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THE LINEAR PROGRAMMING APPROACH TO DETERMINISTIC OPTIMAL CONTROL PROBLEMS

Abstract. Given a deterministic optimal control problem (OCP) with value function, say J^*, we introduce a linear program (P) and its dual (P^*) whose values satisfy $\sup(P^*) \leq \inf(P) \leq J^*(t,x)$. Then we give conditions under which (i) there is *no duality gap*, i.e. $\sup(P^*) = \inf(P)$, and (ii) (P) is *solvable* and it is *equivalent* to the (OCP) in the sense that $\min(P) = J^*(t,x)$.

1. Introduction. A time-honored approach to optimal control problems (OCPs) is via mathematical programming problems on suitable spaces. For instance, this approach can be used to obtain Pontryagin's maximum principle; see e.g. [3]. Another class of results has also been obtained for both deterministic and stochastic OCPs using convex programming methods [2, 5, 6].

This paper is concerned with the *linear programming* (LP) approach to deterministic, finite-horizon OCPs with value function $J^*(t,x)$—when the initial data is (t,x) [see (2.3)]. In this case, we first introduce a linear program (P) and its dual (P^*) for which

$$\sup(P^*) \leq \inf(P) \leq J^*(t,x), \tag{1.1}$$

where $\sup(P^*)$ and $\inf(P)$ denote the values of (P^*) and (P), respectively. Then we give conditions under which

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(i) there is *no duality gap*, i.e.,

$$\sup(P^*) = \inf(P); \tag{1.2}$$

(ii) the linear program (P) is *solvable*, which means that (P) has an optimal solution (and we write $\min(P)$ instead of $\inf(P)$), and is *equivalent* to the OCP in the sense that

$$\min(P) = J^*(t,x). \tag{1.3}$$

Related literature. In recent papers [8, 9], we have obtained results similar to (1.1)–(1.3) for some discrete-time stochastic control problems on general Borel spaces. Our work is also related to the *convex programming* approach in [2, 5, 6] in that we use (LP) *duality* theory to get (1.1)–(1.3); in fact, to set our OCP we follow closely [5, 6]. Finally, we should mention that for several classes of OCPs (see e.g. [12, 13]) there is a well known, direct way—i.e., without going through the dual program (P^*)—to get (1.3); namely, one simply writes down the associated linear program (P) and then uses continuity/compactness arguments to get a minimizing sequence that converges to the optimal value. But of course, using duality, one gets more information on the OCP. For example, it turns out that the dual (P^*) is associated with the dynamic programming equation (DPE) in a sense to be precised in the Corollary to Theorem 5.1.

Organization of the paper. In Section 2 we introduce the OCP we are interested in, and recall some facts on the dynamic programming equation. Section 3 presents the linear programs (P) and (P^*) associated with the OCP. We also prove the consistency of these programs. In Section 4 we present the proof of (1.1)–(1.2), whereas the equality (1.3) is proved in Section 5. Finally, in Section 6 we introduce a particular approximation to the value function.

2. The optimal control problem

Remark 2.1. *Notation.* (a) If X is a generic metric space, then we denote by $C(X)$ the space of real-valued continuous bounded functions with finite uniform norm $\|\ \|$. If $b : X \to \mathbb{R}$ is a continuous function with $b(\cdot) \ge 1$ (which we call a *bounding function*), then $C_b(X)$ stands for the real vector space of all continuous functions $v : X \to \mathbb{R}$ such that

$$\|v\|_b := \|v/b\| = \sup_{x\in X} |v(x)|/b(x) < \infty.$$

Let $\mathcal{D}_b(X)$ be the *dual* of $C_b(X)$, i.e. the vector space of all bounded linear functionals on $C_b(X)$. If $\xi \in \mathcal{D}_b(X)$ and $v \in C_b(X)$, we denote by $\langle \xi, v\rangle$ the value of ξ at v.

