APPLICATIONES MATHEMATICAE
24,1 (1996), pp. 57–75

S. BYLKA (Warszawa)

ALGORITHM FOR TURNPIKE POLICIES IN THE DYNAMIC LOT SIZE MODEL

Abstract. This article considers optimization problems in a capacitated lot sizing model with limited backlogging. Nothing is assumed about the cost function in the case of finite restrictions of the size on the stock and backlogs. The holding and backlogging costs are functions assumed to be stationary or nearly stationary in time. In both cases, it is shown that there exists an optimal infinite inverse policy and a periodical turnpike policy. Some forward and backward procedures are adopted that determine an optimal infinite inverse policy and a strong turnpike policy relative to the class of standard or batch ordering type policies. Some remarks on the existence of planning and forecast horizons are also given.

1. Introduction. The classic lot size model (Wagner and Whitin [15]) involves the production of a single product, storage in a warehouse of unlimited capacity and without backlogging. Various modifications have been made to this classic model. Some of them include the introduction of upper bounds on production (on size of the order) or inventory and the backlogging or stockouts of orders. In the general case, policies of standard (with Wagner–Whitin property) type and batch ordering type are known to be "suboptimal". However, for several reasons these policies are still attractive and deserve research attention. Under a batch policy, only an integer multiple of base lot size can be ordered. The more restricted order size accommodates easy packaging and transportation.

The classical EOQ (economic order quantity) formula is perhaps the best known decision formula in the production inventory literature. See, for example, Ackoff *et al.* [1] for explicit formulas for linear holding and backlogging cost functions both in backlogging and no backlogging cases.

1991 *Mathematics Subject Classification*: Primary 90B05; Secondary 90C39.
Key words and phrases: lot size models, forecast horizon, turnpike, networks.

The stationarity assumption which underlies the EOQ model is that of the stationary demands and cost parameters over all (infinitely many) periods. An answer to the question of how many initial periods with stationarity are sufficient to assure optimality of the EOQ formula for the amount to be produced in period one, in the case without backlogging, is given by Chand, Sethi and Proth [6] for linear holding cost. A general answer is that it is H periods if and only if H is a forecast horizon in the family of all finite problems with stationarity in the first H periods. The minimal horizon for the case of nondecreasing holding cost has been found by Bylka and Sethi [5]. A generalization of the EOQ formula can be given through a turnpike policy (see also [4]). The purpose of this paper is to show how a forecast horizon may be obtained for turnpike policies in the capacitated dynamic lot size model.

Problems of existence of a horizon and a horizontal decision for a family of dynamic programming problems have been considered in operations research for a long time, in many specialized models, more intuitively or more precisely. A horizon was formally defined first in a little known but precursory paper of Łoś [10] as well as in the papers of Hinderer and Hübner [8], Lundin and Morton [12], Lee and Denardo [9] for the deterministic case and Bes and Sethi [3] for the stochastic case. The ideas presented here can be combined with their results. Horizon algorithms typically consist of solving problems of increasing length until some stopping criterion is satisfied. We refer to Bensoussan, Proth and Queyranne [2]; Ryan, Bean and Smith [13]; and Federgruen and Tzur [7] for a detailed literature review of algorithms in dynamic models. The problem of finding planning horizons for a single item capacitated lot size model has been considered by Sandbothe and Thompson [14] and Lotfi and Yoon [11]. They obtain a forecast horizon for stationary cost parameters in the case with limitations on production capacity when stockouts are permitted.

For convenience, we use a graph theory framework in our formulation. In the network presentation of the model, the EOQ formula corresponds to a cycle with minimal average cost, i.e. a turnpike. We present a natural generalization of the EOQ formula in the model with limited backlogging, stockouts and with quantized ordering (see also [16] for the stochastic case). Additionally, we omit the assumption of linearity of the cost functions. We show that the decision given by turnpike policies can be used as rolling horizontal procedure with a finite horizon. Moreover, we formulate a simple and efficient algorithm to obtain the minimal forecast horizon which has linear time complexity. For those familiar with the theory of forecast horizons the paper presents an examination of the assumption of 0-initial inventory and some perturbations of stationarity of initial demands.

The formulation of the model under study is presented in Section 2. In Section 3, the networks for stationary cost of arcs are presented. We use them for finding optimal standard plans and optimal batch ordering plans. In Section 4 we present definitions and study interaction between turnpike policies, optimal infinite inverse policies and rolling horizontal plans. Two algorithms that utilize both the optimality conditions and the turnpike horizon theorems are presented in Sections 5 and 6. The first applies to problems with stationary cost functions. The second treats the nearly stationary case. Computational results with the algorithms are given through examples and in the tables.

2. The dynamic lot size model with limited backlogging. For the purpose of this paper, we introduce the following version of the capacitated dynamic lot size model. As usual, $\mathbb{Z}$ denotes the set of all integers, while $\mathbb{N}$ is the set of positive integers.

- A demand function for a single product is defined by a sequence $d_1, d_2, \ldots$ with $d_t \geq 0$. The demand d_t is assumed to occur in time period t, $t \in N$, and must be satisfied instantaneously.
- The other variables of the model are:
  I_0 = the initial inventory (at the beginning of period 1),
  I_t = the final inventory in period t and also the initial inventory in period $t+1$,
  S_t = the amount of stockouts incurred in period t,
  u_t = the amount of production (or ordering) in period t,
- For an infinite production sequence $u = (u_1, u_2, \ldots)$ we write:
  $u|T = (u_1, \ldots, u_T)$, the T-truncation of u (the production T-plan),
  $T|u = (u_{T+1}, u_{T+2}, \ldots,)$ the "tail" without the first T coordinates.

A single lot

$$0 \leq u_t \leq \overline{u} \tag{1}$$

may be launched in any period t, and inventory is replenished in the same period t: there is no lead time in production or delivery of the product. The inventory balance equations are

$$I_t = \max\{-\beta, I_{t-1} + u_t - d_t\}, \tag{2}$$

$$-\beta \leq I_t \leq B, \tag{3}$$

where $\beta \geq 0$ is the limitation of the size of backlogs and $B \geq 0$ is the limitation of the stock. The shortages of size up to β are allowed and backordered, so the inventory positions may assume negative values not less than $-\beta$. The part of shortages above β is lost (stockouts).

$$S_t = I_t - (I_{t-1} + u_t - d_t). \tag{4}$$

We set $\beta = +\infty$ or $B = +\infty$ in the unlimited case. A production plan $u = (u_1, u_2, \ldots)$ is *feasible* if (1)–(4) are satisfied.

Finally, we assume that some preferences for the final inventory I_T are given by a function V_0 which values final inventories. As an example, if we want to have $I_T = 0$ then we define

$$V_0(I) = \begin{cases} 0 & \text{for } I = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

In order to define the objective function, we need to introduce the following production and inventory cost functions:

- $g_t(u_t)$ = the production cost (including the set up cost) in period t,
- $h_t(I_t)$ = the holding cost if $I_t \geq 0$ or the backlogging cost if $I_t < 0$ in period t,
- $s_t(S_t)$ = the stockout cost.

