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A TWO-DISORDER DETECTION PROBLEM

Abstract. Suppose that the process $X = \{X_n, n \in \mathbb{N}\}$ is observed sequentially. There are two random moments of time θ_1 and θ_2, independent of X, and X is a Markov process given θ_1 and θ_2. The transition probabilities of X change for the first time at time θ_1 and for the second time at time θ_2. Our objective is to find a strategy which immediately detects the distribution changes with maximal probability based on observation of X. The corresponding problem of double optimal stopping is constructed. The optimal strategy is found and the corresponding maximal probability is calculated.

1. Introduction. Suppose that a process $X = \{X_n,\ n \in \mathbb{N}\}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$) is observed sequentially. The process is obtained from three Markov processes by switches between them at two random moments of time, θ_1 and θ_2. Our objective is to detect immediately these moments based on observation of X.

This type of problem arises in quality control. An automaton which produces some details changes its parameters. This causes the details to change their quality. Production can be divided into three grades. Assuming that at the beginning of the production process the quality is highest, from some time θ_1 on the products should be classified to a lower grade, and beginning with θ_2 to the lowest grade. We want to detect the moments of these changes.

Shiryaev (1978) considered the disorder problem for independent random variables with one disorder where the mean distance between disorder time and the moment of its detection was minimized. The probability maximizing approach to the problem was used by Bojdecki (1979), and the stopping time which is in a given neighbourhood of the moment of disorder with

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maximal probability was found. The problem with two disorders was considered by Yoshida (1983) and Szajowski (1992). Yoshida (1983) solved the problem of optimal stopping for the observation of a Markov process X so as to maximize the probability that the distance between θ_i, $i = 1, 2$, and the moment of disorder will not exceed a given number (for each disorder independently). He constructed a strategy which stops the process between the first and the second disorder with maximal probability. References to other papers treating variations of the disorder problem can be found in Szajowski (1992).

In the present paper the probability maximizing approach to optimal stopping developed by Bojdecki (1979) is extended to solve a double stopping problem (see Haggstrom (1967), Nikolaev (1979)) arising in the quickest detection of double disorders. In Section 2 the problem is formulated in a rigorous manner. Section 3 contains the reduction of the problem to an optimal stopping problem for a doubly indexed stochastic sequence. The main result is given in Section 4.

2. The double disorder detection problem. Let $X = \{X_n,\ n \in \mathbb{N}\}$, defined on $(\Omega, \mathcal{F}, P)$, be a potentially observable sequence of r.v.'s with values in $(\mathbb{E}, \mathcal{B})$, where $\mathbb{E}$ is a subset of the real line. Assume that the epochs of distributional changes are $\mathbb{N}$-valued $\mathcal{F}$-measurable r.v. θ_1 and θ_2, independent of X and having the distribution

$$(1) \qquad P(\theta_1 = j) = p_1^{j-1} q_1, \qquad P(\theta_2 = k \mid \theta_1 = j) = p_2^{k-j-1} q_2,$$

where $j = 1, 2, \ldots,\ k = j+1, j+2, \ldots$ and $p_i + q_i = 1,\ i = 1, 2$.

Suppose that on $(\Omega, \mathcal{F}, P)$ Markov processes $X^i = \{(X_n^i, \mathcal{F}_n, P_x^i)\}$, $i = 1, 2, 3$, are defined and we have

$$(2) \qquad X_n = \begin{cases} X_n^1 & \text{if } n < \theta_1, \\ X_n^2 & \text{if } \theta_1 \le n < \theta_2, \\ X_n^3 & \text{if } n \ge \theta_2. \end{cases}$$

The measures P_x^i, $i = 1, 2, 3$, are absolutely continuous with respect to some fixed measure P_x and satisfy the following relations: $P_x^i(dy) = f_x^i(y) P(x, dy)$, where $f_x^i(\cdot) \ne f_x^j(\cdot)$, $i \ne j$ and $f_x^{i+1}(y)/f_x^i(y) < \infty$, $i = 1, 2$, for every $x, y \in \mathbb{E}$. The distribution of θ_i, $i = 1, 2$, is given by (1) and the measures P_x^i, $i = 1, 2, 3$, $x \in \mathbb{E}$, are known. We observe the process $(X_n, \mathcal{F}_n, P_x)$, $n = 0, 1, 2, \ldots$, $x \in \mathbb{E}$, which is a Markov process given θ_1 and θ_2, defined by (2) with $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. On the basis of the distribution of θ_1, θ_2 and measures P_x^i, $i = 1, 2, 3$, $x \in \mathbb{E}$, we calculate the finite-dimensional distributions of the observed process.

