APPLICATIONES MATHEMATICAE
24,2 (1996), pp. 127–140

V. G. GELFREICH (Barcelona)

CONJUGATION TO A SHIFT AND THE SPLITTING OF INVARIANT MANIFOLDS

Abstract. We give sufficient conditions for a diffeomorphism in the plane to be analytically conjugate to a shift in a complex neighborhood of a segment of an invariant curve. For a family of functions close to the identity uniform estimates are established.

As a consequence an exponential upper estimate for splitting of separatrices is established for diffeomorphisms of the plane close to the identity. The constant in the exponent is related to the width of the analyticity domain of the limit flow separatrix. Unlike the previous works the cases of non-area-preserving maps and parabolic fixed points are included.

1. Introduction. Normal forms provide a useful instrument for the investigation of dynamical systems. Traditionally the normal forms are studied in a neighborhood of a fixed point, periodic trajectory or other completely invariant object. On the contrary, we study the dynamics of diffeomorphisms in a neighborhood of a segment of an invariant curve far from fixed points. All trajectories leave this neighborhood in a finite time (number of iterations). We provide sufficient conditions for a diffeomorphism to be analytically conjugate to a shift $(t, E) \mapsto (t + h, E)$ in such a neighborhood. For a family of diffeomorphisms close to the identity the estimates of the conjugating map are uniform with respect to a small parameter ($\varepsilon = h$).

As in the paper [FS90], we apply the normal form to the study of the splitting of invariant manifolds associated with a fixed point. For a family close to the identity the splitting is exponentially small with respect to the parameter [Nei84], i.e., it is $O(e^{-\mathrm{const}/\varepsilon})$, provided there exists a homoclinic

1991 *Mathematics Subject Classification*: 34C37, 58F36, 30D05, 34C35.
Key words and phrases: separatrix splitting, finite-difference equation, normal form.
Supported by a grant from the Interministerial Commission of Science and Technology (Spain).

orbit for all small $\varepsilon \neq 0$. The constant in the estimate is related to the position of the singularity in the complex time of the corresponding limit flow [FS90].

We remove the restrictions which appeared in the paper [FS90] due to the use of the Birkhoff normal form: we do not assume the map to be area-preserving and the fixed point to be hyperbolic.

The present approach to the problem was inspired by Lazutkin's papers [Laz84] and [Laz91]. Namely, in the normal form coordinates one of the invariant manifolds is represented by $E = 0$, an the other is given by a graph of an ε-periodic function $E = E(t)$. The Fourier series argument shows that all Fourier coefficients of this function (except the zero one) are exponentially small provided the function $E(t)$ is analytic in an ε-independent complex strip of the variable t. The presence of a homoclinic orbit implies that the zero Fourier coefficient is also exponentially small.

We postpone the exact formulations and proofs of these results to Sections 3 and 4, and first explain the analytic theory of linear finite-difference equations, which provides a basis for the proofs.

2. On solutions of linear finite-difference equations

2.1. *Solutions of the difference equation* $\Delta_h a = g$. In this section we study the equation

$$\Delta_h a \equiv a(t+h) - a(t) = g(t) \tag{1}$$

in the class $\mathcal{A}(\overline{\Omega})$ of functions analytic in a rectangle

$$\Omega = \{t \in \mathbb{C} : |\operatorname{Re} t| < r_1,\ |\operatorname{Im} t| < r_2\}$$

and continuous in its closure. We will use the supremum norm for this space.

The equation (1) is a first order linear finite-difference equation with respect to the function a. Its general solution can be represented as the sum of a particular solution and an arbitrary h-periodic function. The particular solution can be easily found in the class of smooth functions by a partition of unity. As we will see below the analytical case is not so simple.

The basic idea [Laz91] is to represent the rectangle as

$$\Omega = \Omega_+ \cap \Omega_-,$$

where

$$\Omega_\pm = \{t \in \mathbb{C} : \pm \operatorname{Re} t > -r_1,\ |\operatorname{Im} t| < r_2\},$$

and reduce the problem (1) in $\overline{\Omega}$ to the pair of problems in $\overline{\Omega}_\pm$.

