx}^{k+1}\|_2^2.$$

(i.a) For the right-hand side, we have first

$$\int_0^1 h_1^{k+1,1}w^{k+1}\,dx\le C_{\rm e}\|f\|_\infty\|r^k\|_2\|w_x^{k+1}\|_2.$$

(i.b) Then

$$\int_0^1 h_{2,x}^{k+1,k}w^{k+1}\,dx\le\bigg(\frac{1}{M^2}\|\xi+\varrho_0\|_\infty^{\gamma-1}\|r^k\|_2+\lambda(C_{\rm e}\|\varrho_x^k\|_\infty+\|\varrho_{xx}^k\|_2)\|r_x^k\|_2$$
$$+\lambda(\varrho_0+\|\varrho^{k-1}\|_\infty)\|r_{xx}^k\|_2\bigg)\|w_x^{k+1}\|_2$$
$$\le C_1^{1/2}\|r^k\|_{\dot H^2}\|w_x^{k+1}\|_2.$$

(i.c) The term $\int_0^1\frac{r^k}{\varrho^{k-1}+\varrho_0}w^{k+1}u_{xxxx}^k\,dx$ appears in $\int_0^1 h_3^{k+1,k}w^{k+1}\,dx$. An integration by parts gives

$$\int_0^1\frac{r^k}{\varrho^{k-1}+\varrho_0}w^{k+1}u_{xxxx}^k\,dx\le\int_0^1\bigg|\bigg(\frac{r^k}{\varrho^{k-1}+\varrho_0}w^{k+1}\bigg)_{xx}\bigg||u_{xx}^k|\,dx$$

($|\ |$ is the absolute value). Now we use the Hölder inequality and the fact that H^2 is an algebra to obtain an upper bound for $\|(r^kw^{k+1}/(\varrho^{k-1}+\varrho_0))_{xx}\|_2$. Upper bounds for the other terms of $\int_0^1 h_3^{k+1,k}w^{k+1}\,dx$ are classical, and we have

$$\|h_3^{k+1,k}\|_2\|w^{k+1}\|_2\le C_2\|r^k\|_{H^1}\|w_x^{k+1}\|_2+C_3\|r^k\|_{H^2}\|w_{xx}^{k+1}\|_2,$$

where C_2 and C_3 are positive and depend on $N,Re,\varrho_{\rm m},\varrho_{\rm M},\lambda,\varepsilon$; moreover, C_2 (resp. C_3) depends on $\|u_x^k(t)\|_2$ (resp. $\|u_{xx}^k(t)\|_2$), $\int_0^t C_2^2(\tau)\,d\tau\le C_1\|u_x^k\|_{2,2}^2$ and $\int_0^t C_3^2(\tau)\,d\tau\le C_1\|u_{xx}^k\|_{2,2}^2$

(i.d) For the last term we have

$$\int_0^1 h_4^{k+1,k}w^{k+1}\,dx\le C_{\rm e}(\|r^k\|_\infty\|u_x^k\|_2^2+\|u_{xx}^k\|_2\|w_x^k\|_2)\|w_x^{k+1}\|_2.$$

For t in $[0,t_*]$ an integration with respect to time of the differential inequality gives

$$\begin{aligned}(2.8)\qquad &\|((\varrho^k+\varrho_0)^{1/2}(u^{k+1}-u^k))(t)\|_2^2\\ &\quad+\frac{1}{Re}\|(u^{k+1}-u^k)_x\|_{2,2}^2+\varepsilon\|(u^{k+1}-u^k)_{xx}\|_{2,2}^2\\ &\le C_1t\|(u^k-u^{k-1})_x\|_{\infty,2}^2\\ &\quad+C_1t^{1/2}(N^{1/2}+(1+\|f\|_\infty)t^{1/2})\|\varrho^k-\varrho^{k-1}\|^2_{L^\infty(0,t_*;\dot H^2)},\end{aligned}$$

and for t_* small enough, and a positive constant which we denote again C_1, we have the following upper bound for the right-hand side of (2.8):

$$C_1t^{1/2}(\|\varrho^k-\varrho^{k-1}\|^2_{L^\infty(0,t_*;\dot H^2)}+\|(u^k-u^{k-1})_x\|^2_{\infty,2}).$$