The T-periods optimization problem is to find

$$V_T(I_0) = \min_{u|T} \Big\{ \sum_{t=1}^{T} [g_t(u_t) + h_t(I_t) + s_t(S_t)] + V_0(I_T) \Big\} \tag{6}$$

subject to (1)–(4) with a given initial inventory I_0 and a function V_0. The quantities $(u_1, \ldots, u_t)$ are decision variables. Every plan $u|T$ which realizes this minimum is called an *optimal* T*-plan.* A feasible infinite plan is *optimal* if it realizes the minimum of average cost per period.

Let us denote the data in period t as

$$\xi_t = (d_t, g_t, h_t, s_t).$$

The infinite sequence $\xi = (\xi_1, \xi_2, \ldots)$ will be termed the *dynamic parameter of the problem* $[(\xi|I_0), \overline{u}, (\beta, B), V_0]$. The T-truncation $\xi|T = (\xi_1, \ldots, \xi_T)$ with the initial inventory I_0, the function V_0 and with a given pair (β, B) of uniform lower and upper bounds on inventories define the problem (6), (1)–(4) completely. We refer to the problem as the *dynamic programming problem*, (d.p.p. $(\xi|T, I_0)$ if we investigate finite programs with given other data).

Definition 1 (of types of production plans). Let u be a feasible production plan.

1. Let $I_0 = \sum_{t=1}^{i_0} d_t$ for some $i_0 \geq 0$. If for every $t = 1, 2, \ldots$ there exists $i_t \in \mathbb{N}$ such that $i_0 \leq i_1 \leq i_2 \leq \ldots$ and

$$\sum_{\tau=1}^{t} u_\tau = \sum_{\tau=1}^{i_t} d_\tau$$

then we say that u is a *standard plan.*

2. Let $I_0 = 0$ or let I_0 be a multiple of a quantity q (a *batch* q). If no partial fill is allowed, i.e. $u_t \in \{0, q, 2q, 3q, \ldots\}$ for every $t = 1, 2, \ldots$, then we say that u is *q-batch production plan.*

We say that a function ϕ is a *production decision function* if $\phi(I) \geq I$ for every $I \in [-\beta, B]$.

A T-tuple $f = (f_1, \ldots, f_T)$ of production decision functions is called a *T-policy* whenever f_t is used as the decision function in period t. It is a *feasible policy* if the production plan $(u_1^f, \ldots, u_T^f)$, where

$$u_t^f = f_t(I_{t-1}) - I_{t-1} \qquad \text{for } t = 1, \ldots, T, \tag{7}$$

is feasible. A policy f is called an *optimal* T-period policy for ξ if for every initial inventory I_0 its production sequence $(u_1^f, \ldots, u_T^f)$ is an optimal T-plan.

Every feasible standard production plan or batch production plan can be expressed as a path in a finite directed graph.

We use as assumptions some of the properties given below:

PROPERTY 1. *There exist two numbers $b < 0 < \overline{b}$ such that for every optimal T-plan we have $b \leq I_t \leq \overline{b}$ for each $t \leq T$. Of course, we can take $b = -\beta$, $\overline{b} = B$ if β or B is finite.*

PROPERTY 2 (The Wagner–Whitin Property). *Every optimal production plan is a standard production plan. We then speak of a* standard type problem.

PROPERTY 3 (Batch production or ordering). *All demands of a dynamic parameter ξ as well as $I_0, \overline{u}, \beta$ and B are multiples of a quantity q and every feasible production plan is a q-batch production plan. We then speak of a* q-batch ordering type problem.

PROPERTY 4 (Stationarity). *For every $t = 1, 2, \ldots$ we have $\xi_t = (d_t, g_t, h_t, s_t) = (d, g, h, s)$.*

PROPERTY 5 (Nearly stationarity). *There exists $t_0 \in N$ such that for every $t = t_0 + 1, t_0 + 2, \ldots$ we have $\xi_t = (d_t, g_t, h_t, s_t) = (d, g, h, s)$.*

PROPERTY 6 (Linearity and monotonicity). *The production cost is linear with a given set up cost $K \geq 0$ and a given unit cost $c \geq 0$. The holding inventory cost function $h(x)$ is nonincreasing for $x \leq 0$ (nondecreasing cost of backlogging), nondecreasing for $x \geq 0$, $h(0) = 0$ and $\lim_{|x|\to\infty} h(x) = +\infty$.*

We first observe that Properties 1 and 2 are well known properties of solutions of lot size models. Properties 4 and 6 imply Property 1 (and also Property 2 if additionally h is convex). The set of all dynamic programming problems $[\xi, \overline{u}, (\beta, B), V_0]$ which have Property n will be denoted by Ξ_n for $n = 1, \ldots, 6$.

3. Networks with stationary costs of arcs. Consider an optimization problem with a stationary dynamic parameter $\xi_t = (d, g, h, s)$. It can be represented as a shortest path problem. More precisely, it is the problem of finding a cheapest path of given length in:

1) a network G_ξ whose paths corespond to standard production plans, and

2) a network G_ξ^q whose paths corespond to q-batch production plans.

Assume the problem has Properties 1 and 3 with

$$d = i^* q, \text{ for some natural } i^*, \text{ and } I_0 = i_0 q,\ \overline{u} = \overline{k} q,\ b = mq,\ \overline{b} = Mq \tag{8}$$

for some natural i_0, $\overline{k}$, m and M. We define the network G_ξ^q such that

$$W_\xi^q = \{i \in \mathbb{Z} \mid m \le i \le M\}$$

and

$$E_\xi^q = \{(i,j) \in W_\xi^q \times W_\xi^q \mid 0 \le j - i + i^* \le \overline{k}\}$$

are the set of nodes and the set of arcs, respectively. It is easy to see that in nontrivial cases all these networks have loops, i.e. $(i,i) \in E_\xi^q$ for each node i.

Each arc (i,j) has arc cost value

$$\begin{aligned} &C_\xi^q(i,j) \\ &\quad = \begin{cases} g[(j-i)q + d] + h(jq) & \text{for } j > m, \\ \min\limits_{0 \le k \le m-i+i^*} [g(kq) + s((m-i+i^*-k)q) + h(mq) & \text{for } j = m. \end{cases} \end{aligned} \tag{9}$$

If $P = (i_0, i_1, \ldots, i_T)$ is a path of length T and $i_t > m$ for each t, then it coresponds to the production T-plan

$$u|T = ((i_1 - i_0 + i^*)q, (i_2 - i_1 + i^*)q, \ldots, (i_T - i_{T-1} + i^*)q).$$

Each node i_t corresponds to the inventory $I_t = i_t q$. If $i_t = m$ for some t then the production u_t may be less than $(i_t - i_{t-1} + i^*)q$ because of stockouts. The cost value of the path P is defined as usual to be the sum

$$C_\xi^q(P) = \sum_{t=1}^{T} C_{(t-1)|\xi}^q (i_{t-1}, i_t) \tag{10}$$

and it is equal to the cost of the corresponding production T-plan.

It is easy to see that if $q = d$ then standard production plans correspond to paths in the network G_ξ^d. We have $G_\xi = G_\xi^d$ in the stationary parameter case.

R e m a r k 1. Assume the problem has Property 1. Then:

1) Every problem of standard type or q-batch production type is equivalent to the problem of finding an optimal path of given length in the network G_ξ or G_ξ^q, respectively.