Let $\mathcal{L}$ be the space of all complex-valued Lipschitz functions defined on $\partial\Omega$ which take constant values for $\operatorname{Re} t_1 < -r_1/2$ and $\operatorname{Re} t_1 > r_1/2$ (given a function, left and right values are not necessarily equal). We provide this space with the norm

$$\|\chi\| = \max_t |\chi(t)| + \sup_{t_1 \neq t_2} \frac{|\chi(t_1) - \chi(t_2)|}{|t_1 - t_2|}.$$

LEMMA 1 ([Laz91]). *Let* $\chi \in \mathcal{L}$ *and* $g \in \mathcal{A}(\overline{\Omega})$. *Then the function*

$$h(t) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\chi(\xi) g(\xi)}{\xi - t}\, d\xi$$

is analytic in $\mathbb{C} \setminus \operatorname{supp} \chi$, *has continuous continuations on* $\partial\Omega$ *from inside and from outside*, *and*

$$|h(t)| \le \sup_{t \in \overline{\Omega}} |g| \cdot \|\chi\| (1 + r_1 + r_2). \tag{2}$$

PROPOSITION 2. *There is a continuous linear operator* $\Delta_h^{-1} : \mathcal{A}(\overline{\Omega}) \to \mathcal{A}(\overline{\Omega})$ *such that* $a = \Delta_h^{-1} g$ *is a solution of the equation* (1), *and*

$$\|\Delta_h^{-1}\| \le m_\Omega h^{-1},$$

where the constant m_Ω *depends only on the size of the rectangle* Ω.

Proof. Let χ, $0 \le \chi \le 1$, be a real smooth function of the real argument, such that $\chi(s) = 0$ for $s \le -r_1/2$ and $\chi(s) = 1$ for $s \ge r_1/2$. Let

$$\chi_-(t) = \chi(\operatorname{Re} t) \quad \text{and} \quad \chi_+(t) = 1 - \chi(\operatorname{Re} t).$$

Given $g \in \mathcal{A}(\overline{\Omega})$, the functions

$$g_\pm(t) = \frac{1}{2\pi i \cosh(\varrho t)} \int_{\partial\Omega} \frac{\chi_\pm(\xi) \cosh(\varrho\xi) g(\xi)}{\xi - t}\, d\xi,$$

where $\varrho = r_2^{-1}$, are analytic in $\Omega_\pm$ respectively, have continuous continuations on their closures,

$$g(t) = g_+(t) + g_-(t), \qquad t \in \Omega,$$

and for $t \in \Omega_\pm$ the following estimates hold:

$$|g_\pm(t)| \le \frac{\|\chi_\pm\|(1 + r_1 + r_2) \max_{\xi \in \overline{\Omega}} |\cosh(\varrho\xi)| \cdot \|g\|}{\cosh(\varrho t)} \le \frac{C_{r_1 r_2 \chi} \|g\|}{\cosh(\varrho t)}.$$

Direct substitution shows that the functions

$$a_-(t) = \sum_{k=1}^{\infty} g_-(t - kh) \quad \text{and} \quad a_+(t) = -\sum_{k=0}^{\infty} g_+(t + kh)$$

provide solutions for the equations

$$\Delta_h a_+ = g_+ \quad \text{and} \quad \Delta_h a_- = g_-,$$

respectively. Then we obtain the desired solution

$$a(t) = a_-(t) + a_+(t)$$

of the equation (1). We have the estimate

$$|a(t)| \le \sum_{k=-\infty}^{\infty} \frac{C_{r_1 r_2 \chi}\|g\|}{|\cosh(\varrho(t+kh))|}.$$

Since for $t \in \Omega$,

$$\sum_{k=-\infty}^{\infty} \frac{1}{|\cosh(\varrho(t+kh))|} \le h^{-1} \max_{|\sigma|\le r_2} \int_{i\sigma-\infty}^{i\sigma+\infty} \frac{dt'}{|\cosh(\varrho t')|}$$

$$\le h^{-1} r_2 |\cosh(r_2^{-1} r_1)| \int_{-\infty}^{\infty} \frac{dt}{\sqrt{\sinh^2(t)+1/2}} \le \widetilde{C}_{r_1 r_2}$$

we can take $m_\Omega = C_{r_1 r_2 \chi}\widetilde{C}_{r_1 r_2}$ and the proof is complete. ∎

2.2. *The method of variation of parameters.* In this section we develop a formal theory of systems of two finite-difference equations

(3) $$\vec{u}(t+h) = A(t)\vec{u}(t) + \vec{g}(t).$$

This system can be reduced to a pair of first order linear difference equations described in the previous section in the following way. Let $\vec{u}_1$ and $\vec{u}_2$ be two linearly independent solutions of the homogeneous equation