(ii) We take the scalar product of (2.4) and $\frac{1}{\varrho^k+\varrho_0}w^k_{xx}$. Proceeding as for the estimate (2.8) we obtain

$$\begin{aligned}(2.9)\qquad &\|(u^{k+1}-u^k)_x\|_2^2+\frac{1}{Re}\left\|\frac{(u^{k+1}-u^k)_{xx}}{(\varrho^k+\varrho_0)^{1/2}}\right\|^2_{2,2}+\varepsilon\left\|\frac{(u^{k+1}-u^k)_{xxx}}{(\varrho^k+\varrho_0)^{1/2}}\right\|^2_{2,2}\\ &\le C_1t^{1/2}(\|\varrho^k-\varrho^{k-1}\|^2_{L^\infty(0,t_*;\dot H^2)}+\|(u^k-u^{k-1})_x\|^2_{\infty,2}).\end{aligned}$$

(iii) We take the second spatial derivative of (2.5) and the scalar product of the result and r^{k+1}_{xx}. Then we obtain a differential inequality whose integration with respect to time gives

$$\begin{aligned}&\|r^{k+1}_{xx}\|_2\exp\Big(-C_{\rm e}\int_0^t\|u^{k+1}_{xx}\|_2\,d\tau\Big)\\ &\le C_{\rm e}(\|\varrho^{k+1}_{xx}\|_{\dot H^1}\|w^{k+1}_x\|_{2,2}+\|\varrho^{k+1}_x\|_{\infty,2}\|w^{k+1}_{xx}\|_{2,2})t^{1/2}\\ &\quad+C_{\rm e}\bigg(\|\varrho^{k-1}+\varrho_0\|^{1/4}_{\infty,\infty}\\ &\qquad\times\bigg(\|u^k_{xx}\|^2_{2,2}+\left\|\frac{u^k_{xxxx}}{(\varrho^{k-1}+\varrho_0)^{1/2}}\right\|^2_{2,2}\bigg)^{1/2}\|r^{k+1}_x\|_{\infty,2}\bigg)t^{1/2}\\ &\quad+C_{\rm e}\bigg(\|\varrho^{k+1}\|_{\infty,\infty}\|\varrho^k+\varrho_0\|^{1/2}_{\infty,\infty}\left\|\frac{w^{k+1}_{xxx}}{(\varrho^k+\varrho_0)^{1/2}}\right\|_{2,2}\bigg)t^{1/2}\end{aligned}$$

and the estimate (for t_* small enough and a positive constant denoted again by C_1)

$$\begin{aligned}(2.10)\qquad \|(\varrho^{k+1}-\varrho^k)_{xx}\|_2^2&\le C_1t^{1/2}\|\varrho^k-\varrho^{k-1}\|^2_{L^\infty(0,t_*;\dot H^2)}\\ &\quad+C_1t^{1/2}\|(u^k-u^{k-1})_x\|^2_{\infty,2}.\end{aligned}$$

We take t_* small enough to have $C_1t_*^{1/2}<1/2$. Then by (2.8)–(2.10), (ϱ^k,u^k) is a Cauchy sequence in $L^\infty([0,t_*];\dot H^2_{\rm per})\times C([0,t_*];\dot H^1_{\rm per})$, and the limit (ϱ,u) is a weak solution of (1.1)–(1.2)–(0.3)–(0.4) in $L^\infty([0,t_*];\dot H^1_{\rm per})\times(L^\infty([0,t_*];L^2)\cap L^2([0,t_*];\dot H^2_{\rm per}))$. Moreover,

$$\varrho(x,t)+\varrho_0\ge 0 \tag{2.11}$$

for (x,t) in $\bar I\times[0,t_*]$. ■

III. Regularity of the solution. In this section, we show the regularity of the solution for f $(\neq 0)$ in L^∞, and the global existence for $f=0$. We take initial data satisfying

$$(\varrho_{\rm i},u_{\rm i})\in \dot H^3_{\rm per}\times \dot H^2_{\rm per}\quad\text{and}\quad \frac{(\varrho_{\rm i})_{xxx}}{\varrho_{\rm i}+\varrho_0}\in L^2. \tag{3.1}$$

III.1. *The case $f\neq 0$.* We have

Lemma 3.1. *The solution satisfies*

$$\begin{aligned}\|(\varrho+\varrho_0)^{1/2}u\|_2^2&+\frac{2}{\gamma(\gamma-1)M^2}\|\varrho+\varrho_0\|_\gamma^\gamma\\&+\lambda\|\varrho_x\|_2^2+\frac{1}{Re}\|u_x\|_{2,2}^2+2\varepsilon\|u_{xx}\|_{2,2}^2\\&\le\|(\varrho_{\rm i}+\varrho_0)^{1/2}u\|_2^2+\frac{2}{\gamma(\gamma-1)M^2}\|\varrho_{\rm i}+\varrho_0\|_\gamma^\gamma\\&+\lambda\|(\varrho_{\rm i})_x\|_2^2+C_{\rm e}\varrho_0Re\|f\|_\infty^2t.\end{aligned}\tag{3.2}$$

