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W. SZLENK (Warszawa and Barcelona)

MATHEMATICAL MODEL OF MIXING IN RUMEN

Abstract. A mathematical model of mixing food in rumen is presented. The model is based on the idea of the Baker Transformation, but exhibits some different phenomena: the transformation does not mix points at all in some parts of the phase space (and under some conditions mixes them strongly in other parts), as observed in ruminant animals.

1. Introduction. Some years ago in the Department of Animal Physiology of the Warsaw Agricultural University, the following phenomenon was observed in the digestive process of sheep (for these type of experiments see [3]). Each sheep was given two types of food, which we will call A and B. Every 5 minutes a sample of food was taken from a fixed location in the rumen. After 1 hour the samples were found to be composed almost entirely of either component A or component B; that is, the two substances practically did not mix.

There arises the following question: what is the mixing mechanism in the rumen?

The aim of this paper is to present a mathematical model of the mixing process which could explain the phenomenon in question.

2. Rumen's activity. The rumen (see Figure 2.1) is the first of four stomachs in the body of ruminant animals. In principle, its functions are similar for all ruminants (cattle, sheep, goats).

Food in the rumen floats on the surface of a liquid. Different sections of the rumen contract almost periodically as shown in Figure 2.2.

During contraction, the volume of the section may diminish by half, and the pressure inside increases significantly. One distinguishes two phases in the rumen's activity. In the first phase the food is mainly mixed, in

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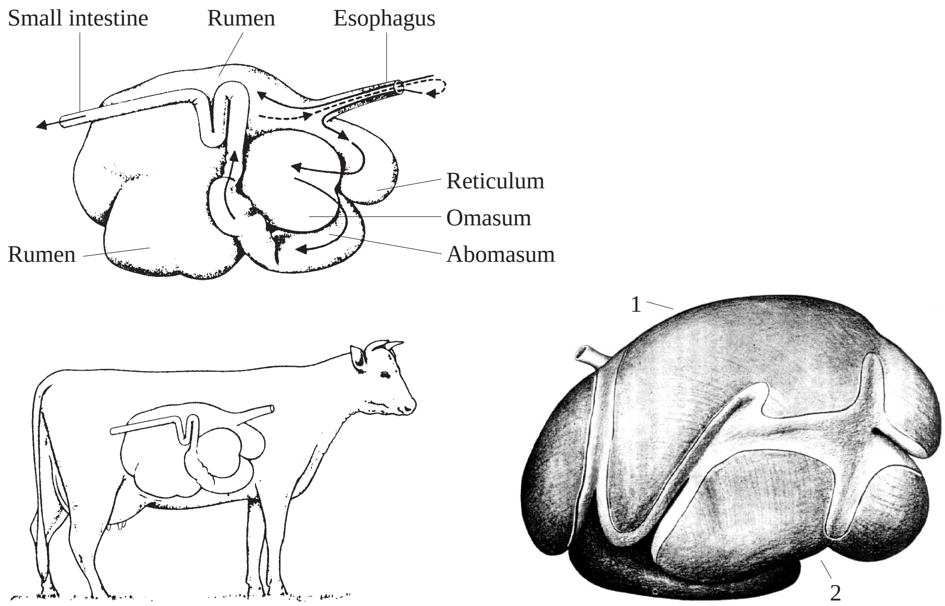

(a) (b)

Fig. 2.1. (a) In the ruminant animal the true stomach (abomasum) is preceded by several other compartments. The first and largest of these, the rumen, serves as a giant fermentation vat that aids in cellulose digestion. (b) The rumen seen from the left: 1. *Saccus dorsalis ruminis*, the upper rumen; 2. *Saccus ventralis ruminis*, the lower rumen.

A B C D

Fig. 2.2. The scheme of motions of the rumen. The dotted parts are squeezed.

the second it is simultaneously mixed and transported to other parts of the digestive system. In this paper we concentrate on the first phase only. Simplifying the process we may divide the first phase into two parts: in the first part—produced by a horizontal force—the rumen becomes longer vertically and narrower horizontally (stages A and B in Figure 2.2). In the second part—produced by a vertical force—the rumen returns to its initial position (stages C and D in Figure 2.2).

One full cycle of the first stage lasts less than 1 minute. Assuming one cycle per minute we are considering 60 iterations of the same action per hour. In the theory of dynamical systems for mixing processes, 60 iterations normally bring the system to its asymptotic state, which means a uniform distribution of matter in the entire phase space.