(4) $$\vec{u}_k(t+h) = A(t)\vec{u}_k(t), \qquad k = 1, 2.$$

Then a solution of the nonhomogeneous equation can be represented in the form

(5) $$\vec{u}(t) = c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)$$

with

(6) $$\Delta_h c_1(t) = \frac{\det(\vec{g}(t); \vec{u}_2(t+h))}{W(t+h)},$$

(7) $$\Delta_h c_2(t) = \frac{\det(\vec{u}_1(t+h); \vec{g}(t))}{W(t+h)},$$

where

(8) $$W(t) = \det(\vec{u}_1(t); \vec{u}_2(t)).$$

Indeed, substituting (5) into the equation (3) gives

$$c_1(t+h)\vec{u}_1(t+h) + c_2(t+h)\vec{u}_2(t+h)$$
$$= A(t)(c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)) + \vec{g}(t)$$
$$= c_1(t)\vec{u}_1(t+h) + c_2(t)\vec{u}_2(t+h) + \vec{g}(t).$$

We gather the terms containing c_k on the left hand side:

$$(\vec{u}_1(t+h); \vec{u}_2(t+h)) \begin{pmatrix} \Delta_h c_1(t) \\ \Delta_h c_2(t) \end{pmatrix} = \vec{g}(t).$$

This system has the determinant equal to $W(t+h)$, and is equivalent to (6) and (7). Conversely, given a solution $\vec{u}(t)$ of the system (3), we can represent it in the form (5) taking

$$c_1(t) = \frac{\det(\vec{u}(t); \vec{u}_2(t))}{W(t)}, \qquad c_2(t) = \frac{\det(\vec{u}_1(t); \vec{u}(t))}{W(t)}.$$

In general it is not easy to find two linearly independent solutions of the system. But in many cases one can find one solution $\vec{u}_1(t)$, and then the second solution can be easily constructed.

First we note that

$$(\vec{u}_1(t+h); \vec{u}_2(t+h)) = A(t)(\vec{u}_1(t); \vec{u}_2(t))$$

and we have

$$W(t+h) = \det(A(t)) \cdot W(t). \tag{9}$$

Provided $\det(A(t)) \neq 0$ this equation can be replaced using the substitution

$$W(t) = \exp w(t) \tag{10}$$

by the first order difference equation

$$\Delta_h w(t) = \log \det(A(t)). \tag{11}$$

In the previous section we developed a method for solving the equations of this form in the class of functions analytic in a rectangle. Note that if $\det A(t) = 1$ the equation (11) is trivial, and we can take $W(t) \equiv 1$.

Using $W(t)$ we can construct the second solution of the homogeneous equation (4). The first equation of the system (4) reads

$$u_{21}(t+h) = A_{11}(t)u_{21}(t) + A_{12}(t)u_{22}(t). \tag{12}$$

The second subscript in $u_{kl}(t)$ refers to the number of the component of a vector $\vec{u}_k(t)$, and $A_{ik}(t)$ denotes the ik-component of the matrix $A(t)$. Using (8) in the form

$$u_{22}(t) = \frac{W(t) + u_{12}(t)u_{21}(t)}{u_{11}(t)} \tag{13}$$

we can eliminate the second component of the vector $\vec{u}_2(t)$:

$$u_{21}(t+h) = A_{11}(t)u_{21}(t) + A_{12}(t)\frac{W(t) + u_{12}(t)u_{21}(t)}{u_{11}(t)}.$$

Taking into account that $u_{11}(t)$ also satisfies the equation (12) we can rewrite the last equation as

$$u_{21}(t+h) = \frac{u_{11}(t+h)}{u_{11}(t)}u_{21}(t) + \frac{A_{12}(t)W(t)}{u_{11}(t)}. \tag{14}$$

The corresponding homogeneous equation has a solution $u_{11}(t)$ and we again use the variation of parameters:

$$u_{21}(t) = c_0(t)u_{11}(t). \tag{15}$$

Then

$$c_0(t+h)u_{11}(t+h) = u_{11}(t+h)c_0(t) + \frac{A_{12}(t)W(t)}{u_{11}(t)}$$

and we have

$$\Delta_h c_0(t) = \frac{A_{12}(t)W(t)}{u_{11}(t)u_{11}(t+h)}. \tag{16}$$

Thus we reduce the problem of construction of the second solution for the homogeneous system to the standard form of the single first order difference equation. The components of the vector $\vec{u}_2$ can be obtained by (15) and (13).