Proof. We proceed as for Proposition 1.1 to obtain

$$\begin{aligned}\frac12\frac{d}{dt}\bigg(\|(\varrho+\varrho_0)^{1/2}u\|_2^2+\frac{2}{\gamma(\gamma-1)M^2}\|\varrho+\varrho_0\|_\gamma^\gamma+\lambda\|\varrho_x\|_2^2\bigg)&\\+\frac{1}{Re}\|u_x\|_2^2+\varepsilon\|u_{xx}\|_2^2=\int_0^1(\varrho+\varrho_0)fu\,dx.&\end{aligned}$$

By the Hölder and Gagliardo–Nirenberg inequalities, we have

$$\int_0^1(\varrho+\varrho_0)fu\,dx\le C_{\rm e}\|\varrho+\varrho_0\|_1\|f\|_\infty\|u_x\|_2\le\frac{1}{2Re}\|u_x\|_2^2+C_{\rm e}^2\varrho_0Re\|f\|_\infty^2$$

and the lemma follows from an integration with respect to time. ■

We have more regularity in

Proposition 3.2.

$$\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}},\varrho_{xxx},u_{xx}\in L^\infty(0,t_*;L^2),\tag{3.3}$$

$$\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}},\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\in L^2(0,t_*;L^2).\tag{3.4}$$

Proof. Set

$$a_0 = \|(\varrho_{\mathrm{i}}+\varrho_0)^{1/2}u_{\mathrm{i}}\|_2^2 + \frac{2}{\gamma(\gamma-1)M^2}\|\varrho_{\mathrm{i}}+\varrho_0\|_\gamma^\gamma + \lambda\|(\varrho_{\mathrm{i}})_x\|_2^2,$$

$$a_1 = C_{\mathrm{e}}\varrho_0 Re\|f\|_\infty^2, \qquad a_2 = \exp(a_0),$$

$$a_3 = \frac{C_{\mathrm{e}}^2}{8}\left(\frac{Re}{\varepsilon}\right)^{1/2} + C_{\mathrm{e}}\varrho_0 Re\|f\|_\infty^2.$$

Interpolating L^∞ between L^2 and H^2 gives

$$\int_0^t \|u_x^{k+1}\|_\infty\,d\tau \le \int_0^t \left(\frac{C_{\mathrm{e}}^2}{8}\left(\frac{Re}{\varepsilon}\right)^{1/2} + \frac{1}{Re}\|u_x(\tau)\|_2^2 + \varepsilon\|u_{xx}(\tau)\|_2^2\right)d\tau$$

and (from Lemma 3.1) we obtain

$$\exp\left(\int_0^t \|u_x^{k+1}\|_\infty\,d\tau\right) \le a_2\exp(a_3 t). \tag{3.5}$$

We take the third spatial derivative of (1.1) and the scalar product of the result and $\varrho_{xxx}/(\varrho+\varrho_0)$. Then

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2\right) + 3\int_0^1 \frac{\varrho_{xxx}^2 u_x}{\varrho+\varrho_0}\,dx + 6\int_0^1 \frac{\varrho_{xx}\varrho_{xxx}u_{xx}}{\varrho+\varrho_0}\,dx + 4\int_0^1 \frac{\varrho_x\varrho_{xxx}u_{xxx}}{\varrho+\varrho_0}\,dx = -\int_0^1 \varrho_{xxx}u_{xxxx}\,dx. \tag{3.6}$$

The scalar product of (1.2) and $u_{xxxx}/(\varrho+\varrho_0)$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_{xx}\|_2^2) + \frac{1}{Re}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \varepsilon\left\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 \\ = \int_0^1\left(f - uu_x - \frac{\varrho_x}{M^2(\varrho+\varrho_0)^{2-\gamma}}\right)u_{xxxx}\,dx - \frac{1}{Re}\int_0^1 \frac{\varrho_x u_{xx}u_{xxx}}{(\varrho+\varrho_0)^2}\,dx \\ + \lambda\int_0^1 \varrho_{xxx}u_{xxxx}\,dx.\end{aligned} \tag{3.7}$$

From these two equations, we deduce a differential inequality whose left-hand side is