2) We can use the network G_ξ or G_ξ^q even if the problem is neither of standard type or q-batch type. Optimal paths correspond to the best standard plans or the best q-batch production plans, respectively.

Assume that we have a network G_ξ^q and i^* satisfies (8). A *decision function* is a function ϕ such that (similarly to (7)) $(i, \max\{\phi(i) - i^*, m\})$ is an arc for each node $i \in W_\xi^q$. Therefore, every T-policy $f = (f_1, \ldots, f_T)$ determines the path $i_0, \ldots, i_T$ such that

$$i_t = \max\{f_t(i_{t-1}) - i^*, m\} \qquad \text{for } t = 1, \ldots, T$$

for any given node i_0.

R e m a r k 2. In fact, the decision in the network is the arc $(i, \max\{\phi(i) - i^*, m\})$. It is convenient to speak of ϕ as a decision function because policies in networks correspond to standard or batch ordering policies in the inventory model.

EXAMPLE 1. Consider the d.p.p. $[(\xi, I_0), \overline{u}, (\beta, B), V_0] \in \Xi_4 \cap \Xi_6$. Specifically, $\beta = 2$, $B = +\infty$, $d = 2$, $I_0 = 0$, V_0 is defined by (5), $s(x) = x + 2$, $g(x) = 5$ if $x > 0$ and $g(0) = 0$ and

$$h(x) = \begin{cases} x & \text{for } x \geq 0, \\ x^2 & \text{for } -2 \leq x \leq 0. \end{cases}$$

We have $[(\xi, I_0), \overline{u}, (\beta, B), V_0] \in \Xi_1$ because Property 1 is satisfied for $b = -2$ and $\overline{b} = 6$. We can use the network G_ξ^1 with $W_\xi^1 = \{i \in \mathbb{Z} \mid -2 \leq i \leq 6\}$. For $\overline{u} = 3$ we have

$$E_\xi^1 = \{(i, j) \mid -2 \leq i \leq 6,\ -2 \leq j \leq 6 \text{ and } |j - i| \leq 1 \text{ or } j = 2\}.$$

with costs of edges

$$C_\xi^1((i, j)) = \begin{cases} |i| & \text{for } j = i - 2,\ j \neq -2, \\ 5 + |i| & \text{for } j \geq i - 1,\ j \neq -2, \\ 6 - i & \text{for } j = -2,\ i < 0, \\ 4 & \text{for } (i, j) = (0, -2). \end{cases}$$

4. Turnpike policies and infinite inverse optimal policies. All notions which will be defined below can be considered with respect to standard policies or batch policies in the model (optimal relatively to the class of standard policies or batch policies, respectively).

DEFINITION 2. An infinite sequence $\pi = (\pi_1, \pi_2, \ldots)$ of decision functions is a (T_0, T^0)-*turnpike policy* if for every $T \geq T_0 + T^0$ there exist

$0 \leq T_1 \leq T_0$ and $0 \leq T^1 \leq T^0$ such that

$$(f_1, \ldots, f_{T_1}, \pi_1, \ldots, \pi_{T-T_1-T^1}, f_{T_1+1}, \ldots, f_{T_1+T^1})$$

is an optimal T-policy for some decision functions $f_1, \ldots, f_{T_1+T^1}$. If additionally $T_0 = 0$ then we say $\pi = (\pi_1, \pi_2, \ldots)$ is a *rolling T^0-horizontal policy.*

The notions defined above can be considered with respect to production policies in an inventory model or to policies in the network. In the stationary case, a sequence of quantities $u^* = (u_1^*, u_2^*, \ldots)$ is a *rolling T^0-horizontal plan* (see also Lee and Denardo [9]) for (ξ, I_0) iff for every $T \geq T^0$ there exists an optimal T-plan $u = (u_1, \ldots, u_T)$ such that $u|(T - T^0) = u^*|(T - T^0)$. Analogously to the notion of rolling T^0-horizontal plan, one can define a rolling T^0-horizontal path for a node i_0 in networks. It is easy to see that every rolling horizontal policy defines rolling horizontal paths for every node i_0.

DEFINITION 3. Let $(f_1, \ldots, f_{T^0})$ be a finite sequence of decision functions and $n \geq 0$ be an integer. A finite sequence $\pi = (\pi_1, \ldots, \pi_\tau)$ of decision functions will be called an (n, T^0)-*strong turnpike policy* if there exists a collection of sequences $(f_1^r, \ldots, f_n^r)$ for $r = 1, \ldots, \tau$ such that for every T if $T = T^0 + n + k\tau + r$ for some nonnegative integer k, then the policy

$$(f_1^r, \ldots, f_n^r, \pi_{\tau-r+1}, \ldots, \pi_\tau, \pi_1, \ldots, \pi_\tau, \ldots, \pi_1, \ldots, \pi_\tau, f_1, \ldots, f_{T^0})$$

is an optimal T-policy.

Also, we write briefly $(\pi_1, \ldots, \pi_\tau, \pi_1, \ldots, \pi_\tau, \ldots,) \equiv (\pi, \pi, \ldots)$.

In the stationary forecast case, infinite inverse policies are very useful.

DEFINITION 4. We say that an infinite sequence $f = (f_1, f_2, \ldots)$ of decision functions is an *infinite inverse optimal policy* for a d.p.p. ξ iff for every $T \in \mathbb{N}$, the policy $(f_T, t_{T-1}, \ldots, f_1)$ is an optimal T-policy for the d.p.p. ξ.

Immediately from the definitions given above we have:

PROPOSITION 1. *If $\pi^* = (\pi_1^*, \ldots, \pi_\tau^*)$ is an (n, T^0)-strong turnpike policy for a d.p.p. ξ then the infinite sequence $\pi = (\pi^*, \pi^*, \ldots)$ is an $(n + \tau, T^0)$-turnpike policy for ξ.*

PROPOSITION 2. *If $\pi = (\pi_1, \pi_2, \ldots)$ is an infinite inverse optimal policy for a d.p.p. ξ, $\xi \in \Xi_4$, and a "tail" of π is periodical, i.e. $\pi_{t+\tau} = \pi_t$ for every $t \geq T^0$, then the finite sequence $\pi^* = (\pi^*_{T^0+\tau}, \ldots, \pi^*_{T^0+1})$ is a $(0, T^0)$-strong turnpike policy for the d.p.p. ξ. If additionally $\tau = 1$ then $\pi = (\pi^*, \pi^*, \ldots)$ is a rolling T^0-horizontal policy.*

Turnpike policies are closely connected with cycles with minimal average cost. In the case without backlogs Chand *et al.* [6] (see also [5]) show that there exists a forecast horizon for the EOQ formula determined by simple cycles with minimal average cost (also named turnpikes).

5. A simple algorithm for finding infinite inverse optimal policies in the stationary case. Assume $[\xi,\overline{u},(\beta,B),V_0]\in\Xi_4\cap\Xi_6$ and $\xi_t=(d,g,h,s)$ for every $t=1,2,\ldots$ As above $d=i^*q$ and $\overline{u}=\overline{k}q$, $b=mq$, $\overline{b}=Mq$ as in (8). We use the network G^q_ξ.

Let

- $V_t(i)$ be the total cost of an optimal t-policy as a function of the initial node i (the initial inventory iq),
- f_0 be the "passive" decision function, i.e. $f_0(i)=\max\{i-i^*,m\}$,
- $L_t(i)$ express the total cost when we use the following t-policy: in the first period f_0 is used (no order is placed), in subsequent periods an optimal $(t-1)$-policy is used.