2.3. *Uniform estimates for solutions of the system of two equations.* In this section we obtain uniform estimates for a solution of the system (3). Namely, we assume that the matrix A depends on the parameter h and is h-close to the identity:

$$A(t;h) = I + hB(t;h), \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

All the functions are assumed to be analytic in the rectangle $\Omega \subset \mathbb{C}$ and continuous in its closure. We make the following assumptions:

A1. The matrix B is uniformly bounded, i.e., there is a constant M_B such that

$$|B_{ik}(t;h)| \le M_B, \qquad i,k = 1,2. \tag{17}$$

A2. There is a uniformly bounded solution $\vec{u}_1(t;h)$ of the homogeneous equation (4) with the first component uniformly separated from zero, i.e., there are positive constants M_{u_1} and m_u such that

$$|u_{11}(t;h)|, |u_{12}(t;h)| \le M_{u_1}, \qquad |u_{11}(t;h)| \ge m_u > 0. \tag{18}$$

PROPOSITION 3. *There is a constant h_0 such that for $0 < h < h_0$ there exist a uniformly bounded solution $\vec{u}_2(t;h)$ of the homogeneous equation* (4), *a constant M_W such that*

$$M_W^{-1} \le |\det(\vec{u}_1;\vec{u}_2)| \le M_W,$$

and a continuous linear operator $L : (\mathcal{A}(\overline{\Omega}))^2 \to (\mathcal{A}(\overline{\Omega}))^2$, with the norm bounded by $\|L\| \le M_L h^{-1}$, such that $\vec{u} = L\vec{g}$ is a solution of the nonhomogeneous system (3).

Proof. Let $h_0 = \min\{M_{u_1}^{-1} m_u, 1\}/(3M_B)$. Due to Proposition 2 there is a solution of the equation (11) in $\mathcal{A}(\overline{\Omega})$ such that

$$|w(t;h)| \le m_\Omega h^{-1} \|\log\det A(t;h)\|.$$

Then (10) implies that

$$M_W^{-1} \le |W(t;h)| \le M_W$$

with

$$M_W = \exp(m_\Omega h^{-1} \|\log\det A(t;h)\|) \le \exp(24 m_\Omega M_B).$$

The last inequality is valid due to the following chain of inequalities:

$$\begin{aligned}\left|\frac{\log\det A(t,h)}{h}\right| &\le \left|\frac{\log(1-2M_Bh-2M_B^2h^2)}{h}\right| \\ &\le \frac{2M_Bh+2M_B^2h^2}{1-2M_Bh-2M_B^2h^2} \le 24M_B \quad \text{for } 0<h<h_0,\end{aligned}$$

where we used the fact that $|\ln(1-x)| \le x/(1-x)$ for $x \in (0,1)$.

Applying again the operator Δ_h^{-1} from Proposition 2 we obtain the solution of the equation (16) such that

$$|c_0(t;h)| \le m_\Omega \frac{M_B M_W}{m_u(m_u - 2hM_BM_{u_1})} \le m_\Omega \frac{3M_BM_W}{m_u^2}.$$

Then we construct the second solution of the homogeneous equation (4) by the formulae (15) and (13). They are obviously bounded by

$$\begin{aligned}|u_{21}(t;h)| &\le m_\Omega \frac{3M_BM_W}{m_u^2} M_{u_1}, \\ |u_{22}(t;h)| &\le \frac{M_W}{m_u} + m_\Omega \frac{3M_BM_W}{m_u^2} M_{u_1}.\end{aligned}$$

Thus we obtain a constant M_{u_2} such that

$$|u_{21}(t;h)|, |u_{22}(t;h)| \le M_{u_2}.$$

The right hand sides of the equations (6) and (7) contain functions calculated at the point $t+h$, which can be outside Ω. To estimate these values we iterate once the corresponding equations and denote the new constants by $\widetilde{M}_W$, $\widetilde{M}_{u_1}$ and $\widetilde{M}_{u_2}$, respectively.

Applying the operator Δ_h^{-1} to the equations (6) and (7) we obtain the solutions of these equations such that

$$|c_1(t;h)| \le 2m_\Omega h^{-1} \widetilde{M}_W \widetilde{M}_{u_2} \|g\|, \qquad |c_2(t;h)| \le 2m_\Omega h^{-1} \widetilde{M}_W \widetilde{M}_{u_1} \|g\|.$$

Finally, the equation (5) gives the desired solution of the equation (3), which obviously can be estimated by

$$\|\vec{u}(t;h)\| \le |c_1(t;h)| \cdot \|\vec{u}_1\| + |c_2(t;h)| \cdot \|\vec{u}_2(t;h)\| \le 4m_\Omega h^{-1} \widetilde{M}_W M_{u_1} M_{u_2} \|g\|.$$

This immediately implies the desired estimate for the norm of the operator L. ∎

3. Conjugation to the shift. Let $\vec{x}(t;h)$ be a solution of the finite-difference equation

$$\vec{x}(t+h;h) = \vec{f}(\vec{x}(t;h);h)$$

analytic in a neighborhood of the rectangle Ω. If the following hypotheses are fulfilled the map $\vec{f}$ is conjugate to a shift in a neighborhood of $\vec{x}(\Omega;h)$:

H1. $\vec{f}(\vec{x};h) = \vec{x} + h\vec{g}(\vec{x};h)$, where the function $\vec{g}(\vec{x};h)$ and its derivatives up to the second order are uniformly bounded in a parameter independent neighborhood of $\vec{x}(\Omega;h)$.