(b) Let $\mathcal{M}_b(X)$ be the vector space of all finite signed measures μ on the Borel sets of X such that $\|\mu\|_b := \int b\, d|\mu|$ is finite, where $|\cdot|$ stands for the total variation. Then, identifying $\mu \in \mathcal{M}_b(X)$ with the linear functional $v \to \langle \mu, v\rangle := \int v\, d\mu$ on $C_b(X)$, we see that $\mathcal{M}_b(X) \subset \mathcal{D}_b(X)$ since

$$|\langle \mu, v\rangle| \le \|v\|_b \|\mu\|_b.$$

(c) Let T, $0 < T < \infty$, be the optimization horizon, and $U \subset \mathbb{R}^n$ the control set, which is assumed to be compact. Define $\Sigma := [0,T] \times \mathbb{R}^n$, $S := \Sigma \times U$.

If v is a function on $\mathbb{R}^n$, we consider it to be a function on Σ, S or $\mathbb{R}^n \times U$, defining $v(t,x) := v(x)$, $v(t,x,u) := v(x)$ or $v(x,u) := v(x)$ respectively.

For each $t \in [0,T]$, the set $\mathcal{U}(t)$ of control processes is the set of Borel measurable functions $\mathbf{u} : [t,T] \to U$.

The optimal control problem (OCP). Let $f : S \to \mathbb{R}^n$ be a given function, and consider the controlled system

$$\dot{x}(s) := f(s, x(s), \mathbf{u}(s)), \qquad t < s \le T,\ x(t) = x, \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $\mathbf{u} \in \mathcal{U}(t)$. The OCP is then to minimize

$$J(t,x;\mathbf{u}) := \int_t^T l_0(s, x(s), \mathbf{u}(\mathbf{s}))\, ds + L_0(x(T)) \tag{2.2}$$

over the pairs $(x(\cdot), \mathbf{u}(\cdot))$ that satisfy Definition 2.2. The OCP's *value function* J^* is defined as

$$J^*(t,x) := \inf_{\mathcal{U}(t)} J(t,x;\mathbf{u}). \tag{2.3}$$

DEFINITION 2.2. A pair $(x(\cdot), \mathbf{u}(\cdot))$ is said to be *admissible* for the initial data (t,x) if $\mathbf{u}(\cdot) \in \mathcal{U}(t)$, and $x(\cdot)$ satisfies (2.1). We shall denote by $\mathcal{P}(t,x)$ the family of all admissible pairs, given the initial data (t,x).

Throughout the following we assume (H1)–(H3) below:

(H1) f belongs to $C(S)$ and it is Lipschitz in $x \in \mathbb{R}^n$, uniformly in $(t,u) \in [0,T] \times U$, i.e.

$$\sup_S |f(t,x,u)| \le K \quad \text{and} \quad |f(t,x,u) - f(t,y,u)| \le c|x-y| \quad \forall x,y \in \mathbb{R}^n,$$

where c is some constant independent of (t,u).

(H2) l_0 and L_0 are nonnegative, bounded away from zero, continuous functions on S and $\mathbb{R}^n$ respectively, and there exists a real-valued continuous function $b(x)$ on $\mathbb{R}^n$ such that

$$l_0(t,x,u) \le b(x), \qquad \forall (t,x,u) \in S,$$

$$L_0(x) \le b(x), \qquad \forall x \in \mathbb{R}^n,$$

$$b(x)/l_0(t,x,u) \in C(S), \quad \text{and} \quad b(x)/L_0(x) \in C(\mathbb{R}^n).$$

(H3) There exist $\varepsilon_0 > 0$ and $c > 0$ such that for all $|s-t|, |x-y| < \varepsilon_0$,

$$|b(y) - b(x)| \le c|y-x|b(x),$$
$$|l_0(t,x,u) - l_0(s,y,u)| \le c(|y-x| + |t-s|)b(x),$$
$$|L_0(y) - L_0(x)| \le c|y-x|b(x);$$

without loss of generality we may take c to be the same as in (H1).