We have for $t=1,2,\ldots,$

$$L_t(i)=C^q_\xi(i,f_0(i))+V_{t-1}(f_0(i))=\widetilde{h}(f_0(i)q)+V_{t-1}(f_0(i)),\tag{11}$$

where

$$\widetilde{h}(x)=\begin{cases}h(x) & \text{for } x\ge mq,\\ h(m)+s(mq-x) & \text{for } x<mq,\end{cases}$$

and the total cost

$$\begin{aligned}V_t(i)&=\min_{0\le k\le\overline{k}}C^q_\xi(i,i+k-i^*)+L_t(i+k-i^*)\\&=\min_{0\le j-i\le\overline{k}}[g((j-i)q)+L_t(j-i^*)].\end{aligned}\tag{12}$$

LEMMA 1. *Assume Property* 4. *If for every t, the policy $f^t=(f^t_t,\ldots,f^t_1)$ is an optimal t-policy for ξ then the sequence $(f^1_1,f^2_2,f^3_3,\ldots)$ is an infinite inverse optimal policy for ξ.*

Proof. The policy $k|f^t=(f^t_{t-k},\ldots,f^t_1)$ is an optimal $(t-k)$-policy for $k|\xi=\xi$ for every natural $k<t$. Therefore, for every natural t, the policy $(f^t_t,f^{t-1}_{t-1},\ldots,f^1_1)$ is an optimal t-policy for ξ and the sequence $(f^1_1,f^2_2,f^3_3,\ldots)$ is an infinite inverse optimal policy for ξ. ■

THEOREM 1. *Assume Properties* 4 *and* 6 *are satisfied. If a decision function π_t realizes the minimum in* (12), *i.e.*

$$V_t(i)=g((\pi_t(i)-i)q)+L_t(\pi_t(i)-i^*))\qquad \textit{for every } t=1,2,\ldots,\tag{13}$$

then the sequence $\pi=(\pi_1,\pi_2,\ldots)$ is an infinite inverse optimal policy in the network G^q_ξ.

Proof. It is enough to prove that for every t, the policy $\pi^t=(\pi_t,\ldots,\pi_1)$ is an optimal t-policy for ξ. The proof is by induction with respect to t.

If $t=1$, then π^1 is an optimal 1-policy for ξ because (13) holds for $t=1$. Assume that π^{t-1} is an optimal $(t-1)$-policy. From (13) and (11) we have

$$V_t(i) = g((\pi_t(i)-i)q) + \widetilde{h}((\pi_t(i)-i^*)q) + V_{t-1}(\pi_t(i)-i^*).$$

This implies that for every optimal $(t-1)$-policy $(f_1,\ldots,f_{t-1})$, the policy $(\pi_t, f_1,\ldots,f_{t-1})$ is an optimal t-policy for ξ. From the induction hypothesis $(\pi_t,\pi^{t-1}) = \pi^t$ is also an optimal t-policy. From Lemma 1 the sequence $\pi = (\pi_1,\pi_2,\ldots)$ is an infinite inverse optimal policy for ξ. ■

ALGORITHM 1. Consider the network G_ξ^q with $i^* = (1/q)d$. The integers m, M and $\overline{k}$ are also given. Calculate

$$R(i) = \begin{cases} h((i-i^*)q) & \text{for } i = m+i^*,\ldots,M+i^*,\\ h(mq)+s((m-i+i^*)q) & \text{for } i = m,\ldots,m+i^*-1,\end{cases}$$

and set $Q_0(i) = V_0(i)$.

Step t (for $t=1,2,\ldots$):

1. Compute
$$R_t(i) = \begin{cases} R(i) - Q_{t-1}(i-i^*) & \text{for } i = m+i^*,\ldots,M+i^*,\\ R(i) - Q_{t-1}(m) & \text{for } i = m,\ldots,m+i^*-1.\end{cases}$$

2. Compute
$$\varepsilon_t = \min_{m\le j\le M+i^*} R_t(j).$$

3. Compute, for each $i = m,\ldots,M$,
$$Q_t(i) = -\varepsilon_t + \min_{0\le j-i\le \overline{k}} [g((j-i)q) + R_t(j)].$$

4. Define $J_t(i)$ to be the set of all decisions j such that
$$Q_t(i) = -\varepsilon_t + g((j-i)q) + R_t(j) \qquad \text{for } i = m,\ldots,M.$$

5. Define Π_t to be the set of all decision functions f_t such that
$$f_t(i) \in J_t(i) \qquad \text{for each } i = m,\ldots,M.$$

Every sequence $\pi^T = (\pi_T,\ldots,\pi_1)$ of decision functions such that $\pi_t \in \Pi_t$ for each $t = 1,\ldots,T$ is an optimal T-policy.

If $Q_t(i) = Q_{t'}(i)$ for some $t' < t$ and for each $i = m,\ldots,M$ then set $\tau = t-t'$, $T^0 = t'$ and Stop.

We have $Q_{T+\tau} \equiv Q_T$ for every $T \le T^0$. Otherwise go to Step $t+1$.

THEOREM 2. *Algorithm* 1 *terminates in a finite number of steps. If* $Q_{T+\tau} \equiv Q_T$ *and* $\pi_t \in \Pi_T$ *for* $t = 1,\ldots,T^0+\tau$ *then the sequence is a* $(0,T^0)$*-strong turnpike policy in the network* G_ξ^q.

Proof. Let Δ be the set of all simple cycles in G_ξ^q with minimal average cost, say r^*. There exists a natural number T^* such that for each node i

of the network and for every $T \geq T^*$ there exists an optimal T-path of the following form:

$$P(i,T) = (i, i_1, \ldots, i_{T'}, \delta, \ldots, \delta, i_{T-T''}, \ldots, i_T),$$

where $\delta \in \Delta$ and $T' + T'' < T^*$. Let λ be a multiple of the lengths of all cycles of Δ. For every $T \geq T^*$ and each node i we have

$$C_\xi^q(P(i, T+\lambda)) = C_\xi^q(P(i,T)) + \lambda r^*. \tag{14}$$

On the other hand (see Steps 2 and 3 of Algorithm 1), we have

$$V_T(i) = Q_T(i) + \sum_{t=1}^{T} \varepsilon_t. \tag{15}$$

From (14) and (15), for $T \geq T^*$ we have

$$V_{T+\lambda}(i) = V_T(i) + \lambda r^*$$

and

$$Q_{T+\lambda}(i) = V_{T+\lambda}(i) - \sum_{t=1}^{T+\lambda} \varepsilon_t = Q_T(i) + \lambda r^* - \sum_{t=T+1}^{T+\lambda} \varepsilon_t.$$