Let $\mathcal{S}$ denote the set of all stopping times with respect to the filtration $(\mathcal{F}_n)$, $n = 0, 1, \ldots$, and $\mathcal{T} = \{(\tau, \sigma) : \tau < \sigma,\ \tau, \sigma \in \mathcal{S}\}$. Let us determine a pair of stopping times $(\tau^*, \sigma^*) \in \mathcal{T}$ such that for every $x \in \mathbb{E}$,

$$(3)\quad P_x(\tau^* < \sigma^* < \infty,\ |\theta_1 - \tau^*| \le d_1,\ |\theta_2 - \sigma^*| \le d_2) = \sup_{(\tau,\sigma)\in\mathcal{T}} P_x(\tau < \sigma < \infty,\ |\theta_1 - \tau| \le d_1,\ |\theta_2 - \sigma| \le d_2).$$

This problem will be denoted by $D_{d_1 d_2}$.

3. Reduction of the double "disorder problem" to double optimal stopping of a Markov process. A *compound stopping variable* is a pair (τ, σ) of stopping times such that $\tau < \sigma$ a.e. Define $\mathcal{T}_m = \{(\tau,\sigma) \in \mathcal{T} : \tau \ge m\}$, $\mathcal{T}_{mn} = \{(\tau,\sigma) \in \mathcal{T} : \tau = m,\ \sigma \ge n\}$ and $\mathcal{S}_m = \{\tau \in \mathcal{S} : \tau \ge m\}$. Set $\mathcal{F}_{mn} = \mathcal{F}_n$, $m, n \in \mathbb{N}$, $m \le n$. We define a two-parameter stochastic sequence $\xi(x) = \{\xi_{mn}(x),\ m, n \in \mathbb{N},\ m < n,\ x \in \mathbb{E}\}$ by

$$\xi_{mn}(x) = P_x(|\theta_1 - \tau| \le d_1,\ |\theta_2 - \sigma| \le d_2 \mid \mathcal{F}_{mn}).$$

For every $m, n \in \mathbb{N}$ with $m < n$, we can consider the optimal stopping problem of $\xi(x)$ on $\mathcal{T}_{mn}$. A compound stopping variable (τ^*, σ^*) is said to be *optimal* in $\mathcal{T}_m$ (or $\mathcal{T}_{mn}$) if $E_x \xi_{\tau^*\sigma^*} = \sup_{(\tau,\sigma)\in\mathcal{T}_m} E_x \xi_{\tau\sigma}$ (or $E_x \xi_{\tau^*\sigma^*} = \sup_{(\tau,\sigma)\in\mathcal{T}_{mn}} E_x \xi_{\tau\sigma}$). Define

$$(4)\quad \eta_{mn}(x) = \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}} E(\xi_{\tau\sigma} \mid \mathcal{F}_{mn}),$$

$$(5)\quad \eta_m = E_x(\eta_{m,m+1} \mid \mathcal{F}_m).$$

If we put $\xi_{m\infty} = 0$, then

$$\eta_{mn} = \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}} P_x(|\theta_1 - m| \le d_1,\ |\theta_2 - n| \le d_2 \mid \mathcal{F}_{mn}).$$

From the theory of optimal stopping for double indexed processes (cf. Haggstrom (1967), Nikolaev (1981)) the sequence η_{mn} satisfies

$$\eta_{mn} = \max\{\xi_{mn}, E(\eta_{m,n+1} \mid \mathcal{F}_{mn})\}.$$

Moreover, if $\sigma_m^* = \inf\{n \ge m : \eta_{mn} = \xi_{mn}\}$, then (m, σ_m^*) is optimal in $\mathcal{T}_{mn}$ and $\eta_{mn} = E_x(\xi_{m\sigma_m^*} \mid \mathcal{F}_{mn})$ a.e.

LEMMA 1. *The stopping time σ_m^* is optimal for every stopping problem* (4).