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{xx}\|_2^2 + \lambda\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2\right) + \frac{1}{Re}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \varepsilon\left\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2$$

and (by the Gagliardo–Nirenberg inequalities) the right-hand side has the following upper bound:

$$\frac{3\varepsilon_1}{Re}\bigg\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2 + 4\varepsilon_2\varepsilon\bigg\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2$$
$$+ \lambda(\varepsilon_3 + 3\|u_x\|_\infty + 10C_{\mathrm{e}}\|u_{xx}\|_2)\bigg\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2 + \frac{\varrho_0^2}{4\varepsilon_2\varepsilon}\|f\|_\infty^2$$
$$+ \frac{C_{\mathrm{e}}^4}{4\varepsilon_2\varepsilon}\|(\varrho+\varrho_0)^{1/2}u\|_2^2\|u_x\|_2\|u_{xx}\|_2 + \frac{1}{4\varepsilon_2\varepsilon M^4}\|\varrho+\varrho_0\|_\infty^{2\gamma-1}\|\varrho_x\|_2^2$$
$$+ \frac{C_{\mathrm{e}}^4}{64\varepsilon_1^3 Re}\|\varrho+\varrho_0\|_\infty\bigg\|\frac{1}{\varrho+\varrho_0}\bigg\|_\infty^6\|\varrho_x\|_2^4\|u_{xx}\|_2^2 + \frac{\lambda^3 C_{\mathrm{e}}^{12}}{64\varepsilon_2^2\varepsilon_3\varepsilon^2}\|\varrho+\varrho_0\|_\infty^2\|\varrho_x\|_2^6.$$

Let us take $\varepsilon_2 = 1/6$ and $\varepsilon_2 = 1/8$.

We denote by a_4 and a_5 real numbers depending on C_{e}, a_2, Re, M, λ, ε, γ, ϱ_{M}, ϱ_{m}, and set $a_6 = \frac{2\varrho_0^2}{\varepsilon}\|f\|_\infty^2$. Finally, we define

$$h_1(t) = C_{\mathrm{e}}\|u_{xx}\|_2 + \varepsilon_3 + \frac{1}{\varepsilon}(a_0 + a_1 t) + a_4(a_0^3 + a_1^3 t^3)\exp(7a_3 t),$$
$$h_2(t) = a_6 + a_5(a_0^3 + a_1^3 t^3)\exp(3a_3 t).$$

There exist positive functions g_1 and g_2 depending on time so that

$$\int_0^t h_1(\tau)\,d\tau \le \frac{t^{1/2}}{2}g_1(t), \qquad \int_0^t h_2(\tau)\,d\tau \le \frac{t}{2}g_2(t).$$

The differential inequality is

$$\text{(3.8)}\qquad \frac{d}{dt}\bigg(\|u_{xx}\|_2^2 + \lambda\bigg\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2\bigg) + \frac{2}{Re}\bigg\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2$$
$$+ 2\varepsilon\bigg\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2 \le 2\bigg(\|u_{xx}\|_2^2 + \lambda\bigg\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\bigg\|_2^2\bigg)h_1(t) + 2h_2(t)$$

and (by the Gronwall lemma) we obtain

$$\text{(3.9)}\qquad \|u_{xx}(t)\|_2^2 + \lambda\bigg\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}(t)\bigg\|_2^2$$
$$\le \|(u_{\mathrm{i}})_{xx}\|_2^2\exp(t^{1/2}g_1(t)) + \bigg(\lambda\bigg\|\frac{(\varrho_{\mathrm{i}})_{xxx}}{(\varrho_i+\varrho_0)^{1/2}}\bigg\|_2^2 + tg_2(t)\bigg)\exp(t^{1/2}g_1(t))$$

for $t \in [0, t_*]$. A direct integration of (3.9) with respect to time gives (3.4).

Now we take the third spatial derivative of (1.1) and the scalar product of the result and ϱ_{xxx}. Then the estimate (3.4) and an integration with respect to time show that ϱ_{xxx} belongs to $L^\infty(0, t_*; L^2)$. This completes the proof of the proposition. ■

III.2. *The case $f = 0$.* We have to show the existence of uniform (with respect to time) and strictly positive bounds of $\varrho(x,t) + \varrho_0$. We define $\varrho_{\min} = \min_{x\in I}(\varrho_0 + \varrho_{\mathrm{i}}(x))$ with $0 < \varrho_{\min} < \varrho_0$, and

$$f(\varrho) = \left(\frac{2\lambda}{\gamma(\gamma-1)(\gamma+2)^2M^2}\right)^{1/2}(\varrho_0^{(\gamma+2)/2} - (\varrho+\varrho_0)^{(\gamma+2)/2}).$$

Let us make the following hypothesis:

(H) We suppose the existence of $\mu_1, \mu_2, \mu_3, \mu_4 (\in\,]0,1[)$ so that

$$\sum_{i=1}^{4}\mu_i \le 1,$$
$$\frac{1}{2}\|(\varrho_{\mathrm{i}}+\varrho_0)^{1/2}u_i\|_2^2 \le \mu_1 f(\varrho_{\min}-\varrho_0),$$
$$\frac{2^\gamma}{\gamma(\gamma-1)M^2}\|\varrho_{\mathrm{i}}\|_\gamma^\gamma \le \mu_2 f(\varrho_{\min}-\varrho_0),$$
$$\frac{2^\gamma\varrho_0^\gamma}{\gamma(\gamma-1)M^2} \le \mu_3 f(\varrho_{\min}-\varrho_0),$$
$$\frac{\lambda}{2}\|(\varrho_{\mathrm{i}})_x\|_2^2 \le \mu_4 f(\varrho_{\min}-\varrho_0).$$

We have

LEMMA 3.3.

(3.10) $$\varrho(t)+\varrho_0 \in C^0(\bar I),$$

(3.11) $$\varrho(x,t)+\varrho_0 \ge \varrho_{\min} > 0$$

uniformly for t in $\mathbb{R}^+$. ■

Proof. (i) For (3.10) we proceed as in Lemma 3.1 to obtain

(3.12) $$\frac{1}{\gamma(\gamma-1)M^2}\|(\varrho+\varrho_0)(t)\|_\gamma^\gamma + \frac{\lambda}{2}\|\varrho_x(t)\|_2^2 \le \frac{1}{2}\|(\varrho_{\mathrm{i}}+\varrho_0)^{1/2}u_i\|_2^2 + \frac{1}{\gamma(\gamma-1)M^2}\|\varrho_{\mathrm{i}}+\varrho_0\|_\gamma^\gamma + \lambda\|(\varrho_{\mathrm{i}})_x\|_2^2$$

from which we deduce the estimates (uniform with respect to time)

(3.13) $$\varrho(t)+\varrho_0 \in L^\gamma, \qquad \varrho_x(t)\in L^2.$$

The first one results from the embedding $W^{1,1}(I) \to C^0(\bar I)$, and we shall write $\varrho_{\max} = \|\varrho+\varrho_0\|_\infty$.

(ii) For (3.11) we follow a proof by Eden–Milani–Nicolaenko [E-M-N] for which we need the hypothesis (H). We define $\varrho_*(t) = \min_{x\in I}(\varrho_0 + \varrho(x,t))$. Then $\varrho_*(t) \le \varrho_0$ (else, we should have $\varrho_0 = \int_0^1(\varrho(x,t)+\varrho_0)\,dx = \|\varrho(t)+\varrho_0\|_1 > \varrho_*(t) > \varrho_0$, which is impossible). Since $\varrho(\cdot,t)$ is continuous on

$\bar{I}$, there exist $a_m(t)$ and $x_m(t)$ in $\bar{I}$ so that $\varrho(a_m(t),t) = 0$, $\varrho(x_m(t),t) = \varrho_*(t) - \varrho_0$, and $f(\varrho(a_m(t),t)) = 0$. We can write

$$f(\varrho(x_m(t),t)) - f(\varrho(a_m(t),t)) = \int\limits_{\varrho(a_m(t),t)}^{\varrho(x_m(t),t)} \frac{\partial f}{\partial \varrho}\, d\varrho.$$

Then

$$f(\varrho(x_m(t),t)) = -\left(\frac{\lambda}{2\gamma(\gamma-1)(\gamma+2)^2M^2}\right)^{1/2} \int\limits_{a_m(t)}^{x_m(t)} (\varrho+\varrho_0)^{\gamma/2}\varrho_z\, dz.$$

Now we have

$$\begin{aligned}(3.14)\qquad & 0 \le f(\varrho(x_m(t),t)) \\ & \le \left(\frac{\lambda}{2\gamma(\gamma-1)(\gamma+2)^2M^2}\right)\|\varrho(t)+\varrho_0\|_\gamma^{\gamma/2}\|\varrho_x(t)\|_2 \le f(\varrho_{\min}-\varrho_0).\end{aligned}$$

Therefore $f(\varrho)$ is nonnegative, nonincreasing and has a strictly positive upper bound. Then we obtain the estimate

$$\varrho_{\min} \le \varrho(\cdot,t) + \varrho_0 \tag{3.15}$$

and the proof of the lemma follows. ■

THEOREM 3.4. *In the case* $f = 0$ *the problem* (1.1)–(1.2)–(0.3)–(0.4) *with initial data satisfying* (3.1) *has a unique, global and regular solution.*

Proof.