If a number j^* satisfies $Q_T(j^*) = \min_i Q_T(i)$ then

$$0 = \min_i Q_{T+\lambda}(i) = Q_{T+\lambda}(j^*) = Q_T(j^*)$$

and this implies

$$\lambda r^* - \sum_{t=T+1}^{T+\lambda} \varepsilon_t = 0.$$

Therefore, $Q_T(i) = Q_{T+\lambda}(i)$ for each $i = m, \ldots, M$, $\tau = \lambda$ and the algorithm stops. The second part of the theorem follows from Theorem 1. Namely, the infinite sequence $(\pi_1, \ldots, \pi_{T^0}, \pi^*, \pi^*, \ldots)$, where $\pi^* = (\pi_{T^0+1}, \ldots, \pi_{T^0+\tau})$ and $\pi_t \in \Pi_t$ for every $t = 1, 2, \ldots$, is an infinite inverse optimal policy. In the same manner we can see that for $T = T^0 + k\tau + r$, $1 \leq r \leq \tau$, the policy $(\pi_{T^0+r}, \ldots, \pi_{T^0+1}, \pi^*, \ldots, \pi^*, \pi_{T^0}, \ldots, \pi_1)$ is an optimal T-policy. Thus, π^* is a $(0, T^0)$-strong turnpike policy, which ends the proof of the theorem. ∎

Actually, the proof above gives more, namely:

COROLLARY 1. *If Algorithm* 1 *terminates with* $Q_{T+\tau} \equiv Q_T$ *then the minimal average cost of the infinite horizon problem is equal to*

$$r^* = \frac{1}{\tau} \sum_{t=T^0+1}^{T^0+\tau} \varepsilon_t.$$

Because Algorithm 1 enables us to find all optimal finite stage policies we can solve the problem of the existence of planning and forecast horizons (introduced in Łoś [10] and Lundin and Morton [12]). We can now formulate:

COROLLARY 2. *Let the stationary problem be of standard type* (*Property* 2) *or q-batch ordering type with an arbitrary q* (*Property* 3). *Suppose Algorithm* 1 *terminates in step $T^0+\tau$. Let $I_0 = i_0 q$ be a given initial inventory position. If there exist $f_t \in \Pi_t$ such that*

$$f_t(i_0)q = I_0 + u_1^* \quad \text{for each } t = T^0+1,\ldots,T^0+\tau$$

then u_1^* is a planning horizontal lot size with T^0 as its forecast horizon. On the other hand, if it is not possible to choose such decision functions then there is no finite forecast horizon.

EXAMPLE 1 (continued). Let $\overline{u}=6$ and all other parameters as before.

Case 1. We look for optimal policies in the network $G_\xi = G_\xi^2$. In this case $\overline{k}=3$, $m=-1$ and we can set $M=4$.

TABLE 1

i	-1	0	1	2	3	4	
$R(i)$	8	4	0	2	6	8	
$Q_0(i)$	$+\infty$	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	
$R_1(i)$	$+\infty$	$+\infty$	0	$+\infty$	$+\infty$	$+\infty$	$\varepsilon_1=0$
$Q_1(i)$	5^{*1}	5^{*1}	0	$+\infty$	$+\infty$	$+\infty$	
$R_2(i)$	13	9	5	2	$+\infty$	$+\infty$	$\varepsilon_2=2$
$Q_2(i)$	5^{*2}	5^{*2}	3	0	$+\infty$	$+\infty$	
$R_3(i)$	13	9	5	5	4	$+\infty$	$\varepsilon_3=4$
$Q_3(i)$	$6^{*1,2}$	5^{**3}	1	1	0	$+\infty$	
$R_4(i)$	14	10	5	3	5	6	$\varepsilon_4=3$
$Q_4(i)$	5^{*2}	5^{*2}	2	0	2	3	
$R_5(i)$	13	9	5	4	4	8	$\varepsilon_5=4$
$Q_5(i)$	5^{*2}	$5^{**2,3}$	1	0	0	4	
$R_6(i)$	13	9	5	3	4	6	$\varepsilon_6=3$
$Q_6(i)$	5^{*2}	5^{*2}	2	0	1	3	
$R_{7+2k}(i)$	13	9	5	4	4	7	$\varepsilon_{7+2k}=4$
$Q_{7+2k}(i)$	5^{*2}	$5^{**2,3}$	1	0	0	3	
$R_{8+2k}(i)$	13	9	5	3	4	6	$\varepsilon_{8+2k}=3$
$Q_{8+2k}(i)$	5^{*2}	5^{*2}	2	0	1	3	$\equiv Q_6$

TABLE 2

i	-1	0	1	2	3	4	>4
$J_1(i)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_2(i)$	$\{2\}$	$\{2\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_3(i)$	$\{1,2\}$	$\{0,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_4(i)$	$\{2\}$	$\{0,2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_5(i)$	$\{2\}$	$\{0,2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_6(i)$	$\{2\}$	$\{2\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_{7+2k}(i)$	$\{2\}$	$\{0,2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$
$J_{7+2k}(i)$	$\{2\}$	$\{2\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{i\}$

The steps of Algorithm 1 are presented in Table 1 ($Q_t(i) = x^{*j}$ means that the decision $j > i$ has already been chosen, $Q_t(i) = x^{**j}$ means that additionally $j = i$ has already been chosen). The sets of all decisions used in optimal policies in Case 1 are presented in Table 2.

We have $\Pi_{t+2} = \Pi_t$ for all $t \geq 4$. Let us define $\pi^* = (\pi_1^*, \pi_2^*)$, where the decision functions have been chosen from Π_7 and Π_8 as follows:

$$\pi_1^*(i) = \begin{cases} 2 & \text{for } i = -1, \\ i & \text{for } i \geq 0, \end{cases}$$

and

$$\pi_2^*(i) = \begin{cases} 2 & \text{for } i \in \{-1, 0\}, \\ i & \text{for } i \geq 1. \end{cases}$$

It is easy to see that $(\pi_1, \pi_2^*, \pi_1^*, \pi_2^*, \pi_1^*, \ldots)$, $\pi_1 \in \Pi_1$, is an infinite inverse optimal policy and, moreover, π^* is a $(0,1)$-strong turnpike policy in the network G_ξ. On the other hand, each policy $(\pi_1, \pi_2, \pi_3, \pi_2^*, \pi_2^*, \ldots)$, $\pi_t \in \Pi_t$ for $t = 1, 2, 3$, is also infinite inverse optimal. Therefore, $\widetilde{\pi} = (\widetilde{\pi}_1)$, where $\widetilde{\pi}_1 = \pi_2^*$ is a stationary $(0,3)$-strong turnpike policy in G_ξ. This turnpike policy determines a turnpike of the network—the simple cycle $(0,1,0)$ with minimal average cost $3\frac{1}{2}$. It also determines the standard production $(0,3)$-strong turnpike policy $f^* = (f_1^*)$, where $f_1^*(I) = 2\pi_2^*(\frac{1}{2}I)$. In fact, it is a policy for d.p.p.'s with the initial inventory I_0 which is a multiple of the demand $d = 2$.

Using f^*, for $I_0 = 0$ we obtain the 4-rolling horizon standard production (ordering) plan $(4, 0, 4, 0, \ldots)$.

Case 2. We look for optimal policies in the network G_ξ^1. In this case $\overline{k} = 6$, $m = -2$ and we can set $M = 7$. Steps of Algorithm 1 and the sets of decisions used in optimal policies are presented in Table 3 and Table 4, respectively.