Proof. It suffices to prove $\lim_{n\to\infty} \xi_{mn} = 0$ (Lemma 4.10 of Chow, Robbins & Siegmund (1971), cf. also Bojdecki (1979), Bojdecki (1982)). For $m, n, k \in \mathbb{N}$ with $n \ge k > m$ and every $x \in \mathbb{E}$ we have

$$E_x(\mathbb{I}_{\{|\theta_1 - m| \le d_1, |\theta_2 - n| \le d_2\}} \mid \mathcal{F}_{mn}) = \xi_{mn}(x) \le E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1 - m| \le d_1, |\theta_2 - j| \le d_2\}} \mid \mathcal{F}_n),$$

where $\mathbb{I}_A$ is the characteristic function of the set A. By Levy's theorem,

$$\limsup_{n\to\infty} \xi_{mn}(x) \le E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1-m|\le d_1, |\theta_2-j|\le d_2\}} \mid \mathcal{F}_{n\infty}),$$

where $\mathcal{F}_\infty = \mathcal{F}_{n\infty} = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$.

We have $\lim_{k\to\infty} \sup_{j\ge k} \mathbb{I}_{\{|\theta_1-m|\le d_1, |\theta_2-j|\le d_2\}} = 0$ a.e. and by the dominated convergence theorem,

$$\lim_{k\to\infty} E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1-m|\le d_1, |\theta_2-j|\le d_2\}} \mid \mathcal{F}_\infty) = 0. \ \blacksquare$$

As the next step the optimal stopping problem for η_m should be solved. Define

$$V_m = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_m} E_x(\eta_\tau \mid \mathcal{F}_m). \tag{6}$$

Then $V_m = \max\{\eta_m, E_x(V_{m+1} \mid \mathcal{F}_m)\}$ a.e. and we define $\tau_n^* = \inf\{k \ge n : V_k = \eta_k\}$.

LEMMA 2. *The strategy τ_0^* is the optimal strategy of the first stop.*

Proof. To show that τ_0^* is the optimal first stop strategy we prove that $P_x(\tau_0^* < \infty) = 1$. We argue in the usual manner, i.e. we show $\lim_{m\to\infty} \eta_m(x)$
$= 0$.

We have

$$\begin{aligned}\eta_m &= E_x(\xi_{m\sigma_m^*} \mid \mathcal{F}_m) = E_x(E_x(\mathbb{I}_{\{|\theta_1-m|\le d_1, |\theta_2-\sigma_m^*|\le d_2\}} \mid \mathcal{F}_{m\sigma_m^*}) \mid \mathcal{F}_m)\\ &= E_x(\mathbb{I}_{\{|\theta_1-m|\le d_1, |\theta_2-\sigma_m^*|\le d_2\}} \mid \mathcal{F}_m)\\ &\le E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1-j|\le d_1, |\theta_2-\sigma_j^*|\le d_2\}} \mid \mathcal{F}_m).\end{aligned}$$

Similarly to the proof of Lemma 1 we have

$$\limsup_{m\to\infty} \eta_m(x) \le E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1-j|\le d_1, |\theta_2-\sigma_j^*|\le d_2\}} \mid \mathcal{F}_\infty).$$

Since

$$\lim_{k\to\infty} \sup_{j\ge k} \mathbb{I}_{\{|\theta_1-k|\le d_1, |\theta_2-\sigma_j^*|\le d_2\}} \le \limsup_{k\to\infty} \mathbb{I}_{\{|\theta_1-k|\le d_1\}} = 0,$$

it follows that

$$\lim_{m\to\infty} \eta_m(x) \le \lim_{k\to\infty} E_x(\sup_{j\ge k} \mathbb{I}_{\{|\theta_1-j|\le d_1, |\theta_2-\sigma_j^*|\le d_2\}} \mid \mathcal{F}_\infty) = 0. \ \blacksquare$$

Lemmas 1 and 2 describe the method of solving the "disorder problem" formulated in Section 2.

4. Immediate detection of the first and second disorder. For the sake of simplicity we restrict ourselves to the case $d_1 = d_2 = 0$. It will be easily seen how to generalize the solution to $D_{d_1 d_2}$ for $d_1 > 0$ or $d_2 > 0$.

First we construct multi-dimensional Markov chains such that ξ_{mn} and η_m are the functions of their states. Set (cf. Yoshida (1983), Szajowski (1992))