With a rough approximation the rumen motions can be considered two-dimensional, since the acting forces are basically vertical and horizontal. So we propose to take as the phase space corresponding to the rumen a two-dimensional domain.

3. Construction of the model. As a model of the mixing process in the rumen we propose the following dynamical system.

Let $Q = [0,1] \times [0,1]$ be the phase space (corresponding to the rumen). We define a map $T_0 : Q \to Q$ as follows. Suppose that $\lambda \in (0,2]$ is a given number, let $Q_0' = [0, \lambda^{-1}] \times [0,1]$, $Q_0'' = (\lambda^{-1}, 1) \times [0,1]$.

For $p = (x,y) \in Q_0'$ (i.e. $x \le \lambda^{-1}$) we set

$$T_0(p) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda^{-1} y \end{pmatrix} \in Q. \tag{3.1}$$

If $p \in Q_0''$ (i.e. $x > \lambda^{-1}$), then $\begin{pmatrix} \lambda x \\ \lambda^{-1} y \end{pmatrix}$ does not belong to Q.

To define T_0 in this case we proceed as follows: we rotate the rectangle $Q'' = [1, \lambda] \times [0, \lambda^{-1}]$ (which is the $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ image of Q_0'') by 90° in the counterclockwise direction about the point $(1,0)$ (the resulting rectangle is denoted by Q''') and we translate Q''' by the vector $[-(1-\lambda^{-1}), 0]$, so its lower left vertex is $(0,0)$; we denote the obtained rectangle by Q^{IV}. Then we apply the map $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ to Q^{IV}: let

$$Q^{\mathrm{V}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} Q^{\mathrm{IV}} = [0,1] \times [0, 1-\lambda^{-1}].$$

Finally, let Q^{VI} be Q^{V} shifted by $[0, \lambda^{-1}]$. The rectangles Q^{VI} and Q_0' form the whole phase space Q. The geometrical construction of T_0 is described in Figure 3.1.

Combining the above operations we eventually find that

$$T_0(p) = \begin{pmatrix} 1-y \\ x \end{pmatrix} \qquad \text{for } p \in Q_0'' \quad (\text{i.e. } x > \lambda^{-1}). \tag{3.2}$$

The dynamical system (Q, T_0) describes the action of the rumen. However, it is not very convenient to analyze, so we define another dynamical system as follows.

Let $R = [0,1] \times [0, \lambda^{-1}]$. It is easy to see that for each $p \in Q$ one of the points $T_0(p)$, $T_0^2(p)$, $T_0^3(p)$ belongs to R. Let $i = i(p)$, $i \in \{1,2,3\}$, be the smallest integer such that $T^i(p) \in R$. Now we define a derived map $T : R \to R$ by $T(p) = T^i(p)$ for $i = i(p)$, $p \in R$.

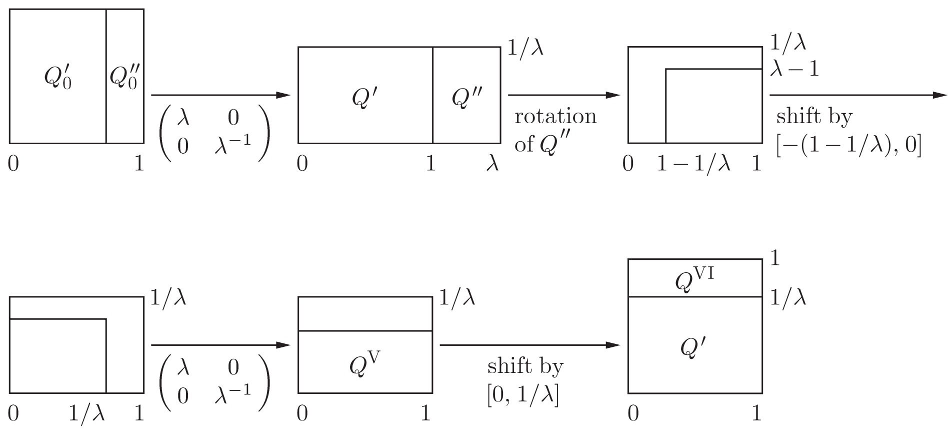

Fig. 3.1. The scheme of construction of the map T_0

The map T can be represented in the following form:

$$
(3.3)\qquad T(x,y) = \begin{cases} (\lambda x, \lambda^{-1} y) & \text{if } x \le \lambda^{-1} \quad (i(p)=1),\\ (\lambda(1-y), \lambda^{-1}x) & \text{if } x > \lambda^{-1} \text{ and } y > 1-\lambda^{-1} \quad (i(p)=2),\\ (\lambda(1-x), \lambda^{-1}(1-y)) & \text{if } x > \lambda^{-1} \text{ and } y \le 1-\lambda^{-1} \quad (i(p)=3). \end{cases}
$$

