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ON THE LIMIT DISTRIBUTIONS OF kTH ORDER STATISTICS FOR SEMI-PARETO PROCESSES

Abstract. Asymptotic properties of the kth largest values for semi-Pareto processes are investigated. Conditions for convergence in distribution of the kth largest values are given. The obtained limit laws are represented in terms of a compound Poisson distribution.

1. Introduction. Pillai [5] has discussed semi-Pareto processes, of which Pareto processes form a proper sub-class. He has examined asymptotic properties of the maximum and minimum of the first n observations. We here obtain conditions for convergence in distribution of the kth largest values for semi-Pareto processes.

We say that a random variable X has *semi-Pareto distribution* and write $X \sim P_S(\alpha, p)$ if its survival function is of the form

$$\overline{F}_X(x) = 1 - F_X(x) = P(X > x) = \frac{1}{1+\psi(x)}, \quad x \ge 0, \tag{1}$$

where $\psi(x)$ satisfies the functional equation

$$\psi(x) = \frac{1}{p}\psi(p^{1/\alpha}x),$$

where $\alpha > 0$ and $0 < p < 1$.

The *autoregressive semi-Pareto model* ARSP(1) is built using a sequence of independent identically distributed (i.i.d.) random variables in the following manner ([5]). Let $\{\varepsilon_n, n \ge 1\}$ be i.i.d. $P_S(\alpha, p)$ random variables and

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for each $n = 1, 2, \ldots$ define

$$X_n = \begin{cases} p^{-1/\alpha} X_{n-1} & \text{with probability } p, \\ \min(p^{-1/\alpha} X_{n-1}, \varepsilon_n) & \text{with probability } 1-p. \end{cases} \tag{2}$$

The process defined by (2) will be called an ARSP(1) process.

The ARSP(1) process is clearly Markovian. If the initial distribution is $X_0 \sim P_S(\alpha, p)$, then $X_n \sim P_S(\alpha, p)$ and the process is strictly stationary.

In particular, if $\{\varepsilon_n, n \geq 1\}$ is a sequence of i.i.d. random variables with common distribution of the Pareto form

$$P(\varepsilon_1 > x) = [1 + (x/\sigma)^{1/\gamma}]^{-1}, \qquad x \geq 0, \tag{3}$$

where $\sigma > 0$ and $\gamma > 0$, we obtain the *autoregressive Pareto* (ARP(1)) *process* ([7]).

2. Level crossing processes. Let $\{X_n, n \geq 1\}$ be an ARSP(1) process. For each $n \geq 1$, let $M_n^{(1)} \geq M_n^{(2)} \geq \ldots \geq M_n^{(n)}$ be the order statistics of $X_1, \ldots, X_n$. The problem is to study the limiting behaviour of the kth order statistics $M_n^{(k)}$ for any fixed $k \geq 1$ as $n \to \infty$. The asymptotic distribution of $M_n^{(k)}$ will be obtained by considering the number of exceedances of a level x by $X_1, \ldots, X_n$.

For any $x > 0$, we define the *level crossing process* $Z_n(x)$ associated with $\{X_n\}$ by

$$Z_n(x) = \begin{cases} 1 & \text{if } X_n > x, \\ 0 & \text{if } X_n \leq x, \end{cases} \tag{4}$$

(cf. [1, 5, 7]). The two-state stochastic process $\{Z_n(x), n \geq 1\}$ turns out to be a Markov chain with transition matrix

$$P = \frac{1}{1+\psi(x)} \begin{bmatrix} p + \psi(x) & 1-p \\ (1-p)\psi(x) & 1 + p\psi(x) \end{bmatrix}.$$

The obvious relation

$$P(M_n^{(k)} \leq x) = P\Big(\sum_{j=1}^{n} Z_j(x) < k\Big), \qquad -\infty < x < \infty, \tag{5}$$

will play a role in this paper.

3. Asymptotic distributions of kth order statistics. Suppose that, for $\tau > 0$, there exists a sequence $\{u_n = u_n(\tau)\}$ such that

$$\lim_{n\to\infty} n\overline{F}_X(u_n(\tau)) = \tau, \tag{6}$$

where $\overline{F}_X$ is given by (1).

We shall investigate properties of the random variable

$$S_n(\tau) = \sum_{j=1}^{n} Z_j(u_n(\tau))$$

for some fixed $\tau > 0$, as $n \to \infty$, and as consequences, we shall obtain limiting distributional results for the kth order statistics. The main tool is a result which gives conditions for the convergence in distribution of sums of 0-1 Markov chains to a compound Poisson distribution (cf. [2, 4, 6]).