$$\begin{aligned}
\Pi^1_n(x) &= P_x(\theta_1 > n \mid \mathcal{F}_n), \qquad \Pi^2_n(x) = P_x(\theta_2 > n \mid \mathcal{F}_n),\\
\Pi_{mn}(x) &= P_x(\theta_1 = m,\ \theta_2 > n \mid \mathcal{F}_{mn}) \quad \text{for } m,n = 1,2,\ldots,\ m<n,\\
H(t,u,\alpha,\beta) &= \alpha p_1 f^1_t(u) + [p_2(\beta-\alpha)+q_1\alpha] f^2_t(u)\\
&\qquad + [1-\beta+q_2(\beta-\alpha)] f^3_t(u),\\
\Pi^1(t,u,\alpha,\beta) &= p_1\alpha f^1_t(u)(H(t,u,\alpha,\beta))^{-1},\\
\Pi^2(t,u,\alpha,\beta) &= \{p_1\alpha f^1_t(u) + [\alpha q_1 + (\beta-\alpha)p_2] f^2_t(u)\}(H(t,u,\alpha,\beta))^{-1},\\
\Pi(t,u,\alpha,\beta,\gamma) &= p_2\gamma f^2_t(u)(H(t,u,\alpha,\beta))^{-1}.
\end{aligned}$$

The following auxiliary results will be needed in the proof of the main theorem.

LEMMA 3. *For each $x \in \mathbb{E}$ and $m,n = 1,2,\ldots$ with $m<n$, and each Borel function $u:\mathbb{R}\to\mathbb{R}$,*

$$\Pi^1_{n+1}(x) = \Pi^1(X_n, X_{n+1}, \Pi^1_n(x), \Pi^2_n(x)), \tag{7}$$

$$\Pi^2_{n+1}(x) = \Pi^2(X_n, X_{n+1}, \Pi^1_n(x), \Pi^2_n(x)), \tag{8}$$

$$\Pi_{m,n+1}(x) = \Pi(X_n, X_{n+1}, \Pi^1_n(x), \Pi^2_n(x), \Pi_{mn}(x)), \tag{9}$$

with the boundary condition $\Pi^1_0(x) = \Pi^2_0(x) = 0$,

$$\Pi_{mm}(x) = \frac{q_1 f^2_{X_{m-1}}(X_m)}{p_1 f^1_{X_{m-1}}(X_m)} \Pi^1_n(x)$$

and

$$E_x(u(X_{n+1}) \mid \mathcal{F}_n) = \int_{\mathbb{E}} u(y) H(X_n, y, \Pi^1_n(x), \Pi^2_n(x))\, P_{X_n}(dy). \tag{10}$$

Proof. (7), (8) and (10) are proved in Yoshida (1983) and Szajowski (1992). The formula (9) follows from the Bayes formula:

$$P_x(\theta_1 = j,\ \theta_2 = k \mid \mathcal{F}_n)$$
$$= \begin{cases}
P_x(\theta_1 = j,\ \theta_2 = k) p_1^n \prod_{s=1}^n f^1_{x_{s-1}}(x_s) \\
\quad \times (S_n(x_0,x_1,\ldots,x_n))^{-1} & \text{if } j > n,\\
P_x(\theta_1 = j, \theta_2 = k) \prod_{s=1}^{j-1} f^1_{x_{s-1}}(x_s) \prod_{t=j}^{n} f^2_{x_{t-1}}(x_t) \\
\quad \times (S_n(x_0,x_1,\ldots,x_n))^{-1} & \text{if } j \le n < k,\\
P_x(\theta_1 = j, \theta_2 = k) \prod_{s=1}^{n} f^1_{x_{s-1}}(x_s) \prod_{t=j}^{k-1} f^2_{x_{t-1}}(x_t) \\
\quad \times \prod_{u=k}^{n} f^3_{x_{u-1}}(x_u)(S_n(x_0,x_1,\ldots,x_n))^{-1} & \text{if } k \le n,
\end{cases}$$

where

$$S_n(x_0,x_1,\ldots,x_n)$$
$$= \sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\Big\{p_1^{j-1}q_1p_2^{k-j-1}q_2\prod_{s=1}^{j-1}f^1_{x_{s-1}}(x_s)\prod_{t=j}^{k-1}f^2_{x_{t-1}}(x_t)\prod_{u=k}^{n}f^3_{x_{u-1}}(x_u)\Big\}$$
$$+\sum_{j=1}^{n}\Big\{p_1^{j-1}q_1p_2^{n-j}\prod_{s=1}^{j-1}f^1_{x_{s-1}}(x_s)\prod_{t=j}^{n}f^2_{x_{t-1}}(x_t)\Big\}+p_1^n\prod_{s=1}^{n}f^1_{x_{s-1}}(x_s).$$