The dynamic programming equation (DPE). We write partial derivatives as $D_0 := \partial/\partial t$ and $D_i := \partial/\partial x_i$ for $i = 1, \dots, n$. Let b be as in (H2) and define $C_b^1(\Sigma)$ as the Banach space consisting of all the functions $\varphi \in C_b(\Sigma)$ with partial derivatives $D_i\varphi$ in $C_b(\Sigma)$ for all $i = 0, 1, \dots, n$, with

$$\|\varphi\|_b^1 := \|\varphi\|_b + \sum_{i=0}^{n} \|D_i\varphi\|_b < \infty. \tag{2.4}$$

For each $\varphi \in C_b^1(\Sigma)$, define $A\varphi \in C_b(S)$ by

$$A\varphi(t,x,u) := D_0\varphi(t,x) + f(t,x,u) \cdot \nabla_x\varphi(t,x), \tag{2.5}$$

where $\nabla_x\varphi$ is the x-gradient of φ. Then $A : C_b^1(\Sigma) \to C_b(S)$ is a linear operator and it is obviously bounded, since

$$\|A\varphi\|_b \le (1 + \|f\|)\|\varphi\|_b^1 \quad \forall \varphi \in C_b^1(\Sigma). \tag{2.6}$$

Definition 2.3. A function φ in $C_b^1(\Sigma)$ is said to be a smooth subsolution to the *dynamic programming equation* (DPE) if

$$A\varphi + l_0 \ge 0 \quad \text{on } [0,T) \times \mathbb{R}^n \times U, \quad \text{and} \quad \varphi(T,x) \le L_0(x) \quad \forall x \in \mathbb{R}^n.$$

If φ is in $C_b^1(\Sigma)$ and $(x(\cdot), \mathbf{u}(\cdot)) \in \mathcal{P}(t,x)$, then

$$\frac{d}{dt}\varphi(t,x(t)) = A\varphi(t,x(t),\mathbf{u}(t)),$$

so that

$$\int_t^T A\varphi(s,x(s),\mathbf{u}(s))\,ds = \varphi(T,x(T)) - \varphi(t,x). \tag{2.7}$$

Therefore, if φ is a smooth subsolution to the DPE, then $\varphi(t,x) \le J(t,x;\mathbf{u})$, and we see that φ and the value function are related by the inequality

$$\varphi(t,x) \le J^*(t,x). \tag{2.8}$$

3. The linear programming formulation. We will use the linear programming terminology of [1], Chapter 3.

Dual pairs. Let b be the function in (H2)–(H3) and define the vector space $\widetilde{C}(S) := C_b(S) \times C_b(\mathbb{R}^n)$, which consists of all pairs $\widetilde{l} = (l, L)$ of functions $l \in C_b(S)$ and $L \in C_b(\mathbb{R}^n)$. (Note that condition (H2) implies that

$(l_0, L_0) \in \widetilde{C}(S))$. Moreover, let $\mathcal{D}_b(S)$ and $\mathcal{D}_b(\mathbb{R}^n)$ be the dual spaces of $C_b(S)$ and $C_b(\mathbb{R}^n)$ respectively, and define $\widetilde{\mathcal{D}}(S)$ as the vector space consisting of pairs $\widetilde{\xi} = (\xi_1, \xi_2)$ of functionals $\xi_1 \in \mathcal{D}_b(S)$ and $\xi_2 \in \mathcal{D}_b(\mathbb{R}^n)$. Then $(\widetilde{C}(S), \widetilde{\mathcal{D}}(S))$ is a dual pair with respect to the bilinear form

$$\langle \widetilde{\xi}, \widetilde{l} \rangle := \langle \xi_1, l \rangle + \langle \xi_2, L \rangle.$$

Let $\mathcal{M}_b(S) \subset \mathcal{D}_b(S)$ and $\mathcal{M}_b(\mathbb{R}^n) \subset \mathcal{D}_b(\mathbb{R}^n)$ be the spaces of measures introduced in Remark 2.1. Then each admissible pair $(x(\cdot), \mathbf{u}(\cdot)) \in \mathcal{P}(t, x)$ defines a pair of measures $\widetilde{M^{\mathbf{u}}} = (M^{\mathbf{u}}, N^{\mathbf{u}})$ in $\mathcal{M}_b(S) \times \mathcal{M}_b(\mathbb{R}^n)$ by setting, for $\widetilde{l} \in \widetilde{C}(S)$,

$$\langle \widetilde{M^{\mathbf{u}}}, \widetilde{l} \rangle = \langle M^{\mathbf{u}}, l \rangle + \langle N^{\mathbf{u}}, L \rangle = \int_t^T l(s, x(s), \mathbf{u}(s))\, ds + L(x(T)). \tag{3.1}$$