H2. $\vec{x}(t;h)$ has uniformly bounded derivative with respect to t and the first component of the derivative is bounded away from zero in Ω.

H3. There is a positive constant m_1 such that

$$\|\vec{x}(t_1;h) - \vec{x}(t_2;h)\| \geq m_1|t_1 - t_2|, \quad \forall t_1, t_2 \in \Omega.$$

THEOREM 4. *Under the above conditions there is a positive constant h_0 such that for $0 < h < h_0$ there exists a one-parameter analytic family of solutions of the equation*

$$\vec{x}(t+h, E; h) = \vec{f}(\vec{x}(t, E; h); h) \tag{19}$$

such that $\vec{x}(t,0;h) = \vec{x}(t;h)$, the substitution $(t,E) \mapsto \vec{x}(t,E;h)$ is a diffeomorphism of $\Omega \times \{E \in \mathbb{C} : |E| < E_0\}$ onto its image where E_0 is a positive constant independent on h; the derivatives of this diffeomorphism are bounded uniformly with respect to h.

Remark 1. The proof also works for an individual diffeomorphism not necessarily close to the identity.

Remark 2. If the map is area-preserving the substitution $(t,E) \mapsto \vec{x}(t,E;h)$ can be chosen to preserve area. Indeed, the Jacobian of the substitution $J(t,E;h)$ is h-periodic in t. Then we obtain an area-preserving substitution introducing a new parameter $\widetilde{E}$ instead of E by the formula $\widetilde{E} = \int_0^E J(t,E';h)\,dE'$ (comp. with [Laz91]).

Proof of Theorem 4. For simplicity of notation we drop the explicit dependence of the functions on the parameter h. On the other hand, all constants are chosen to be independent on h. We look for the family of solutions to the equation (19) in the form

$$\vec{x}(t,E) = \vec{x}(t) + \vec{u}(t,E).$$

We substitute this into the equation and rewrite it to separate the part linear in $\vec{u}$:

$$\begin{aligned}\vec{u}(t+h,E) &= D\vec{f}(\vec{x}(t))\vec{u}(t,E) \\ &\quad + (\vec{f}(\vec{x}(t,E)) - \vec{f}(\vec{x}(t)) - D\vec{f}(\vec{x}(t))\vec{u}(t,E)).\end{aligned} \tag{20}$$

Since the function

$$\vec{u}_1(t) = \frac{d\vec{x}}{dt}(t)$$

is a solution of the homogeneous linear equation

$$\vec{u}(t+h) = D\vec{f}(\vec{x}(t))\vec{u}(t)$$

we can apply the operator L, which is the inverse of the linear part, in order to obtain the equation in the form

$$\vec{u}(t,E) = \vec{N}(\vec{u}).$$

Here the nonlinear operator $\vec{N}$ is defined by

$$\vec{N}(\vec{u})(t,E) = E\vec{u}_2(t) + L(\vec{f}(\vec{x}(t,E)) - \vec{f}(\vec{x}(t)) - D\vec{f}(\vec{x}(t))\vec{u}(t,E)),$$

where $\vec{u}_2(t)$ is the second solution of the homogeneous linear equation defined in Proposition 3. A fixed point of the operator $\vec{N}$ is a solution of the equation (20). To find the fixed point of $\vec{N}$ we consider the ball $\|\vec{u}\| < r$ in the space $(\mathcal{A}(\overline{\Omega}))^2$. It is clear that

$$\|\vec{N}(\vec{u})\| \le C_1|E| + C_2\|\vec{u}\|^2,$$

where $C_1 = \|\vec{u}_2\|$ and $C_2 = \|L\| \cdot \|D^2\vec{f}\|$ and

$$\|D^2\vec{f}\| = \max_{k=1,2} \left\| \begin{pmatrix} \frac{\partial^2 f_k}{\partial x_1 \partial x_1} & \frac{\partial^2 f_k}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f_k}{\partial x_2 \partial x_1} & \frac{\partial^2 f_k}{\partial x_2 \partial x_2} \end{pmatrix} \right\|.$$

Provided $|E| < r/(2C_1)$ and $r < 1/(2C_2)$ the ball is invariant with respect to the nonlinear operator $\vec{N}$. Taking $\|\vec{u}\|, \|\vec{v}\| \le r$ we obtain

$$\|\vec{N}(\vec{u}) - \vec{N}(\vec{v})\| \le C_3 r \|\vec{u} - \vec{v}\|$$

with $C_3 = 2\|L\| \cdot \|D^2\vec{f}\|$. The nonlinear operator is a contraction provided $r < 1/(2C_3)$.