We have $\Pi_{t+3} = \Pi_t$ for all $t \geq 8$. Let us define $\pi^* = (\pi_1^*)$, where the decision function has been chosen from $\Pi_8 \cap \Pi_9 \cap \Pi_{10}$ as follows:

$$\pi_1^*(i) = \begin{cases} 4 & \text{for } i = -2, \\ 5 & \text{for } i = -1, 0, \\ i & \text{for } i \geq 1. \end{cases}$$

It is easy to see that $(\pi_1, \pi_2, \ldots, \pi_6, \pi_1^*, \pi_1^*, \ldots)$, where $\pi_t \in \Pi_t$ for each $t = 1, \ldots, 6$, is an infinite inverse optimal policy for every $T > 6$. Moreover, π^* is a stationary $(0,6)$-strong turnpike policy in the network G_ξ^1. This turnpike policy determines the simple cycle $(1, -1, 3, 1)$ with minimal average cost $3\frac{1}{3}$. It also determines a $(0,7)$-strong turnpike 1-batch production policy $f^* = (f_1^*)$, where $f_1^*(I) = \pi_1^*(I)$, given in (16). In fact, it is a policy for d.p.p.'s with the initial inventory I_0 which are integers. For $I_0 = 0$ we have the 7-rolling horizon 1-batch production plan $(5, 0, 0, 6, 0, 0, 6, 0, 0, \ldots)$.

TABLE 3

i	-2	-1	0	1	2	3	4	5	6	7	ε_t
$R(i)$	8	7	4	1	0	1	2	3	4	5	
$Q_0(i)$	∞	∞	0	∞	∞	∞	∞	∞	∞	∞	
$R_1(i)$	∞	∞	∞	∞	0	∞	∞	∞	∞	∞	0
$Q_1(i)$	5^{*2}	5^{*2}	5^{*2}	5^{*2}	0	∞	∞	∞	∞	∞	
$R_2(i)$	13	12	9	6	5	6	2	∞	∞	∞	2
$Q_2(i)$	5^{*4}	5^{*4}	5^{*4}	4	3	4	0	∞	∞	∞	
$R_3(i)$	13	12	9	6	5	5	5	7	4	∞	4
$Q_3(i)$	$6^{*2,3,4}$	$6^{*2,3,4}$	5^{**6}	2	1	1	1	3	0	∞	
$R_4(i)$	14	14	10	7	5	3	3	4	5	8	3
$Q_4(i)$	$5^{*3,4}$	$5^{*3,4}$	$5^{*3,4}$	4	2	0	0	1	2	5	
$R_5(i)$	13	12	9	6	4	5	4	3	4	6	3
$Q_5(i)$	$6^{*2,4}$	5^{*5}	5^{*5}	3	1	2	1	0	1	3	
$R_6(i)$	14	13	10	6	5	4	3	5	5	5	3
$Q_6(i)$	5^{*4}	5^{*4}	5^{*4}	3	2	1	0	2	2	2	
$R_7(i)$	13	12	9	6	5	4	4	4	5	5	4
$Q_7(i)$	$5^{*3,4}$	$5^{**3,4,5}$	$5^{*3,4,5}$	2	1	0	0	0	1	1	
$R_{8+3k}(i)$	13	12	9	6	5	3	3	3	4	5	3
$Q_{8+3k}(i)$	$5^{*3,4}$	$5^{*3,4,5}$	$5^{*3,4,5}$	3	2	0	0	0	1	2	
$R_{9+3k}(i)$	13	12	9	6	5	4	4	3	4	5	3
$Q_{9+3k}(i)$	$6^{*2,4}$	5^{*5}	5^{*5}	3	2	1	1	0	1	2	
$R_{10+3k}(i)$	14	13	10	6	5	4	4	4	5	5	4
$Q_{10+3k}(i)$	$5^{*3,4}$	$5^{*3,4,5}$	$5^{*3,4,5}$	2	1	0	0	0	1	1	

TABLE 4

i	-2	-1	0	1	2	≥ 3
$J_1(i)$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{i\}$
$J_2(i)$	$\{4\}$	$\{4\}$	$\{4\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_3(i)$	$\{2,3,4\}$	$\{2,3,4\}$	$\{0,6\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_4(i)$	$\{3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_5(i)$	$\{2,4\}$	$\{5\}$	$\{5\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_6(i)$	$\{4\}$	$\{4\}$	$\{4\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_7(i)$	$\{3,4\}$	$\{3,4,5\}$	$\{0,3,4,5\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_{8+3k}(i)$	$\{3,4\}$	$\{5\}$	$\{5\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_{9+3k}(i)$	$\{3,4\}$	$\{3,4,5\}$	$\{3,4,5\}$	$\{1\}$	$\{2\}$	$\{i\}$
$J_{10+3k}(i)$	$\{3,4\}$	$\{3,4,5\}$	$\{3,4,5\}$	$\{1\}$	$\{2\}$	$\{i\}$

6. Turnpike policies for nearly constant dynamic parameter. In this section we assume $[\xi, \overline{u}, (\beta, B), V_0] \in \Xi_1 \cap \Xi_5$. Now we consider a more general situation, where the dynamic parameter is stationary except in a few first periods, i.e. there exists a positive integer t_0 such that $t_0|\xi = (\xi_{t_0+1}, \xi_{t_0+2}, \ldots)$ is stationary. This is more interesting in practice because a few first nonstationary periods can simulate an adaptation of a stationary problem. Note that the networks G_ξ^q with the dynamic cost of paths (as in (10)) may be used. Therefore, we rewrite all formulas of Step t of

Algorithm 1 to obtain the following algorithm for finding optimal T-policies for a nearly stationary dynamic parameter:

ALGORITHM 2. Consider the network G^q_ξ, $\xi_t = (d_t, g_t, h_t, s_t)$ with $\xi_t = (d, g, h, s)$ for all $t > t_0$. Define $i^* = (1/q)d$ and $i^*_t = (1/q)d_t$ for $t = 1, \dots, t_0$. The integers m, M and $\overline{k}$ are also given.

If $T > t_0$ then we use Algorithm 1 for $G^q_{t_0|\xi}$ to calculate $Q_t(i)$ and $J_t(i)$ for $t = 1, \dots, T - t_0$. Set $Q^{t_0}_0(i) = Q_{T-t_0}(i)$ and perform the algorithm in the same way as in Algorithm 1 using the following formulas:

S t e p t (for $t = 1, \dots, t_0$):

0. Calculate

$$R^t(i) = \begin{cases} h_{t_0-t+1}((i - i^*_{t_0-t+1})q) & \text{for } i \geq m + i^*_{t_0-t+1}, \\ h_{t_0-t+1}(mq) + s_{t_0-t+1}((m - i + i^*_{t_0-t+1})q) & \text{for } i < m + i^*_{t_0-t+1}. \end{cases}$$

1. Compute

$$R^{t_0}_t(i) = \begin{cases} R^t(i) - Q^{t_0}_{t-1}(f_{0,t_0-t+1}(i)) & \\ & \text{for } i = m + i^*_{t_0-t+1}, \dots, M + i^*_{t_0-t+1}, \\ R^t(i) - Q^{t_0}_{t-1}(m) & \text{for } i = m, \dots, m + i^*_{t_0-t+1} - 1. \end{cases}$$

2. Compute

$$\varepsilon^{t_0}_t = \min_j R^{t_0}_t(j).$$

3. Compute, for each $i = m, \dots, M$,

$$Q^{t_0}_t(i) = -\varepsilon^{t_0}_t + \min_{0 \leq j-i \leq \overline{k}} [g_{t_0-t+1}((j-i)q) + R^{t_0}_t(j)].$$