That is, $N^{\mathbf{u}}$ is the Dirac measure at $x(T)$, and $M^{\mathbf{u}}$ satisfies

$$M^{\mathbf{u}}(A \times B \times C) = \int_{[t,T] \cap A} I_B(x(s)) I_C(\mathbf{u}(s))\, ds,$$

where A, B and C are arbitrary Borel sets in $[t, T]$, $\mathbb{R}^n$ and U respectively. Note that condition (H1) implies that for each controlled process $x(t)$, $0 < t < T$, defined by (2.1) belongs to a compact set. Thus $\langle \widetilde{M^{\mathbf{u}}}, \widetilde{l} \rangle$ is well defined and finite for each $\widetilde{l}$. Furthermore, if $\varphi \in C_b^1(\Sigma)$, we may write (2.7) as

$$\langle (M^{\mathbf{u}}, N^{\mathbf{u}}), (-A\varphi, \varphi_T) \rangle = \varphi(t, x), \tag{3.2}$$

where $\varphi_T(x) := \varphi(T, x)$, for $x \in \mathbb{R}^n$, denotes the restriction of φ to $\{T\} \times \mathbb{R}^n$. On the other hand, from (2.2)–(2.3),

$$J^*(t, x) = \inf_{\mathcal{U}(t)} \langle (M^{\mathbf{u}}, N^{\mathbf{u}}), (l_0, L_0) \rangle. \tag{3.3}$$

We shall consider $\widetilde{C}(S)$ and $\widetilde{\mathcal{D}}(S)$ to be endowed with the norms

$$\|\widetilde{l}\|_* = \|(l, L)\|_* = \max\{\|l\|_b, \|L\|_b\}$$

and

$$\|\widetilde{\xi}\|_* = \|(\xi_1, \xi_2)\|_* = \max\{\|\xi_1\|_b, \|\xi_2\|_b\}.$$

In addition to $(\widetilde{\mathcal{D}}(S), \widetilde{C}(S))$, we also consider the dual pair $(\mathcal{D}_b^1(\Sigma), C_b^1(\Sigma))$, where $\mathcal{D}_b^1(\Sigma)$ is the dual of $C_b^1(\Sigma)$.

Let $\mathcal{L}_2 : C_b^1(\Sigma) \to \widetilde{C}(S)$ be the linear map defined by

$$\mathcal{L}_2 \varphi := (-A\varphi, \varphi_T), \qquad \varphi \in C_b^1(\Sigma). \tag{3.4}$$

By (2.6), $\mathcal{L}_2$ is continuous. We now define $\mathcal{L}_1 : \widetilde{\mathcal{D}}(S) \to \mathcal{D}_b^1(\Sigma)$ as follows. First, for every $\widetilde{\xi} = (\xi_1, \xi_2) \in \widetilde{\mathcal{D}}(S)$, let $T_{\widetilde{\xi}}$ be defined on $C_b^1(\Sigma)$ as $T_{\widetilde{\xi}}(\varphi) =$

$\langle \widetilde{\xi}, \mathcal{L}_2 \varphi \rangle$. Since $\mathcal{L}_2$ is a continuous linear map, so is $T_{\widetilde{\xi}}$. Therefore, there exists a unique $\nu_{\widetilde{\xi}} \in \mathcal{D}_b^1(\Sigma)$ such that

$$T_{\widetilde{\xi}}(\varphi) = \langle \nu_{\widetilde{\xi}}, \varphi \rangle \quad (= \langle \widetilde{\xi}, \mathcal{L}_2 \varphi \rangle). \tag{3.5}$$

As this holds for every $\widetilde{\xi} \in \widetilde{\mathcal{D}}(S)$, we define $\mathcal{L}_1 : \widetilde{D}(S) \to \mathcal{D}_b^1(\Sigma)$ as

$$\mathcal{L}_1 \widetilde{\xi} := \nu_{\widetilde{\xi}} \tag{3.6}$$

and note that $\mathcal{L}_1$ is the *adjoint* of $\mathcal{L}_2$, i.e., from (3.5),

$$\langle \mathcal{L}_1 \widetilde{\xi}, \varphi \rangle = \langle \widetilde{\xi}, \mathcal{L}_2 \varphi \rangle \quad \forall \widetilde{\xi} \in \widetilde{\mathcal{D}}(S), \ \varphi \in C_b^1(\Sigma). \tag{3.7}$$

Moreover, from (3.7), (3.4) and (2.5), a direct calculation shows that

$$\|\mathcal{L}_1 \widetilde{\xi}\|_b^1 = \sup\{|\langle \mathcal{L}_1 \widetilde{\xi}, \varphi \rangle| : \|\varphi\|_b^1 \le 1\} \le (2 + \|f\|) \|\widetilde{\xi}\|_*.$$

Thus, $\mathcal{L}_1$ is a continuous linear map.