THEOREM 1. *Let* $\{Y_{n,j}, j = 1, \ldots, n\}$, $n = 1, 2, \ldots$, *be a sequence of two-state* 0 *and* 1 *homogeneous Markov chains*, *with transition matrices*

$$\begin{bmatrix} 1-(1-\pi)\varrho_n & (1-\pi)\varrho_n \\ (1-\pi)(1-\varrho_n) & (1-\pi)\varrho_n + \pi \end{bmatrix}, \tag{7}$$

where $0 \le \varrho_n \le 1$ *and* $0 \le \pi \le 1$, *and initial probabilities*

$$P(Y_{n,1} = 1) = 1 - P(Y_{n,1} = 0) = \varrho_n.$$

If

$$\lim_{n\to\infty} n\varrho_n = \lambda, \qquad \lambda > 0,$$

then for $k = 0, 1, 2, \ldots$,

$$\lim_{n\to\infty} P\Big(\sum_{j=1}^{n} Y_{n,j} = k\Big) = T(k, (1-\pi)\lambda, 1-\pi),$$

where

$$T(k,\lambda,r) = \begin{cases} e^{-\lambda} & \text{for } k = 0, \\ \displaystyle\sum_{m=1}^{k} C_{k-1}^{m-1}(1-r)^{k-m} r^m \frac{\lambda^m}{m!} e^{-\lambda} & \text{for } k = 1, 2, \ldots \end{cases} \tag{8}$$

Therefore, *the limit law for* $\sum_{j=1}^{n} Y_{n,j}$ *is of the compound Poisson type.*

Consider now, for some $\tau > 0$, the sequence of Markov chains

$$Y_{n,j} = Z_j(u_n(\tau)), \qquad j = 1, \ldots, n. \tag{9}$$

The transition matrices for the sequence (9) are of the form (7) with

$$\pi = p, \qquad \varrho_n = \frac{1}{1 + \psi(u_n(\tau))}.$$

Note that, by the condition (6), we have

$$\lim_{n\to\infty} n\varrho_n = \lim_{n\to\infty} n\overline{F}_X(u_n(\tau)) = \tau > 0. \tag{10}$$

Finally, it follows from Theorem 1 that

$$\lim_{n\to\infty} P(S_n(\tau) = k) = T(k, (1-p)\tau, 1-p), \qquad k = 0, 1, 2, \ldots \tag{11}$$

We shall use the results (11) to study the limit laws for the kth order statistics of semi-Pareto processes.

THEOREM 2. *Let* $\{X_n, n \geq 1\}$ *be a strictly stationary* $ARSP(1)$ *process. Suppose that*, *for* $\tau > 0$, *there exists a sequence* $\{u_n(\tau), n \geq 1\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\psi(u_n(\tau)) = \frac{1}{\tau}, \tag{12}$$

where ψ *is given by* (1). *Then*, *for each* $k = 0, 1, 2, \ldots$,

$$\lim_{n\to\infty} P(M_n^{(k)} \leq u_n(\tau)) = \sum_{j=0}^{k-1} T(j, (1-p)\tau, 1-p), \tag{13}$$

where the function $T(k, \lambda, r)$ *is defined by* (8).

Proof. From (5) we have

$$P(M_n^{(k)} \leq u_n(\tau)) = P\Big(\sum_{j=1}^{n} Z_j(u_n(\tau)) < k\Big), \quad k = 1, \ldots, n,$$

where $Z_j(x)$ are defined by (4). Thus, by (10)–(12), we obtain the desired result.

The case $k = 1$ of Theorem 2 shows that

$$\lim_{n\to\infty} P(M_n^{(1)} \leq u_n(\tau)) = \exp(-(1-p)\tau), \tag{14}$$

In particular, if $\{X_n, n \geq 1\}$ is a strictly stationary Pareto process with $\overline{F}_{X_n}$ given by (3), then we have $\psi(x) = (x/\sigma)^{1/\gamma}$, and hence (12) holds with $u_n(\tau) = \sigma n^\gamma x$, $\tau = x^{-1/\gamma}$, $x > 0$. Thus, from (14) we obtain the result which is due to Yeh *et al.* ([7], Equation (3.8)):

$$P(M_n^{(1)} \leq \sigma n^\gamma x) = \begin{cases} \exp(-(1-p)x^{-1/\gamma}) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

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