Define

$$
\begin{aligned}
A &= \{p=(x,y)\in R : x \le \lambda^{-1}\},\\
B &= \{p=(x,y)\in R : x > \lambda^{-1},\ y > 1-\lambda^{-1}\},\\
C &= \{p=(x,y)\in R : x > \lambda^{-1},\ y \le 1-\lambda^{-1}\}.
\end{aligned}
$$

For convenience we set

$$
T_1 = T|_A, \qquad T_2 = T|_B.
$$

4. Analysis of the model. Now we will study the dynamical system (R,T).

THEOREM 4.1. *Suppose that λ satisfies the following condition*:

$$
(4.1)\qquad \sqrt[k+2]{2} < \lambda < \sqrt[k+1]{2}
$$

for some natural number $k \ge 1$. Then there exists a union of rectangles $U = \bigcup_{i=1}^{4k} Q_i$ such that $T^{4k}|_{Q_i} = \mathrm{id}$. Therefore, the map T does not mix points in Q.

Proof. The image $T(B)$ is the rectangle $[\lambda-1,1]\times[1/\lambda^2,1/\lambda]$. Let r be the largest natural number such that $1/\lambda^r > \lambda-1$, and set

$$
B_i = [1/\lambda^{i+1}, 1/\lambda^i] \times [1/\lambda^2, 1/\lambda], \qquad i\in\{1,\dots,r-1\}.
$$

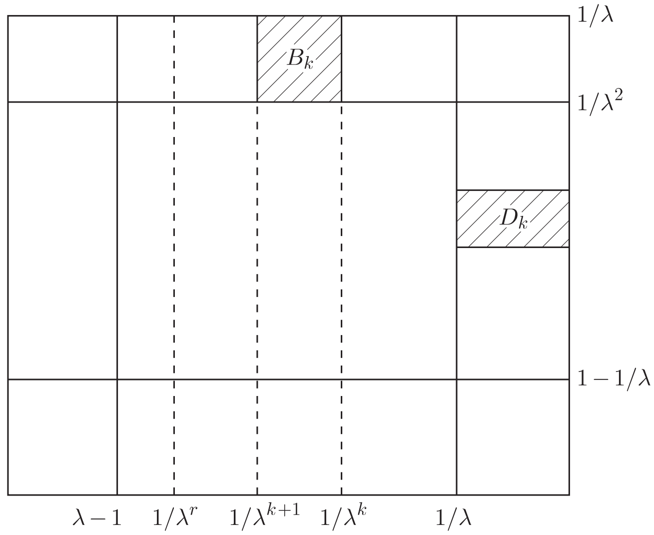

Fig. 4.1. The location of the sets B_k and D_k

Since $\lambda < \sqrt{2} < (1+\sqrt{5})/2$, it follows that $1-1/\lambda < 1/\lambda^2$ and we have the situation of Figure 4.1.

Define $D_i = T^{-1}(B_i) = T_2^{-1}(B_i)$. Then $D_k = (1/\lambda, 1) \times (1-1/\lambda^{k+1}, 1-1/\lambda^{k+2})$.

We will show that the set $P = T_1^k(B_k) \cap D_k$ is not empty. Indeed, the set $T_1^k(B_k)$ is the rectangle $(1/\lambda, 1) \times (1/\lambda^{k+2}, 1/\lambda^{k+1})$. Since $\lambda > \sqrt[k+2]{2}$, we have $\lambda^{k+2} - \lambda - 1 > 0$, which implies

$$\frac{1}{\lambda^{k+2}} < 1 - \frac{1}{\lambda^{k+1}} < \frac{1}{\lambda^{k+1}} < 1 - \frac{1}{\lambda^{k+2}}.$$

Therefore the rectangles D_k and $T_1^k(B_k)$ intersect as shown in Figure 4.2.