(i) *Regularity of the solution.* The scalar product of (1.4) and $-\varrho_{xxx}$ gives

$$\begin{aligned}(3.16)\qquad & \int\limits_0^1 ((\varrho+\varrho_0)u_t)_x\varrho_{xx}\, dx + \frac{1}{M^2}\int\limits_0^1 (\varrho+\varrho_0)^{\gamma-1}\varrho_{xx}\, dx \\ & \qquad + \lambda\int\limits_0^1 (\varrho+\varrho_0)\varrho_{xxx}^2\, dx \\ & = -\frac{\gamma-1}{M^2}\int\limits_0^1 (\varrho+\varrho_0)^{\gamma-2}\varrho_x^2\varrho_{xx}\, dx \\ & \qquad + \int\limits_0^1 (\varrho+\varrho_0)uu_x\varrho_{xxx}\, dx - \frac{1}{Re}\int\limits_0^1 u_{xx}\varrho_{xxx}\, dx \\ & \qquad + \varepsilon\int\limits_0^1 \varrho_{xxx}u_{xxxx}\, dx.\end{aligned}$$

Define

$$0 < \alpha \le \frac{1}{4C_e^2 \varrho_{\max}} \left(\frac{\lambda+\varepsilon}{\varrho_0}\right)^{1/2}. \tag{3.17}$$

Combining (3.6)–(3.7) and (3.15) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_{xx}\|_2^2 + (\lambda+\varepsilon\alpha)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + 2\alpha\int_0^1 ((\varrho+\varrho_0)u)_x \varrho_{xx}\,dx\right) \\
&\quad + \frac{\alpha(2-\gamma)(\gamma-1)}{3M^2}\left\|\frac{\varrho_x}{(\varrho+\varrho_0)^{(3-\gamma)/4}}\right\|_2^2 + \frac{1}{M^2}\|(\varrho+\varrho_0)^{(\gamma-1)/2}\varrho_{xx}\|_2^2 \\
&\quad + \lambda\|(\varrho+\varrho_0)^{1/2}\varrho_{xxx}\|_2^2 + \frac{1}{Re}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \varepsilon\left\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 \\
&= -3(\lambda+\alpha\varepsilon)\int_0^1 \frac{\varrho_{xxx}^2 u_x}{\varrho+\varrho_0}\,dx - 6(\lambda+\alpha\varepsilon)\int_0^1 \frac{\varrho_{xx}\varrho_{xxx}u_{xx}}{\varrho+\varrho_0}\,dx \\
&\quad - \int_0^1 u u_x u_{xxxx}\,dx - \frac{1}{M^2}\int_0^1 (\varrho+\varrho_0)^{\gamma-2}\varrho_x u_{xxxx}\,dx - \frac{1}{Re}\int_0^1 \frac{\varrho_x u_{xx} u_{xxx}}{(\varrho+\varrho_0)^2}\,dx \\
&\quad - \alpha\int_0^1 u\varrho_{xx}^2\,dx + \alpha\int_0^1 (((\varrho+\varrho_0)u)_{xx})^2\,dx - \frac{\alpha}{Re}\int_0^1 u_{xx}\varrho_{xxx}\,dx \\
&\quad - 3\alpha\int_0^1 u u_x \varrho_x \varrho_{xx}\,dx - \alpha\int_0^1 (\varrho+\varrho_0)u u_{xx}\varrho_{xx}\,dx - \alpha\int_0^1 (\varrho+\varrho_0)u_x^2\varrho_{xx}\,dx \\
&\quad - \alpha\int_0^1 (\varrho+\varrho_0)\varrho_{xx} u u_x\,dx - 4(\lambda+\alpha\varepsilon)\int_0^1 \frac{\varrho_x \varrho_{xxx} u_{xxx}}{\varrho+\varrho_0}\,dx.
\end{aligned} \tag{3.18}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be positive numbers to be defined later. We add

$$(\varepsilon_1 - (\varepsilon_2\|u_x\|_2^2 + \varepsilon_3\|u_{xx}\|_2^2))\left(2\alpha\int_0^1 ((\varrho+\varrho_0)u)_x\varrho_{xx}\,dx\right)$$

to each side of the previous equation. By the choice of α, we have

$$\begin{aligned}
0 &\le \frac{1}{2}\left(\|u_{xx}\|_2^2 + (\lambda+\varepsilon\alpha)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2\right) \\
&< \|u_{xx}\|_2^2 + (\lambda+\varepsilon\alpha)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 - 2\alpha\int_0^1 (\varrho+\varrho_0)u\varrho_{xxx}\,dx \\
&< \frac{3}{2}\left(\|u_{xx}\|_2^2 + (\lambda+\varepsilon\alpha)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2\right).
\end{aligned} \tag{3.19}$$