We have

$$\begin{aligned}\Pi_{m,n+1}(x) &= P_x(\theta_1=m,\ \theta_2>n+1\mid\mathcal{F}_{n+1})\\ &= p_2f^2_{X_n}(X_{n+1})\Pi_{mn}(x)S_n(x_0,x_1,\ldots,x_{n+1})\\ &\quad\times(S_{n+1}(x_0,x_1,\ldots,x_n))^{-1}\end{aligned}$$

and

$$S_{n+1}(x_0,x_1,\ldots,x_{n+1})=H(X_n,X_{n+1},\Pi^1_n(x),\Pi^2_n(x))S_n(x_0,x_1,\ldots,x_n).$$

Hence

$$\Pi_{m,n+1}(x)=\frac{p_2f^2_{X_n}(X_{n+1})\Pi_{mn}(x)}{H(X_n,X_{n+1},\Pi^1_n(x),\Pi^2_n(x))}. \ \blacksquare$$

By the above we have

$$\begin{aligned}\xi_{mn}(x) &= P_x(\theta_1=j,\ \theta_2=k\mid\mathcal{F}_{mn})\\ &= \frac{p_1^{j-1}q_1p_2^{k-j-1}q_2\prod_{s=1}^{j-1}f^1_{x_{s-1}}(x_s)\prod_{t=j}^{n-1}f^2_{x_{t-1}}(x_t)f^3_{X_{n-1}}(X_n)}{S_n(x_0,x_1,\ldots,x_n)}\\ &= \frac{q_2}{p_2}\Pi_{mn}(x)\frac{f^3_{X_{n-1}}(X_n)}{f^2_{X_{n-1}}(X_n)}.\end{aligned}$$

We can observe that $(X_n,X_{n+1},\Pi^1_n(x),\Pi^2_n(x),\Pi_{mn}(x))$ for $n=m+1$, $m+2,\ldots$ is a function of $(X_{n-1},X_n,\Pi^1_{n-1}(x),\Pi^2_{n-1}(x),\Pi_{m,n-1}(x))$ and X_{n+1}. Moreover, the conditional distribution of X_{n+1} given $\mathcal{F}_n$ (cf. (10)) depends on X_n, $\Pi^1_n(x)$ and $\Pi^2_n(x)$ only. These facts imply that $\{(X_n,X_{n+1},\Pi^1_n(x),\Pi^2_n(x),\Pi_{mn}(x))\}_{n=m+1}^{\infty}$ form a homogeneous Markov process (see Chapter 2.15 of Shiryaev (1978)). This allows us to reduce the problem (4) for each m to the optimal stopping problem for the Markov process $Z_m(x)=\{(X_{n-1},X_n,\Pi^1_n(x),\Pi^2_n(x),\Pi_{mn}(x)),\ m,n\in\mathbb{N},\ m<n,\ x\in\mathbb{E}\}$ with the reward function

$$h(t,u,\alpha,\beta,\gamma)=\frac{q_2}{p_2}\gamma\frac{f^3_t(u)}{f^2_t(u)}.$$

LEMMA 4. *The solution of the optimal stopping problem* (4) *for* $m = 1, 2, \ldots$ *has the form*

$$\sigma_m^* = \inf\left\{n > m : \frac{f^3_{X_{n-1}}(X_n)}{f^2_{X_{n-1}}(X_n)} \ge R^*(X_n)\right\} \tag{11}$$

where $R^*(t) = p_2 \int_{\mathbb{E}} r^*(t,s) f_t^2(s)\, P_t(ds)$. *Here* $r^* = \lim_{n\to\infty} r_n$, *where* $r_0(t,u) = f_t^3(u)/f_t^2(u)$ *and*

$$r_{n+1}(t,u) = \max\left\{\frac{f_t^3(u)}{f_t^2(u)}, p_2 \int_{\mathbb{E}} r_n(u,s) f_u^2(s)\, P_u(ds)\right\}. \tag{12}$$

The function $r^*(t,u)$ *satisfies the equation*

$$r^*(t,u) = \max\left\{\frac{f_t^3(u)}{f_t^2(u)}, p_2 \int_{\mathbb{E}} r^*(u,s) f_u^2(s)\, P_u(ds)\right\}. \tag{13}$$

The value of the problem is

$$\eta_m = E_x(\eta_{m,m+1} \mid \mathcal{F}_m) = \frac{q_2}{p_2}\frac{q_1}{p_1} \Pi_n^1(x) R^*(X_m). \tag{14}$$