4. Define $J^{t_0}_t(i)$ to be the set of all decisions j such that

$$Q^{t_0}_t(i) = -\varepsilon^{t_0}_t + g_{t_0-t+1}((j-i)q) + R^{t_0}_t(j) \qquad \text{for } i = m, \dots, M.$$

and

$$\Pi^{t_0}_t = \{f \mid f(i) \in J^{t_0}_t(i),\ i = m, \dots, M\},$$

5. Define Π_t to be the following set of decision functions:

$$\Pi_t = \begin{cases} \{f \mid f(i) \in J_t(i),\ i = m, \dots, M\} & \text{if } t \leq T - t_0, \\ \{f \mid f(i) \in J^{t_0}_{t+t_0-T}(i),\ i = m, \dots, M\} & \text{if } t > T - t_0. \end{cases}$$

Every sequence $\pi^T = (\pi_T, \dots, \pi_1)$ of decision functions such that $\pi_t \in \Pi_t$ for each $t = 1, \dots, T$ is an optimal T-policy.

THEOREM 3. *Let* $[\xi, \overline{u}, (\beta, B), V_0] \in \Xi_3 \cap \Xi_5$ *for some* q *and* $(t_0|\xi)$ *be stationary for some* t_0. *Then there exists a* (t_0, T^0)*-strong turnpike policy.*

P r o o f. We use Algorithm 1 for the network $G^q_{t_0|\xi}$. From Theorem 2, there exist two positive integers T^0 and τ such that the algorithm terminates with $Q_{T^0+\tau} \equiv Q^0_T$. Pick $\pi_t \in \Pi_t$ for $t = T^0 + 1, \dots, T^0 + \tau$. Then $\pi^* = (\pi_{T^0+\tau}, \dots, \pi_{T^0+1})$ is a $(0, T^0)$-strong turnpike policy for $(t_0|\xi)$.

For every $T > T^0$ if $T = T^0 + k\tau - r$ for some $1 \le r \le \tau$, then

$$(\pi_{T^0+r}, \dots, \pi_{T^0+1}, \pi^*, \dots, \pi^*, \pi_{T^0}, \pi_{T^0-1}, \dots, \pi_1)$$

is an optimal T-policy.

For each $r = 1, \dots, \tau$ and $Q_0^{t_0} \equiv Q_{T^0+r}$ we compute, using Algorithm 2, the sets of policies Π_t^r. Let us choose $f_t^r \in \Pi_t^r$ for $t = 1, \dots, t_0$. For every $T > T^0$ if $T = T^0 + k\tau - r$ for some $1 \le r \le \tau$, then

$$(f_{t_0}^r, \dots, f_1^r, \pi_{T^0+r}, \dots, \pi_{T^0+1}, \pi^*, \dots, \pi^*, \pi_{T^0}, \pi_{T^0-1}, \dots, \pi_1)$$

is an optimal T-policy in the network G_ξ^q with dynamic arc cost. Therefore $\pi^* = (\pi_{T^0+\tau}, \dots, \pi_{T^0+1})$ is a (t_0, T^0)-strong turnpike policy. ∎

As a corollary, we have the following forecast horizon result (see also Federgruen and Tzur [7] and Chand, Sethi and Proth [6]):

COROLLARY 3. *Let the problem be of standard type or q-batch type with arbitrary q. Assume the dynamic parameter ξ to be nearly stationary with stationary tail $(t_0|\xi)$. Let Algorithm 1 be stopped with $Q_{T^0+\tau} \equiv Q_T^0$ and let the sets of policies Π_t^r, $t = 1, \dots, t_0$ and $r = 1, \dots, \tau$, be constructed by Algorithm 2 for the network G_ξ^q. If for a given initial inventory $I_0 = i_0 q$ there exist $f^r \in \Pi_{t_0}^r$ such that*

$$(f^r(i_0) - i_0)q = u_1^* \quad \text{for each } r = 1, \dots, \tau$$

then u_1^* is a planning horizontal lot size with $t_0 + T^0$ as its forecast horizon. On the other hand, if it is not possible to choose such decision functions then there is no finite forecast horizon.

EXAMPLE 2. Let us consider the family of d.p.p.'s with all stationary parameters the same as in Example 1. The dynamic parameter $\xi_t = (d_t, g_t, h_t, s_t)$ is as follows: for $t \le 4$ we have $g_t(0) = 0$ and if $x > 0$ then $g_t(x) = K_t$, where K_t are given in Table 5.

TABLE 5

t	1	2	3	4
K_t	$5\frac{1}{2}$	$5\frac{1}{2}$	5	5
d_t	2	3	2	1

We specify the values of d_t by the data in Table 5 and all other parameters the same as in Example 1. Note that if we change the dynamic parameter ξ to $(4|\xi)$ then we have the same stationary problem as in Example 1.

We look for optimal policies in the network G_{ξ^1}. In this case $\overline{k} = 6$, $m = -2$ and we can set $M = 7$. Using Algorithm 1 we can find strong turnpike policies in $G_{4|\xi}^1$ (see Tables 3 and 4 in Example 1). Let us perform

Algorithm 2 for three cases with respect to $Q_0^{t_0}$, i.e. $Q_0^4 = Q_8$, $Q_0^4 = Q_9$ and $Q_0^4 = Q_{10}$. Steps of Algorithm 2 and the sets of decisions used in optimal policies for the first period are presented in Table 6 and Table 7, respectively.