R e m a r k 3.5. The quantity

$$\left(\|u_{xx}\|_2^2 + (\lambda+\alpha\varepsilon)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 - 2\alpha\int_0^1 (\varrho+\varrho_0)u\varrho_{xxx}\,dx\right)^{1/2}$$

is a norm on $\dot{H}^3_{\mathrm{per}} \times \dot{H}^2_{\mathrm{per}}$ equivalent to $(\|u_{xx}\|_2^2 + \|\varrho_{xxx}/(\varrho+\varrho_0)^{1/2}\|_2^2)^{1/2}$. ■

Now, by Lemma 3.3, for $u \in H^4_{\mathrm{per}}$, the use of Poincaré's inequality gives

$$\|u\|_2 \le C_{\mathrm{e}}\|u_x\|_2 \le C_{\mathrm{e}}^2\|u_{xx}\|_2, \tag{3.20}$$

$$\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 > \frac{1}{2}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \frac{1}{2C_{\mathrm{e}}^2\varrho_{\max}^2}\|u_{xx}\|_2^2, \tag{3.21}$$

$$\|(\varrho+\varrho_0)^{1/2}\varrho_{xxx}\|_2^2 > \frac{1}{2}\|(\varrho+\varrho_0)^{1/2}\varrho_{xxx}\|_2^2 + \frac{\varrho_{\min}^2}{2}\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2, \tag{3.22}$$

$$\|(\varrho+\varrho_0)^{(\gamma-1)/2}\varrho_{xx}\|_2^2 \le C_{\mathrm{e}}\varrho_{\max}^{\gamma}\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2. \tag{3.23}$$

Set

$$X = \|u_{xx}\|_2^2 + (\lambda+\alpha\varepsilon)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2, \quad Y = -2\alpha\int_0^1(\varrho+\varrho_0)u\varrho_{xxx}\,dx.$$

By (3.20)–(3.22) and the Gagliardo–Nirenberg inequalities, we can get an upper bound of the right-hand side of (3.18). Let a_7, a_8, a_9 be positive numbers depending on Re, M, λ, ε, $\varrho_{\min}$, $\varrho_{\max}$, and

$$a_{11} = \inf\left(\frac{1}{Re\varrho_{\max}}, \frac{\lambda\alpha}{(\lambda+\alpha\varepsilon)\varrho_{\max}}\right).$$

After some calculations we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(X+Y) &+ (a_{11} - (a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2))X \\ &+ (\varepsilon_1 - (\varepsilon_2\|u_x\|_2^2 + \varepsilon_3\|u_{xx}\|_2^2))Y \\ &+ \frac{(\lambda+\varepsilon\alpha)}{2}\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \frac{\varepsilon}{2}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|2^2 \\ &\le a_9 + (\varepsilon_1 - (\varepsilon_2\|u_x\|_2^2 + \varepsilon_3\|u_{xx}\|_2^2))Y.\end{aligned} \tag{3.24}$$

Now we use the following lemma whose proof will be given later.

LEMMA 3.6. *The right-hand side of* (3.24) *has the following upper bound*:

$$\begin{aligned}&\frac{\lambda+\alpha\varepsilon}{4}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \left(\frac{a_{11}}{2} + a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2\right)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 \\ &\quad + \frac{2\alpha^4\varepsilon_1^2a_0^2\varrho_{\max}^6C_{\mathrm{e}}^2}{(\lambda+\alpha\varepsilon)\varrho_{\min}^2}\left(\frac{\varepsilon_2^2}{a_8^2} + \frac{\varepsilon_3^2}{a_8^2}\right) + \frac{4\alpha^2\varepsilon_1a_0\varrho_{\max}^2}{a_{11}^2} + a_9.\end{aligned} \tag{3.25}$$

We take $\varepsilon_1 = a_{11}/2$, $\varepsilon_2 = 3a_7/2$, $\varepsilon_3 = 3a_8/2$. By Lemma 3.5, the inequality (3.24) reads

$$\frac{1}{2}\frac{d}{dt}(X+Y) + \frac{1}{2}(a_{11} - 3(a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2))(X+Y) \le a_9 + \frac{2\alpha^2 a_0 \varrho_{\max}^2}{a_{11}} + \frac{9\alpha^4 a_{11}^2 a_0^2 \varrho_{\max}^6 C_{\rm e}^2}{8(\lambda+\alpha\varepsilon)\varrho_{\min}^2} = \frac{a_{10}}{2}. \tag{3.26}$$