Proof. For any Borel function $u : \mathbb{E} \times \mathbb{E} \times [0,1]^3 \to [0,1]$ define two operators

$$\begin{aligned} T_x u(t,s,\alpha,\beta,\gamma) = E_x(u(X_n, X_{n+1}, \Pi^1_{n+1}(x), \Pi^2_{n+1}(x), \Pi_{m,n+1}(x)) \mid \\ X_{n-1} = t,\ X_n = s,\ \Pi_n^1(x) = \alpha,\ \Pi_n^2(x) = \beta,\ \Pi_{mn}(x) = \gamma) \end{aligned}$$

and

$$Q_x u(t,s,\alpha,\beta,\gamma) = \max\{u(t,s,\alpha,\beta,\gamma), T_x u(t,s,\alpha,\beta,\gamma)\}.$$

By the well-known theorem from the theory of optimal stopping (see Shiryaev (1978), Ch. 2, and Nikolaev (1981)) we conclude that the solution of (4) is the Markov time

$$\begin{aligned} \sigma_m^* = \inf\{n > m : h(X_{n-1}, X_n, \Pi_n^1(x), \Pi_n^2(x), \Pi_{mn}(x)) \\ = h^*(X_{n-1}, X_n, \Pi_n^1(x), \Pi_n^2(x), \Pi_{mn}(x))\}, \end{aligned}$$

where $h^* = \lim_{k\to\infty} Q_x^k h(t,u,\alpha,\beta,\gamma)$. Then

$$\begin{aligned} T_x h(t,u,\alpha,\beta,\gamma) &= E_x\bigg(\frac{q_2}{p_2} \Pi_{m,n+1}(x) \frac{f^3_{X_n}(X_{n+1})}{f^2_{X_n}(X_{n+1})} \,\bigg|\, \\ &\qquad X_{n-1} = t,\ X_n = u,\ \Pi_n^1(x) = \alpha,\ \Pi_n^2(x) = \beta,\ \Pi_{mn}(x) = \gamma\bigg) \\ &= \frac{q_2}{p_2} \gamma p_2 E\left(\frac{f_u^2(X_{n+1})}{H(u, X_{n+1}, \alpha, \beta)} \frac{f_u^3(X_{n+1})}{f_u^2(X_{n+1})} \,\middle|\, \mathcal{F}_n\right) \\ &\overset{(10)}{=} q_2 \gamma \int_{\mathbb{E}} \frac{f_u^3(s)}{H(u,s,\alpha,\beta)} H(u,s,\alpha,\beta)\, P_u(ds) = q_2\gamma \end{aligned}$$

and

$$Q_x h(t,u,\alpha,\beta,\gamma) = \frac{q_2}{p_2}\gamma \max\left\{\frac{f_t^3(u)}{f_t^2(u)}, p_2\right\}. \tag{15}$$

Define $r_n(t,u)$ as in the statement of the lemma. We show that

$$Q_x^l h(t,u,\alpha,\beta,\gamma) = \frac{q_2}{p_2}\gamma r_l(t,u) \tag{16}$$

for $l = 1,2,\ldots$ By (15) we have $Q_x h = (q_2/p_2)\gamma r_1$. Assume (16) for $l \le k$. By (10) we have

$$\begin{aligned} T_x Q_x^k h(t,u,\alpha,\beta,\gamma) &= E_x\bigg(\frac{q_2}{p_2}\Pi_{m,k+1}(x) r_k(X_n, X_{n+1}) \,\bigg|\\ &\qquad X_{n-1} = t,\ X_n = u,\ \Pi_n^1(x) = \alpha,\ \Pi_n^2(x) = \beta, \Pi_{mn}(x) = \gamma\bigg)\\ &= \frac{q_2}{p_2}\gamma p_2 \int_{\mathbb{E}} r_k(u,s) f_u^2(s)\, P_u(ds). \end{aligned}$$

It is easy to show (see Shiryaev (1978)) that

$$Q_x^{k+1}h = \max\{h, T_x Q_x^k h\} \quad \text{for } k = 1,2,\ldots$$

Hence we get $Q_x^{k+1}h = (q_2/p_2)\gamma r_{k+1}$ and (16) is proved for $l = 1,2,\ldots$ This gives

$$h^*(t,u,\alpha,\beta,\gamma) = \frac{q_2}{p_2}\gamma \lim_{k\to\infty} r_k(t,u) = \frac{q_2}{p_2}\gamma r^*(t,u)$$

and

$$\begin{aligned} \eta_{mn} &= \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}} E_x(\xi_{\tau,\sigma} \mid \mathcal{F}_{mn})\\ &= h^*(X_{n-1}, X_n, \Pi_n^1(x), \Pi_n^2(x), \Pi_{mn}(x)). \end{aligned}$$