Then the iteration procedure

$$\vec{u}^0 = 0, \qquad \vec{u}^n = \vec{N}(\vec{u}^{n-1}), \quad n \ge 1,$$

is uniformly convergent provided

$$2C_1|E| < r < \min\left\{\frac{1}{2C_2}, \frac{1}{2C_3}\right\}$$

and the limit is unique. In particular, these conditions are satisfied for

$$|E| \le |E_0| = \frac{1}{4C_1} \min\left\{\frac{1}{2C_2}, \frac{1}{2C_3}\right\}.$$

This procedure also converges in the smaller ball $r = 4|E|C_1$ if $|E| < E_0$, which implies

$$\vec{u}(t,E) = E\vec{u}_2(t) + O(E^2)$$

and

$$\vec{x}(t,E) = \vec{x}(t) + E\vec{u}_2(t) + O(E^2).$$

Consequently,

$$\left.\frac{\partial \vec{x}}{\partial E}\right|_{E=0} = \vec{u}_2, \qquad \left.\frac{\partial \vec{x}}{\partial t}\right|_{E=0} = \vec{u}_1$$

Since $\det(\vec{u}_1;\vec{u}_2)$ is bounded away from zero, the substitution $(t,E) \mapsto \vec{x}(t,E)$ is a diffeomorphism in a small neighborhood of the point $(t,0)$. The implicit function theorem provides a uniform estimate for the size of this neighborhood. To check that this map is a global diffeomorphism we suppose that $\vec{x}(t_1,E_1) = \vec{x}(t_2,E_2)$ for different values of the arguments. Then we note that

$$\begin{aligned} &|\vec{x}(t_1,E_1) - \vec{x}(t_2,E_2)| \\ &\qquad \ge |\vec{x}(t_1) - \vec{x}(t_2)| - |\vec{x}(t_1,E_1) - \vec{x}(t_1)| - |\vec{x}(t_2,E_2) - \vec{x}(t_2)| \\ &\qquad \ge m_1|t_1 - t_2| - \sup\left|\frac{\partial \vec{x}}{\partial E}\right|(|E_1| + |E_2|). \end{aligned}$$

Consequently, the supposition implies the inequality

$$|t_1 - t_2| \le \sup\left|\frac{\partial \vec{x}}{\partial E}\right| \cdot \frac{2E_0}{m_1}.$$

Decreasing E_0 if necessary, we obtain a contradiction, since it was proved that the map is a diffeomorphism in the neighborhood of $(t,E) = (t_1,0)$. ∎

4. Application to the splitting of separatrices. We assume that the diffeomorphism $\vec{f}(\vec{x};\varepsilon) = \vec{x} + \varepsilon\vec{g}(\vec{x};\varepsilon)$ has two invariant manifolds associated with a fixed point, and these manifolds can be parameterized by analytic solutions $\vec{x}_-(t;\varepsilon)$ and $\vec{x}_+(t;\varepsilon)$, respectively, of the system of finite-difference equations

$$\vec{x}(t+\varepsilon;\varepsilon) = \vec{x}(t;\varepsilon) + \varepsilon\vec{g}(\vec{x}(t;\varepsilon);\varepsilon). \tag{21}$$

We also assume that the segments corresponding to $t \in \Omega$ are ε-close for some constant r_1 and $r_2 \equiv \varrho$. In particular, this can happen if the invariant manifolds are close to the separatrix $\vec{x}_0(t)$ of the system of differential equations

$$\frac{d\vec{x}_0}{dt} = \vec{g}(\vec{x}_0;0),$$

and the "unperturbed" separatrix $\vec{x}_0(t)$ is analytic and bounded in the strip $|\mathrm{Im}(t)| < \varrho$.

THEOREM 5. *Let* $0 < \varepsilon < \varepsilon_0$. *If there is a homoclinic point, e.g.,* $\vec{x}_-(t_1) = \vec{x}_+(t_2)$ *for some* $t_1(\varepsilon)$, $t_2(\varepsilon)$, $|t_1(\varepsilon)|, |t_2(\varepsilon)| < r_1$, *then the splitting of invariant manifolds is exponentially small with respect to* ε.