The use of the estimate (3.12) allows us to establish the existence of positive numbers c_1, c_2, τ_0 so that

$$0 < c_1 \le \frac{1}{\tau_0}\int_t^{t+\tau_0} (a_{11} - 3(a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2))\,d\tau \le c_2. \tag{3.27}$$

The assumptions of the uniform Gronwall lemma [G-T] are satisfied, and we obtain

$$0 < (X+Y)(t) \le \left((X+Y)(0) - \frac{a_{10}}{c_1}\right)\exp(2\tau_0(c_1+c_2) - c_1 t) + \frac{a_{10}}{c_1}\exp(2\tau_0(c_1+c_2)). \tag{3.28}$$

(ii) *Global existence.* We can define an upper bound of the right-hand side of (3.27) (denoted again by $a_{10}/2$) so that $(X+Y)(0) < a_{10}/c_1$, and a positive number

$$t_* = 2\tau_0(c_1+c_2) + \ln\left(\frac{a_{10}}{c_1} - (X+Y)(0)\right) - \ln\left(\frac{a_{10}\exp(2\tau_0(c_1+c_2))}{c_1} - (X+Y)(0)\right). \tag{3.29}$$

By Remark 3.5 and the estimate (3.28), for R positive and $(\varrho_{\rm i}, u_{\rm i})$ in $B_{\dot H^3_{\rm per}\times \dot H^2_{\rm per}}(0,R)$, we have $(\varrho, u)(t)$ in $B_{\dot H^3_{\rm per}\times \dot H^2_{\rm per}}(0,R)$ for t in $[0, t_*]$. Then we have the same estimate for t in $[t_*, 2t_*]$ and by induction, we obtain the estimate for t in $\mathbb{R}_+$, and the assertion of the theorem. ∎

It remains to show Lemma 3.6. It is easy to establish the following estimate:

$$Y \le 2\alpha\int_0^1 (\varrho+\varrho_0)^{3/2}|u|\left|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right| dx \le 2\alpha\|\varrho+\varrho_0\|_\infty \|(\varrho+\varrho_0)^{1/2}u\|_2 \left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2.$$

By Lemma 3.3 we have

$$
\begin{aligned}
&(\varepsilon_1 - (\varepsilon_2\|u_x\|_2^2 + \varepsilon_3\|u_{xx}\|_2^2))Y \\
&\qquad \le 2\alpha a_0 \varrho_{\max}(\varepsilon_1 + (\varepsilon_2\|u_x\|_2^2 + \varepsilon_3\|u_{xx}\|_2^2))\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2.
\end{aligned}
$$

We use Hölder's inequality to obtain the following upper bound for the right-hand side:

$$
\begin{aligned}
(3.30)\qquad &\left(\frac{a_{11}}{2} + a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2\right)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \frac{4\alpha^2\varepsilon_1 a_0 \varrho_{\max}^2}{a_{11}^2} \\
&\qquad + \alpha^2\varepsilon_1 a_0\varrho_{\max}^2\left(\frac{\varepsilon_2^2}{a_7}\|u_x\|_2^2 + \frac{\varepsilon_3^2}{a_8}\|u_{xx}\|_2^2\right).
\end{aligned}
$$

Interpolation gives

$$
\|u_x\|_2, \|u_{xx}\|_2 \le \frac{C_{\mathrm{e}}^{1/2}\varrho_{\max}^{1/2}}{\varrho_{\min}}\left\|\frac{u_{xxxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^{1/2}.
$$

Thus (3.30) has the following upper bound:

$$
\begin{aligned}
&\frac{\lambda+\alpha\varepsilon}{4}\left\|\frac{u_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 + \left(\frac{a_{11}}{2} + a_7\|u_x\|_2^2 + a_8\|u_{xx}\|_2^2\right)\left\|\frac{\varrho_{xxx}}{(\varrho+\varrho_0)^{1/2}}\right\|_2^2 \\
&\qquad + \frac{2\alpha^4\varepsilon_1^2 a_0^2\varrho_{\max}^6 C_{\mathrm{e}}^2}{(\lambda+\alpha\varepsilon)\varrho_{\min}^2}\left(\frac{\varepsilon_2^2}{a_8^2} + \frac{\varepsilon_3^2}{a_8^2}\right) + \frac{4\alpha^2\varepsilon_1 a_0\varrho_{\max}^2}{a_{11}^2}
\end{aligned}
$$

and the conclusion of the lemma follows. ∎

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Received on 7.12.1995;
revised version on 15.2.1996