We have

$$T_x h^*(t,u,\alpha,\beta,\gamma) = \frac{q_2}{p_2}\gamma p_2 \int_{\mathbb{E}} r^*(u,s) f_u^2(s)\, P_u(ds) = \frac{q_2}{p_2}\gamma R^*(u)$$

and σ_m^* has the form (11). By (5) and (10) we obtain

$$\eta_m(x) = f(X_{n-1}, X_n, \Pi_n^1(x), \Pi_n^2(x)) = E(\eta_{m,m+1} \mid \mathcal{F}_m) \tag{17}$$

$$= E\bigg(\frac{q_2}{p_2}\Pi_{m,m+1} r^*(X_m, X_{m+1}) \,\bigg|\, \mathcal{F}_m\bigg) \tag{18}$$

$$= \frac{q_2}{p_2}\Pi_{mm} \int_{\mathbb{E}} r^*(X_m, s) f_{X_m}^2(s)\, P_{X_m}(ds). \ \blacksquare \tag{19}$$

By Lemmas 4 and 3 the optimal stopping problem (6) has been transformed to the optimal stopping problem for the homogeneous Markov process

$$W = \{(X_{m-1}, X_m, \Pi_m^1(x), \Pi_m^2(x)),\ m \in \mathbb{N},\ x \in \mathbb{E}\}$$

with the reward function

$$f(t,u,\alpha,\beta)=\frac{q_1q_2}{p_1p_2}\cdot\frac{f_t^2(u)}{f_t^1(u)}\alpha R^*(u).$$

LEMMA 5. *The solution of the optimal stopping problem* (6) *for* $n=1,2,\ldots$ *has the form*

$$\tau_n^*=\inf\{k\ge n:(X_{k-1},X_k,\Pi_k^1(x),\Pi_k^2(x))\in B^*\} \tag{20}$$

where $B^*=\{(t,u,\alpha,\beta):f_t^2(u)/f_t^1(u)\ge p_1\int_{\mathbb{E}}v^*(u,s)\,P_u(ds)\}$. *Here* $v^*(t,u)=\lim_{n\to\infty}v_n(t,u)$, *where* $v_0(t,u)=R^*(u)$ *and*

$$v_{n+1}(t,u)=\max\left\{\frac{f_t^2(u)}{f_t^1(u)},p_1\int_{\mathbb{E}}v_n(u,s)f_u^1(s)\,P_u(ds)\right\}. \tag{21}$$

The function $v^*(t,u)$ *satisfies the equation*

$$v^*(t,u)=\max\left\{\frac{f_t^2(u)}{f_t^1(u)},p_1\int_{\mathbb{E}}v^*(u,s)f_u^1(s)\,P_u(ds)\right\}. \tag{22}$$

The value of the problem is $V_n=v^*(X_{n-1},X_n)$.

Proof. For any Borel function $u:\mathbb{E}\times\mathbb{E}\times[0,1]^2\to[0,1]$ define two operators

$$T_xu(t,s,\alpha,\beta)=E_x(u(X_n,X_{n+1},\Pi_{n+1}^1(x),\Pi_{n+1}^2(x))\mid X_{n-1}=t,\ X_n=s,\ \Pi_n^1(x)=\alpha,\ \Pi_n^2(x)=\beta)$$

and

$$Q_xu(t,s,\alpha,\beta)=\max\{u(t,s,\alpha,\beta),T_xu(t,s,\alpha,\beta)\}.$$

As in the proof of Lemma 4 we conclude that the solution of (6) is the Markov time

$$\tau_m^*=\inf\{n>m:f(X_{n-1},X_n,\Pi_n^1(x),\Pi_n^2(x))=f^*(X_{n-1},X_n,\Pi_n^1(x),\Pi_n^2(x))\},$$

where $f^*=\lim_{k\to\infty}Q_x^kf(t,u,\alpha,\beta)$. We have

$$\begin{aligned}T_xh(t,u,\alpha,\beta)&=E_x\left(\frac{q_1q_2}{p_1p_2}\Pi_{n+1}^1(x)\frac{f_{X_n}^2(X_{n+1})}{f_{X_n}^1(X_{n+1})}R^*(X_{n+1})\,\middle|\,X_{n-1}=t,\ X_n=u,\ \Pi_n^1(x)=\alpha,\ \Pi_n^2(x)=\beta\right)\\&=\frac{q_1q_2}{p_1p_2}\gamma p_1E\left(\frac{f_u^1(X_{n+1})}{H(u,X_{n+1},\alpha,\beta)}\cdot\frac{f_u^2(X_{n+1})}{f_u^1(X_{n+1})}R^*(X_{n+1})\,\middle|\,\mathcal{F}_n\right)\end{aligned}$$