Remark 3.1. *Notation.* Given a real vector space X with a positive cone X^+ we write $x \ge 0$ whenever $x \in X^+$. Let $\widetilde{C}(S)^+ := \{\widetilde{l} = (l, L) \in \widetilde{C}(S) : l \ge 0, \ L \ge 0\}$ be the natural positive cone in $\widetilde{C}(S)$, and

$$\widetilde{\mathcal{D}}(S)^+ := \{\widetilde{\xi} = (\xi_1, \xi_2) \in \widetilde{\mathcal{D}}(S) : \langle \widetilde{\xi}, \widetilde{l} \rangle \ge 0 \ \forall \widetilde{l} \in \widetilde{C}(S)^+\}$$

the corresponding dual cone.

Linear programs. Let $\widetilde{l}_0$ be the pair $(l_0, L_0) \in \widetilde{C}(S)$, and let $\nu^0 := \delta_{(t,x)} \in \mathcal{D}_b^1(\Sigma)$ be the Dirac measure concentrated at the initial condition (t, x) of (2.1), that is, $\langle \nu^0, \varphi \rangle = \varphi(t, x)$ for $\varphi \in C_b^1(\Sigma)$. Consider now the following linear program (P) and its dual (P^*).

(P) *minimize* $\langle \widetilde{\xi}, \widetilde{l}_0 \rangle$, subject to:

$$\mathcal{L}_1 \widetilde{\xi} = \nu^0, \quad \widetilde{\xi} \in \widetilde{\mathcal{D}}(S)^+. \tag{3.8}$$

(P^*) *maximize* $\langle \nu^0, \varphi \rangle$ $[= \varphi(t, x)]$, subject to:

$$\mathcal{L}_2 \varphi \le \widetilde{l}_0, \quad \varphi \in C_b^1(\Sigma), \tag{3.9}$$

where the latter inequality is understood componentwise, i.e.,

$$-A\varphi \le l_0 \quad \text{and} \quad \varphi_T \le L_0.$$

Recall that $\varphi_T(\cdot) := \varphi(T, \cdot)$ is the restriction of φ to $\{T\} \times \mathbb{R}^n$. Let $F(P)$ (resp. $F(P^*)$) be the set of feasible solutions to (P) (resp. (P^*)); i.e. $F(P)$ (resp. $F(P^*)$) is the set of pairs $\widetilde{\xi} = (\xi_1, \xi_2)$ in $\widetilde{\mathcal{D}}(S)$ that satisfy (3.8) (resp. the set of functions $\varphi \in C_b^1(\Sigma)$ that satisfy (3.9)).

Consistency. The linear program (P) is said to be *consistent* if $F(P)$ is nonempty, and similarly for $F(P^*)$. The program (P^*) is consistent, since e.g. $\varphi(\cdot) \equiv 0$ is in $F(P^*)$. On the other hand, (P) is also consistent since $F(P)$ contains the set of all pairs $\widetilde{M}^{\mathbf{u}} = (M^{\mathbf{u}}, N^{\mathbf{u}}) \ge 0$ such that

$(x(\cdot),\mathbf{u}(\cdot))\in\mathcal{P}(t,x)$; see (3.1). Indeed, by (3.7), the equality $\mathcal{L}_1\widetilde{M^{\mathbf{u}}}=\nu^0$ in (3.8) holds if and only if

$$\langle\widetilde{M^{\mathbf{u}}},\mathcal{L}_2\varphi\rangle=\langle(M^{\mathbf{u}},N^{\mathbf{u}}),(-A\varphi,\varphi_T)\rangle=\varphi(t,x)\quad\forall\varphi\in C_b^1(\Sigma),$$

which is the same as (3.2) for $(\xi_1,\xi_2)=(M^{\mathbf{u}},N^{\mathbf{u}})$.