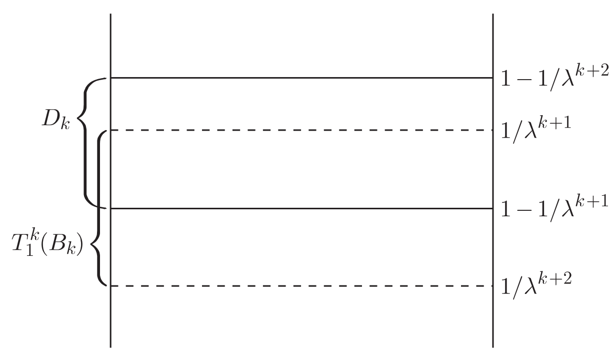

Fig. 4.2. The mutual position of the sets D_k and $T_1^k(B_k)$

Set $S = T_1^k \circ T_2$. Since the set P is not empty, $S^2(p)$ makes sense for $p = (x, y) \in P$ and

$$\begin{aligned}
(4.2)\qquad S^2(x,y) &= T_1^k \circ T_2 \circ T_1^k \circ T_2(x,y) = T_1^k \circ T_2 \circ T_1^k\left(\lambda(1-y), \frac{1}{\lambda}x\right) \\
&= T_1^k \circ T_2\left(\lambda^{k+1}(1-y), \frac{1}{\lambda^{k+1}}x\right) \\
&= T_1^k\left(\lambda\left(1-\frac{1}{\lambda^{k+1}}x\right), \lambda^k(1-y)\right) = (\lambda^{k+1}-x, 1-y).
\end{aligned}$$

This means that S^2 is the symmetry with respect to the point $q = (\lambda^{k+1}/2, 1/2)$. In view of assumption (4.1) the following inequalities hold:

$$\frac{1}{\lambda} < \frac{\lambda^{k+1}}{2} < 1, \qquad 1 - \frac{1}{\lambda^{k+1}} < \frac{1}{2} < \frac{1}{\lambda^{k+1}}.$$

Therefore the point q belongs to the rectangle P. Thus the set $Q = P \cap S^{-2}(P)$ is a non-empty rectangle and $S^4(p)$ makes sense for $p \in Q$. Obviously

$$S^4(p) = p \quad \text{ for } p \in Q.$$

Therefore

$$T^{4(k+1)}|_{T^i(Q)} = S^4|_{T^i(Q)} = \mathrm{id}$$

for $i \in \{0, \ldots, 4(k+1)\}$. The set $U = \bigcup_{i=0}^{4k-3} T^i(Q)$ fulfils the requirement of the Theorem. ■

The general description of the dynamics of (R, T) is as follows. For a given n we can split the space R into a finite union of rectangles P_j such that T^n restricted to P_j is of the following form: either

$$T^n(p) = (A_j^n - \lambda^{-k_n}x, B_j^n - \lambda^{k_n}y)$$

or

$$T^n(p) = (A_j^n - \lambda^{-k_n}y, B_j^n - \lambda^{k_n}x),$$

where A_j^n, B_j^n are some numbers associated with the rectangle P_j, and x_n is an integer depending on n only. The numbers k_n may oscillate between $-\infty$ and $+\infty$. For some points p and some iterations n the number k_n may be equal to 0, in spite of the fact that p does not belong to the set U. On some parts of R we may observe a random behaviour of T^n. Now we will study one such case.

Assume that λ satisfies the following equation:

$$(4.3)\qquad \lambda^{r+2} - \lambda^{r+1} - 1 = 0$$

for some natural r. The equation (4.3) has exactly one root in the interval $(1, 2)$.

This equation implies the following equalities:

$$\lambda - 1 = \frac{1}{\lambda^{r+1}}, \qquad 1 - \frac{1}{\lambda} = \frac{1}{\lambda^{r+2}}. \tag{4.4}$$

Set

$$E = \left[\left(0, \frac{1}{\lambda^{r+1}}\right) \times \left(0, \frac{1}{\lambda}\right)\right] \cup \bigcup_{i=1}^{r+1} \left[\left(\frac{1}{\lambda^{r-i}}, \frac{1}{\lambda^{r-i+1}}\right) \times \left(0, \frac{1}{\lambda^{i+1}}\right)\right].$$

In view of (4.4) the set E is T-invariant. We will show that the dynamical system (E, T) is an ergodic Markov chain.

Denote by E_0 the set $(0, 1/\lambda^{r+1}) \times (0, 1/\lambda)$ and let

$$E_i = \left(\frac{1}{\lambda^{r-i}}, \frac{1}{\lambda^{r-i+1}}\right) \times \left(0, \frac{1}{\lambda^{i+1}}\right), \qquad i \in \{1, \ldots, r+1\}.$$

To any $p \in E$ we assign a sequence $i_n = i_n(p)$, $n \in \mathbb{Z}$, $i_n \in \{0, 1, \ldots, r+1\}$, in the following way: i_n is the index of the set E_i to which $T^n(p)$ belongs. We obtain a space of sequences

$$X = [(x_n)_{-\infty}^{+\infty}, x_n \in \{0, 1, \ldots, r+1\}] \subset \prod_{k=-\infty}^{+\infty} \{0, 1, \ldots, r+1\}_k$$

on which T acts as the left shift σ.