That is, there is an analytic coordinate system (t, E), *with derivatives bounded uniformly in* ε, *such that in these coordinates one invariant manifold has the equation* $E = 0$ *and the other*

$$E = \Theta_0(\varepsilon) + \Theta_1(\varepsilon) \sin \frac{2\pi t}{\varepsilon} + O(e^{-4\pi(\varrho-\delta)/\varepsilon})$$

with $|\Theta_0(\varepsilon)|$, $|\Theta_1(\varepsilon)| \le \mathrm{const} \cdot e^{-2\pi(\varrho-\delta)/\varepsilon}$.

Remark 3. The coefficients $\Theta_0(\varepsilon)$ and $\Theta_1(\varepsilon)$ can be identically zero, so, generally speaking, the theorem gives only an upper bound for the splitting.

Proof of Theorem 5. We can assume that the curve $\vec{x}_-(t)$, $t \in \Omega$, satisfies the assertions of Theorem 4. Then in an ε-independent neighborhood of this segment there are coordinates (t, E) such that $x_-(t)$ corresponds to $E = 0$ and the map in these coordinates is the shift $(t, E) \mapsto (t + \varepsilon, E)$. The functions

$$T(s) = t(\vec{x}_+(s)) - s, \qquad \Theta(s) = E(\vec{x}_+(s))$$

are ε-periodic analytic functions in the strip $|\mathrm{Im}\, s| \le \varrho - \delta'$ for an arbitrary positive constant δ'.

It is easy to check using a Fourier series argument that if an ε-periodic function $f(s)$ is analytic in the strip $|\mathrm{Im}\, s| < b$, then its Fourier coefficients can be estimated as follows:

$$|f_k| \le e^{-|k|2\pi b/\varepsilon} \sup_{|\mathrm{Im}\, s| < b} |f(s)|.$$

This means that all Fourier coefficients are exponentially small except the zero one. Thus we have the representation of the second invariant manifold in the form

$$E = \Theta_0(\varepsilon) + \Theta_1(\varepsilon) \sin \frac{2\pi(s - t_0(\varepsilon))}{\varepsilon} + O(e^{-4\pi(\varrho-\delta')/\varepsilon}),$$
$$t = s + T_0(\varepsilon) + O(e^{-2\pi(\varrho-\delta')/\varepsilon}),$$

and these estimates can be differentiated with respect to s. The function $\Theta(s)$ has a zero:

$$\Theta(t_2) = E(\vec{x}_+(t_2)) = E(\vec{x}_-(t_1)) = 0.$$

In particular, this implies that $|\Theta_0|$ is exponentially small with respect to ε, otherwise the function Θ would have no zeros.

The implicit function theorem implies that the invariant manifold can be represented as $E = E(t)$. Making the change of variables $(t, E) \mapsto (t + t_0(\varepsilon) - T_0(\varepsilon), E)$ we obtain the desired representation of the manifold. ∎

5. Discussion. The theorems from the last two sections can be applied to a wide class of diffeomorphisms close to identity. In this section we give several examples, and compare the results of the present paper with the results of other authors.

In the paper [FS90] the following theorem was proved. Let $F_\varepsilon : U \to \mathbb{R}^2$ be a family of diffeomorphisms with $U \subset \mathbb{R}^2$ and $0 < \varepsilon < \varepsilon_0$ having the form

$$F_\varepsilon(\vec{x}) = \Lambda\vec{x} + \varepsilon^\alpha \vec{f}(\vec{x}) + \varepsilon^{\alpha+1}\vec{g}(\vec{x},\varepsilon) \tag{22}$$

with $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\vec{f}(0) = \vec{g}(0,\varepsilon) = 0$, $D\vec{f}(0) = D_x\vec{g}(0,\varepsilon) = 0$, $\alpha > 0$ and satisfying

- F_ε preserves area,
- F_ε is real-analytic in U and depends analytically on ε,
- $\lambda = 1 + a\varepsilon^\alpha + O(\varepsilon^{\alpha+1})$, $a \neq 0$.

Then the origin is a hyperbolic fixed point of F_ε (if ε_0 is small enough). Let W^{u} and W^{s} be the corresponding invariant manifolds.

- For all $\varepsilon \in (0,\varepsilon_0)$ there is a homoclinic point, $\vec{q}_\varepsilon$, associated with the origin such that the pieces of W^{u} and W^{s} from the origin to $\vec{q}_\varepsilon$ are contained in a compact subset of U.

Under these hypotheses the vector field given by

$$\dot{x}_1 = x_1 + \frac{1}{a} f_1(x_1, x_1), \qquad \dot{x}_2 = -x_2 + \frac{1}{a} f_2(x_1, x_2),$$

is conservative, has the origin as a hyperbolic point and has an analytic homoclinic orbit $\vec{\sigma}$ such that for ε small enough, the real invariant manifolds of F_ε are ε-close to $\vec{\sigma}(\mathbb{R})$.