The latter also implies that, from (3.3),

$$J^*(t,x)=\inf_{\mathcal{U}(t,x)}\langle\widetilde{M^{\mathbf{u}}},\widetilde{l}_0\rangle\ge\inf_{F(P)}\langle\widetilde{\xi},\widetilde{l}_0\rangle=:\inf(P),$$

i.e. the value function J^* and the value, $\inf(P)$, of (P) are related by

$$J^*(t,x)\ge\inf(P).$$

Furthermore, denoting by $\sup(P^*)$ the value of (P^*), *weak duality* yields [1]

$$\inf(P)\ge\sup(P^*);$$

hence,

$$J^*(t,x)\ge\inf(P)\ge\sup(P^*).\tag{3.10}$$

4. Absence of duality gap. In this section we prove that there is *no duality gap* (see (4.1)) and that (P) is *solvable*. More precisely, we have the following theorem.

THEOREM 4.1. *If the hypotheses* (H1)–(H3) *hold*, *then there is no duality gap and* (P) *is solvable*, *i.e.*

$$\sup(P^*)=\inf(P),\tag{4.1}$$

and there exists an optimal solution $\widetilde{\xi}^*\in\widetilde{\mathcal{D}}(S)$ *for* (P), *so that*

$$\sup(P^*)=\min(P)=\langle\widetilde{\xi}^*,\widetilde{l}_0\rangle.$$

Proof. We use Theorems 3.10 and 3.22 of [1], which state that if (P) is consistent with a finite value, and the set

$$D:=\{(\mathcal{L}_1\widetilde{\xi},\langle\widetilde{\xi},\widetilde{l}_0\rangle):\widetilde{\xi}\in\widetilde{\mathcal{D}}(S)^+\}\tag{4.2}$$

is closed in $\mathcal{D}_b^1(\Sigma)\times\mathbb{R}$, then there is no duality gap between (P) and (P^*), and (P) is solvable. Thus, since we have seen that (P) is consistent, it suffices to show that the set D in (4.2) is closed. Let Γ be a directed set, and let $\{\widetilde{\xi}_\gamma=(\xi_{1\gamma},\xi_{2\gamma}):\gamma\in\Gamma\}$ be a net in $\widetilde{\mathcal{D}}(S)^+$ such that $(\mathcal{L}_1\widetilde{\xi}_\gamma,\langle\widetilde{\xi}_\gamma,\widetilde{l}_0\rangle)$ converges to (ν,r) in $\mathcal{D}_b^1(\Sigma)\times\mathbb{R}$, i.e.

$$r=\lim_{\Gamma}\langle\widetilde{\xi}_\gamma,\widetilde{l}_0\rangle\tag{4.3}$$

and

$$\nu=\lim_{\Gamma}\mathcal{L}_1\widetilde{\xi}_\gamma\tag{4.4}$$

in the weak topology $\sigma(\mathcal{D}_b^1(\Sigma), C_b^1(\Sigma))$. We wish to show that (ν, r) is in D, i.e. there exists $\widetilde{\xi} = (\xi_1, \xi_2) \in \widetilde{\mathcal{D}}(S)^+$ such that

$$r = \langle \widetilde{\xi}, \widetilde{l}_0 \rangle \quad \text{and} \quad \nu = \mathcal{L}_1 \widetilde{\xi}. \tag{4.5}$$

By (4.3), given $\varepsilon > 0$, there exists $\gamma(\varepsilon) \in \Gamma$ such that, for all $\gamma \geq \gamma(\varepsilon)$,

$$r - \varepsilon \leq \langle \widetilde{\xi}_\gamma, \widetilde{l}_0 \rangle = \langle \xi_{1\gamma}, l_0 \rangle + \langle \xi_{2\gamma}, L_0 \rangle \leq r + \varepsilon. \tag{4.6}$$