Proposition 4.2. *The dynamical system (X, σ) is an ergodic Markov chain.*

Proof. Let $\mu(A)$ be the normalized Lebesgue measure of a set $A \subset E$, that is, $\mu(A) = |A|/|E|$. Given n, we have to show that

$$\begin{aligned} \mu\{x_n = i_n \mid x_{n-1} = i_{n-1}, \ldots, x_1 = i_1, x_0 = i_0\} \\ = \mu\{x_n = i_n \mid x_{n-1} = i_{n-1}\} = p_{i_{n-1}, i_n} \end{aligned} \tag{4.5}$$

for any sequence $i_0, \ldots, i_n \in \{0, 1, \ldots, r+1\}$

Note that the set $A = \{x_0 = i_0, x_1 = i_1, \ldots, x_{n-1} = i_{n-1}\}$ is a rectangle of maximal in E (i.e. its height is $1/\lambda^{i_0+1}$) and $T^{n-1}(A)$ is a rectangle of maximal width (i.e. its base is $(1/\lambda^{r-i_0}, 1/\lambda^{r-i_0+1})$). Let A_1 be the set $\{x_{i_0} = i_0, x_{i_1} = i_1, \ldots, x_{n-1} = i_{n-1}, x_n = i_n\}$. If $i_{n-1} = 1, \ldots, r+1$, then $\mu\{A_1 \mid A\} = 1$ if $i_n = i_{n-1} + 1 \pmod{(r+1)}$, and $\mu(A_1 \mid A) = 0$ otherwise. If $i_{n-1} = 0$, then since $T^{n-1}(A) = (0, 1/\lambda^{r+1}) \times I$, where I is an interval, we have $\mu\{A_1 \mid A\} = 1/\lambda$ if $i_n = 0$, and $\mu\{A_1 \mid A\} = 1 - 1/\lambda$ if $i_n = 1$, otherwise $\mu\{A_1 \mid FA\} = 0$. So the dynamical system (X, σ) is a Markov chain with transition matrix

$$M = \begin{pmatrix} \lambda^{-1} & 1-\lambda^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

All the entries of the matrix $M^{2(r+1)}$ are positive, so the Markov chain is ergodic. ∎

In view of the Friedman–Ornstein Theorem [1] we have

COROLLARY 4.3. *The dynamical system* (E, T) *is a* K*-system with* K*-partition* $\xi = \{E_i\}_{i=0}^{r+1}$ (see [2] and [6]).

COROLLARY 4.4. *If* $\lambda = 2$, *then* (E, T) *is a Bernoulli system*: *then* $B = \emptyset$ *and the partition* $\xi = \{A, C\}$ *is a* B*-partition.*

The condition (4.3) is not the only condition which implies the existence of a Markov partition for an invariant set E of the system (R, T).

In the dynamical system (R, T) we have at least two invariant sets, U and E. The dynamical system (U, T) is periodic (i.e. there exists an m such that $T^m = \mathrm{id}$), and the system (E, T) is random (i.e. its trajectories behave like realizations of a stochastic process).

There arises a question: what is the typical behaviour in the sense of Lebesgue measure of the system (R_λ, T_λ) for $\lambda \in (0, 2)$?

5. Conclusions. One can construct a lot of mathematical models describing the rumen's actions. The more realistic the model is, the more difficult it is to study it qualitatively. The model presented above is a simplest one. It is a very rough approximation to reality. However, it does exhibit the phenomenon that the food is not well mixed in the rumen. The model is based on the general idea of the Baker Transformation ([2], [5], [6]) with some modification coming from physiology of the rumen's actions. The model has a complicated nature, the mixing process described by it is different in different parts of the rumen. Coexistence of a periodic process and a random one seems to be strange, but it occurs to be the reality of some physiological processes. The model without its biological background is an artificial dynamical system. So we see that sometimes even a seemingly artificial mathematical construction may have some application. There also arises a philosophical question: why Nature chose this type of mixing in rumen, what is the reason for it. But that is not a mathematical problem.

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Wiesław Szlenk
Institute of Applied Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland

and

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain

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