TABLE 6

i	-2	-1	0	1	2	3	4	5	6	7	ε_t
$R^1(i)$	7	4	1	0	1	2	3	4	5	6	
$R^2(i)$	8	7	4	1	0	1	2	3	4	5	
$R^3(i)$	9	8	7	4	1	0	1	2	3	4	
$R^4(i)$	8	7	4	1	0	1	2	3	4	5	
$Q_0^4(i)$	5	5	5	3	2	0	0	0	1	2	
$R_1^4(i)$	12	9	6	5	4	4	3	4	5	7	3
$Q_1^4(i)$	5^{*4}	5^{*4}	3	2	1	1	0	1	2	4	
$R_2^4(i)$	13	12	9	6	4	3	3	4	4	6	3
$Q_2^4(i)$	$5^{*3,4}$	$5^{*3,4}$	$5^{*3,4}$	3	1	0	0	1	1	3	
$R_3^4(i)$	14	13	12	9	6	5	4	3	3	4	3
$Q_3^4(i)$	$6\frac{1}{2}^{*4}$	$5\frac{1}{2}^{*5}$	$5\frac{1}{2}^{*5,6}$	$5\frac{1}{2}^{5,6}$	3	2	1	0	0	1	
$R_4^4(i)$	$14\frac{1}{2}$	$13\frac{1}{2}$	$10\frac{1}{2}$	$6\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	5	5	5	5	5
$Q_4^4(i)$	$5\frac{1}{2}^{*4}$	$5\frac{1}{2}^{*4,5}$	$5\frac{1}{2}^{*4,5,6}$	$1\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	0	0	0	0	
$Q_0^4(i)$	6	5	5	3	2	1	1	0	1	2	
$R_1^4(i)$	13	10	6	5	4	4	4	5	5	7	4
$Q_1^4(i)$	$5^{*2,3,4}$	$5^{*2,3,4}$	2	1	0	0	0	1	1	3	
$R_2^4(i)$	13	12	9	6	2	2	2	3	4	6	2
$Q_2^4(i)$	$5^{*2,3,4}$	$5^{*2,3,4}$	$5^{*2,3,4}$	4	0	0	0	1	2	4	
$R_3^4(i)$	14	13	12	9	6	5	5	2	3	4	2
$Q_3^4(i)$	$8\frac{1}{2}^{*4}$	$5\frac{1}{2}^{*5}$	$5\frac{1}{2}^{*5}$	$5\frac{1}{2}^{5}$	4	3	3	0	1	2	
$R_4^4(i)$	$16\frac{1}{2}$	$15\frac{1}{2}$	$12\frac{1}{2}$	$6\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	6	6	7	5	5
$Q_4^4(i)$	6^{*2}	6^{*2}	6^{*2}	$1\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	1	1	2	0	
$Q_0^4(i)$	5	5	5	2	1	0	0	0	1	1	
$R_1^4(i)$	12	9	6	5	3	3	3	4	5	7	3
$Q_1^4(i)$	$5^{*3,4,5}$	$5^{*3,4,5}$	3	2	0	0	0	1	2	4	
$R_2^4(i)$	13	12	9	6	3	3	2	3	4	6	2
$Q_2^4(i)$	5^{*4}	5^{*4}	5^{*4}	4	1	1	0	1	2	4	
$R_3^4(i)$	14	13	12	9	6	5	5	3	4	4	3
$Q_3^4(i)$	$7\frac{1}{2}^{*4}$	$5\frac{1}{2}^{*5}$	$5\frac{1}{2}^{*5}$	$5\frac{1}{2}^{5}$	3	2	2	0	1	1	
$R_4^4(i)$	15	$14\frac{1}{2}$	$11\frac{1}{2}$	$6\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	5	5	6	5	5
$Q_4^4(i)$	$5\frac{1}{2}^{*4}$	$5\frac{1}{2}^{*4,5}$	$5\frac{1}{2}^{*4,5}$	$1\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	0	0	1	0	

TABLE 7

i	-2	-1	0	1	2	3	≥ 4
$J_4^4(i)[Q_0^4 = Q_8]$	$\{4\}$	$\{4,5\}$	$\{4,5,6\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{i\}$
$J_4^4(i)[Q_0^4 = Q_9]$	$\{2\}$	$\{2\}$	$\{2\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{i\}$
$J_4^4(i)[Q_0^4 = Q_{10}]$	$\{4\}$	$\{4,5\}$	$\{4,5\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{i\}$

The sequence $\pi^* = (\pi_1^*)$ where the decision function has been chosen from $\Pi_8 \cap \Pi_9 \cap \Pi_{10}$ (see (16)), with each collection of three sequences of decision functions $(f_4^r, f_3^r, f_2^r, f_1^r)$, $r = 1,2,3$, $f_t^r \in \Pi_t^r$ for $t = 1,\ldots,4$, is a $(4,11)$-strong turnpike policy in G_ξ^1. From Corollary 4 we conclude that there are no planning and forecast horizons for the initial inventory $I_0 = 0$.

7. Concluding remarks. We have shown the existence of strong turnpike policies in the capacitated dynamic lot size model with limited backlogging and stockouts. Algorithms 1 and 2 construct all decision functions for turnpike polices. We also formulate (Corollary 3 and Corollary 4) an answer about planning and forecast horizons. These results extend those obtained by Chand, Sethi and Proth [6] and Sandbothe and Thompson [14] for the capacitated case. In the not capacitated stationary case, Algorithm 1 gives strong turnpike policies such that the indicated re-order quantities are the same as in the EOQ formulas. By Corollary 3 we can obtain a forecast horizon for this re-order quantity. In general, it may not be a minimal forecast horizon.

References

[1] R. L. Ackoff, E. L. Arnoff and C. Churchman, *Introduction to Operations Research*, Wiley, New York, 1957.

[2] A. Bensoussan, J. M. Proth and M. Queyranne, *A planning horizon algorithm for deterministic inventory management with piecewise linear concave costs*, Naval Res. Logist. 38 (1991), 729–742.

[3] C. Bes and S. Sethi, *Concepts of forecast and decision horizons*: *applications to dynamic stochastic optimization problems*, Math. Oper. Res. 13 (1988), 295–310.

[4] S. Bylka, *Horizon theorems for the solution of the dynamic lot-size model*, in: Proc. Second Internat. Sympos. on Inventories, Budapest, Publ. House Hungar. Acad. Sci., 1982, 649–660.

[5] S. Bylka and S. Sethi, *Existence and derivation of forecast horizons in a dynamic lot size models with nondecreasing holding costs*, Production and Operations Management 1 (1992), 212–224.

[6] S. Chand, S. P. Sethi and J. M. Proth, *Existence of forecast horizons in undiscounted discrete time lot-size model*, Oper. Res. 38 (1990), 884–892.

[7] A. Federgruen and M. Tzur, *The dynamic lot-sizing model with backlogging: A simple $O(n \log n)$ algorithm and minimal forecast horizon procedure*, Naval Res. Logist. 40 (1993), 459–478.

[8] K. Hinderer and G. Hübner, *An improvement of J. F. Shapiro's turnpike theorem for the horizon of finite stage discrete dynamic programs*, in: Trans. Seventh Prague Conf. on Information Theory 1974, Vol. A, Acad. Publ. House, Praha, 1977, 245–255.

[9] C. Y. Lee and E. V. Denardo, *Rolling planning horizons: error bounds for the dynamic lot-size model*, Math. Oper. Res. 11 (1986), 423–432.

[10] J. Łoś, *Horizons in dynamic programs*, in: Proc. Fifth Berkeley Sympos. on Mathematical Statistics and Probability, California University Press, Berkeley, Calif., 1967, 479–490.

[11] V. Lotfi and Y.-S. Yoon, *An algorithm for the single-item capacitated lot size model with concave production and holding costs*, J. Oper. Res. Soc. 45 (1994), 934–941.

[12] R. Lundin and T. Morton, *Planning horizon for the dynamic lot size model*: *Zabel vs. protective procedures and computational results*, Oper. Res. 23 (1975), 711–735.

[13] S. M. Ryan, J. C. Bean and L. Smith, *A tie-breaking rule for discrete infinite horizon optimization*, ibid. 40 (1992), Suppl. 2, S117–S126.

[14] R. A. Sandbothe and G. L. Thompson, *A forward algorithm for the capacitated lot size model with stockouts*, ibid. 38 (1990), 474–486.

[15] H. M. Wagner and T. M. Whitin, *Dynamic version of the economic lot size model*, Manag. Sci. 5 (1959), 89–96.

[16] Y.-S. Zheng and F. Chen, *Inventory policies with quantized ordering*, Naval Res. Logist. 39 (1992), 285–305.

Stanisław Bylka
Institute of Computer Science
Polish Academy of Sciences
21 Ordona Street
01-237 Warszawa, Poland
E-mail: bylka@wars.ipipan.waw.pl

Received on 29.6.1995;
revised version on 12.10.1995 and 12.4.1996