Therefore, for any $\gamma \geq \gamma(\varepsilon)$ and $l \in C_b(S)$,

$$\begin{aligned} |\langle \xi_{1\gamma}, l \rangle| &\leq \langle \xi_{1\gamma}, |l| \rangle \leq \|l\|_b \langle \xi_{1\gamma}, b \rangle \\ &\leq \|l\|_b \langle \xi_{1\gamma}, l_0 \rangle \|b/l_0\| \quad \text{by (H2)} \\ &\leq \|l\|_b \|b/l_0\| (r + \varepsilon) \quad \text{by (4.6);} \end{aligned}$$

that is, $\{\xi_{1\gamma} : \gamma \geq \gamma(\varepsilon)\}$ is a bounded family in $\mathcal{D}_b(S)$. Similarly, $\{\xi_{2\gamma} : \gamma \geq \gamma(\varepsilon)\}$ is a bounded family in $\mathcal{D}_b(\mathbb{R}^n)$, since for all $\gamma \geq \gamma(\varepsilon)$ and $L \in C_b(\mathbb{R}^n)$,

$$\begin{aligned} |\langle \xi_{2\gamma}, L \rangle| &\leq \langle \xi_{2\gamma}, |L| \rangle \leq \|L\|_b \langle \xi_{2\gamma}, b \rangle \\ &\leq \|L\|_b \langle \xi_{2\gamma}, L_0 \rangle \|b/L_0\| \quad \text{by (H2)} \\ &\leq \|L\|_b \|b/L_0\| (r + \varepsilon) \quad \text{by (4.6).} \end{aligned}$$

Thus, $\{\widetilde{\xi}_\gamma : \gamma \geq \gamma(\varepsilon)\}$ is bounded and, therefore, there exists a directed set $\Gamma' \subset \Gamma$ and a pair $\widetilde{\xi} = (\xi_1, \xi_2)$ such that $\{\widetilde{\xi}_\gamma : \gamma \in \Gamma'\}$ converges to $\widetilde{\xi}$. This convergence, together with (4.3), yields $\langle \widetilde{\xi}, \widetilde{l}_0 \rangle = r$, whereas the continuity of $\mathcal{L}_1$ and (4.4) give

$$\mathcal{L}_1 \widetilde{\xi} = \mathcal{L}_1 (\lim_{\Gamma'} \widetilde{\xi}_\gamma) = \lim_{\Gamma'} \mathcal{L}_1 \widetilde{\xi}_\gamma = \nu.$$

That is, (4.5) holds. ∎

5. Equivalence of (P) **and the OCP.** In this section we prove that the original OCP (2.1)–(2.3) and the linear program (P) are *equivalent* in the sense of the following theorem.

THEOREM 5.1. *Assume* (H1)–(H3). *Then* $\min(P) = J^*(t, x)$.

Moreover, from (4.1) and Theorem 5.1, we obtain $J^*(t, x) = \sup(P^*)$. In other words:

COROLLARY. *Under* (H1)–(H3), *the value function* J^* *is the supremum of the smooth subsolutions to the DPE.*

In the proof of Theorem 5.1 we use the following key result, which is proved in the next section.

THEOREM 5.2. *For every* $\varepsilon > 0$ *there exist functions* $\widetilde{J}_\varepsilon$, L_ε *and* γ_ε, *with* $\widetilde{J}_\varepsilon \in C^1_b(\Sigma)$, *such that*

$$\|\widetilde{J}_\varepsilon - J^*\|_b \to 0 \quad \text{as } \varepsilon \to 0, \quad \widetilde{J}_\varepsilon(T,x) = L_\varepsilon(x), \tag{5.1}$$

$$\|L_0 - L_\varepsilon\|_b \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.2}$$

$$A\widetilde{J}_\varepsilon + l_0 \ge \gamma_\varepsilon, \tag{5.3}$$

where

$$\|\gamma_\varepsilon\|_b \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.4}$$

Proof of Theorem 5.1. From (3.10) and the solvability of (P) (Theorem 4.1), we know that $\min(P) \le J^*(t,x)$. Suppose that $\min(P) < J^*(t,x)$. Then there exists $\widetilde{\xi} \in F(P)$ such that