Let $\vec{\sigma}$ be analytic and bounded by a constant in a strip $\Pi_\varrho = \{t \in \mathbb{C} : |\mathrm{Im}(t)| < \varrho\}$ and $\vec{f}$, $\vec{g}$ be analytic in a neighborhood of $\vec{\sigma}(\Pi_\varrho)$.

The main result of [FS90] was that under these conditions the splitting distance can be bounded from above by $O(e^{-2\pi(\varrho-\delta)/\log\lambda})$ for any $\delta > 0$.

In [FS90] it was proven that the separatrices can be represented in a parametric form $\vec{x} = \vec{x}_\pm(t,\varepsilon)$, using solutions of finite-difference equations

$$\vec{x}_\pm(t+h,\varepsilon) = F_\varepsilon(\vec{x}_\pm(t,\varepsilon)),$$

$h = \log\lambda$, satisfying the boundary conditions

$$\vec{x}_\pm(t,\varepsilon) \xrightarrow[\mathrm{Re}(t)\to\pm\infty]{} 0,$$

respectively. This parameterization can be chosen close to the homoclinic solution of the differential equation

$$\vec{x}_\pm(t,\varepsilon) = \vec{\sigma}(t) + O(\varepsilon) \tag{23}$$

in a half-strip $\pm \operatorname{Re}(t) < T_0$, $|\operatorname{Im}(t)| < \varrho$, respectively. This estimate holds for any T_0, but the constant in the upper bound for the error term depends on T_0.

The estimate (23) enables one to use Theorem 5 and to obtain the same upper bound for the splitting. Moreover, it is not too difficult to see that one does not need an area-preserving property of the map to prove the estimate (23). So this assumption can be omitted.

In the recent paper Fontich [Fon95] showed that the Poincaré map (time period map) of a rapidly forced system on the plane can be reduced to the form (22). As a consequence, he obtained an exponentially small upper bound for the splitting. Theorem 5 enables us to obtain the upper bound without the condition of constant zero divergence, used in that paper.

A standard-like map $(x, y) \mapsto (\widetilde{x}, \widetilde{y})$, where

$$\widetilde{x} = x + \widetilde{y}, \qquad \widetilde{y} = y + \varepsilon f(x),$$

and $f(x)$ is an analytic function, can be transformed to the form (22) provided $f(0) = 0$ and $f'(0) > 0$. In this case the origin is a hyperbolic fixed point. Then the theorem from [FS90] can be applied.

If $f'(0) = 0$ and $f''(0) \neq 0$ the origin is a parabolic fixed point. In this case, following the papers [Laz84, Laz87] one can establish the existence of invariant manifolds associated with the origin. These invariant manifolds can be parameterized by a solution of the finite-difference equation with $h = \sqrt{\varepsilon}$, close to a homoclinic solution $\vec{\sigma}(t)$ of the system

$$\dot{x} = y, \qquad \dot{y} = f(x),$$

and the estimate (23) holds. Again we can use Theorem 5, but with ε replaced by $\sqrt{\varepsilon}$.

References

[Fon95] E. Fontich, *Rapidly forced planar vector fields and splitting of separatrices*, J. Differential Equations 119 (1995), 310–335.

[FS90] E. Fontich and C. Simó, *The splitting of separatrices for analytic diffeomorphisms*, Ergodic. Theory Dynam. Systems 10 (1990), 295–318.

[Laz84] V. F. Lazutkin, *Splitting of separatrices for Chirikov's standard map*, VINITI no. 6372/84, 1984 (in Russian).

[Laz87] –, *Separatrices splitting for a standard family of the area-preserving maps*, in: M. Sh. Birman (ed.), *Wave Propagation. Scattering Theory*, Topics in Math. Phys. 12, Leningrad State University, 1987, 32–41 (in Russian).

[Laz91] —, *Exponential splitting of separatrices and an analytical integral for the semi-standard map*, preprint, Université Paris VII, 1991.

[Nei84] A. I. Neishtadt, *The separation of motion in systems with rapidly rotating phase*, Prikl. Mat. Mekh. 48 (1984), 197–204, (in Russian).

Vassiliĭ G. Gelfreich
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via, 585
08071 Barcelona, Spain
E-mail: gelf@maia.ub.es

Current address:
Steklov Mathematical Institute
St. Petersburg Branch
Fontanka 27
191011 St. Petersburg, Russia

Received on 29.5.1995;
revised version on 7.12.1995