$$\stackrel{(10)}{=} \frac{q_1 q_2}{p_1 p_2} \alpha p_1 \int_{\mathbb{E}} \frac{f_u^2(s)}{H(u,s,\alpha,\beta)} H(u,s,\alpha,\beta) R^*(s)\, P_u(ds)$$
$$= \frac{q_1 q_2}{p_1 p_2} \alpha p_1 \int_{\mathbb{E}} R^*(s) f_{X_n}^2(s)\, P_{X_n}(ds)$$

and

$$Q_x f(t,u,\alpha,\beta) = \frac{q_1 q_2}{p_1 p_2} \alpha \max\left\{\frac{f_t^2(u)}{f_t^1(u)}, p_1 \int_{\mathbb{E}} R^*(s) f_u^2\, P_u(ds)\right\} \tag{23}$$
$$= \frac{q_1 q_2}{p_1 p_2} \alpha v_1(t,u).$$

We show that

$$Q_x^l f(t,u,\alpha,\beta) = \frac{q_1 q_2}{p_1 p_2} \alpha v_l(t,u) \tag{24}$$

for $l = 1, 2, \ldots,$ where $v_l(t,u)$ is defined as in the statement of the lemma. By (23) we have $Q_x f = \frac{q_1 q_2}{p_1 p_2} \alpha v_1$. Assume (24) for $l \le k$. By (10) we have

$$T_x Q_x^k f(t,u,\alpha,\beta) = E_x\bigg(\frac{q_1 q_2}{p_1 p_2} \mathit{\Pi}_{k+1}^1(x) v_k(X_n, X_{n+1}) \,\bigg|$$
$$X_{n-1} = t,\ X_n = u,\ \mathit{\Pi}_n^1(x) = \alpha,\ \mathit{\Pi}_n^2(x) = \beta\bigg)$$
$$= \frac{q_1 q_2}{p_1 p_2} \alpha p_1 \int_{\mathbb{E}} v_k(u,s) f_u^1(s)\, P_u(ds).$$

Hence $Q_x^{k+1} f = \frac{q_1 q_2}{p_1 p_2} \alpha v_{k+1}$ and (24) is proved for $l = 1, 2, \ldots$ This gives

$$f^*(t,u,\alpha,\beta) = \frac{q_1 q_2}{p_1 p_2} \alpha \lim_{k\to\infty} v_k(t,u) = \frac{q_1 q_2}{p_1 p_2} \alpha v^*(t,u)$$

and

$$V_m = \frac{q_1 q_2}{p_1 p_2} \mathit{\Pi}_m^1 v^*(X_{m-1}, X_m).$$

We have

$$T_x f^*(t,u,\alpha,\beta) = \frac{q_1 q_2}{p_1 p_2} \alpha p_1 \int_{\mathbb{E}} v^*(u,s) f_u^1(s)\, P_u(ds).$$

It follows that τ_n^* has the form (20). The value of the problem (6) and (3) is

$$v_0(x) = E_x(V_1 \mid \mathcal{F}_0) = \frac{q_1 q_2}{p_1 p_2} E_x \mathit{\Pi}_1^1(x) v^*(x, X_1)$$
$$= \frac{q_1 q_2}{p_2} \int_{\mathbb{E}} v^*(x,s) f_x^1(s)\, P_x(ds). \ \blacksquare$$

By Lemmas 4 and 5 the solution of the problem D_{00} can be formulated as follows.

THEOREM 4.1. *The compound stopping time* $(\tau^*, \sigma^*_{\tau^*})$, *where* σ^*_m *is given by* (11) *and* $\tau^* = \tau^*_0$ *is given by* (20) *is a solution of the problem* D_{00}. *The value of the problem is*

$$P_x(\tau^* < \sigma^* < \infty,\ \theta_1 = \tau^*,\ \theta_2 = \sigma^*_{\tau^*}) = \frac{q_1 q_2}{p_2} \int_{\mathbb{E}} v^*(x,s) f^1_x(s)\, P_x(ds).$$

Remark 1. The problem can be extended to optimal detection of more than two successive disorders. The distribution of θ_1, θ_2 may be more general. The general *a priori* distributions of disorder moments lead to more complicated formulae, since the corresponding Markov chains are not homogeneous.

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