$$\langle \widetilde{\xi}, \widetilde{l}_0 \rangle < J^*(t,x). \tag{5.5}$$

Thus, from (5.3),

$$\begin{aligned}
\langle \widetilde{\xi}, \widetilde{l}_0 \rangle &\ge \langle \xi_1, -A\widetilde{J}_\varepsilon + \gamma_\varepsilon \rangle + \langle \xi_2, L_\varepsilon \rangle + \langle \xi_2, L_0 - L_\varepsilon \rangle \\
&\ge \langle \xi_1, -A\widetilde{J}_\varepsilon \rangle + \langle \xi_2, L_\varepsilon \rangle - \|\gamma_\varepsilon\|_b \|\xi_1\|_b - \|\xi_2\|_b \|L_0 - L_\varepsilon\|_b \\
&= \langle \widetilde{\xi}, \mathcal{L}_2 \widetilde{J}_\varepsilon \rangle - \|\gamma_\varepsilon\|_b \|\xi_1\|_b - \|\xi_2\|_b \|L_0 - L_\varepsilon\|_b \\
&= \langle \mathcal{L}_1 \widetilde{\xi}, \widetilde{J}_\varepsilon \rangle - \|\gamma_\varepsilon\|_b \|\xi_1\|_b - \|\xi_2\|_b \|L_0 - L_\varepsilon\|_b \\
&= \widetilde{J}_\varepsilon(t,x) - \|\gamma_\varepsilon\|_b \|\xi_1\|_b - \|\xi_2\|_b \|L_0 - L_\varepsilon\|_b \quad \text{by (3.8)}.
\end{aligned}$$

From (5.1)–(5.2) and (5.4), it follows that $J^*(t,x) \le \langle \widetilde{\xi}, \widetilde{l}_0 \rangle$, which contradicts (5.5). ■

6. Approximation of the value function. In this section we prove the approximation Theorem 5.2. We will do this via several lemmas, from which we obtain a particular approximation to the optimal cost function. We first extend our control problem to a larger time interval.

Put

$$\begin{aligned}
f(t,x,u) &:= f(0,x,u) \quad \text{and} \quad l_0(t,x,u) := l_0(0,x,u) \quad \text{if } t < 0; \\
f(t,x,u) &:= f(T,x,u) \quad \text{and} \quad l_0(t,x,u) := l_0(T,x,u) \quad \text{if } t > T.
\end{aligned}$$

For each $\varepsilon > 0$, define $\Sigma_\varepsilon := [-\varepsilon, T+\varepsilon] \times \mathbb{R}^n$, $S_\varepsilon := \Sigma_\varepsilon \times U$, and $\mathcal{U}_\varepsilon(t)$ as the set of Borel measurable functions $\mathbf{u} : [t, T+\varepsilon] \to U$, $-\varepsilon \le t < T+\varepsilon$.

Note that, thus defined, the extensions of l_0 and f to Σ_ε and S_ε satisfy (H1) and (H2).

Define

$$J_\varepsilon(t,x;\mathbf{u}) := \int_t^{T+\varepsilon} l_0(r, x(r), \mathbf{u}(r))\, dr + L_0(x(T+\varepsilon)),$$

where

$$\begin{aligned} \dot{x}(r) &= f(r, x(r), \mathbf{u}(r)), \qquad t < r \le T + \varepsilon, \\ x(t) &= x. \end{aligned} \tag{6.1}$$

The value function J_ε^* is defined as

$$J_\varepsilon^*(t,x) := \inf_{\mathcal{U}_\varepsilon(t)} J_\varepsilon(t,x;\mathbf{u}).$$

Note that $\varepsilon = 0$ yields the original OCP.

We shall now establish properties of the value function J_ε^*. Below, C stands for a generic constant whose values may be different in different formulas.

Lemma 6.1. *There exists C such that for all $\varepsilon < 1$,*

$$J_\varepsilon^*(t,x) \le Cb(x) \quad \forall (t,x) \in \Sigma_\varepsilon.$$

Proof. From (H3) it follows that

$$b(y) \le b(x)\,(1 + c|y - x|) \le b(x) e^{c|x-y|} \tag{6.2}$$

for all $|x - y| < \varepsilon_0$. By induction, one can show the validity of (6.2) for all $x, y \in \mathbb{R}^n$. From (6.1) and (H1) we obtain, for each $\mathbf{u} \in \mathcal{U}_\varepsilon(t)$ and $r \ge t$,

$$|x(r) - x| \le K|r - t|. \tag{6.3}$$

Then, by (H2) and (6.2)–(6.3),

$$\begin{aligned} J_\varepsilon(t,x;\mathbf{u}) &\le \int_t^{T+\varepsilon} b(x(r))\,dr + b(x(T+\varepsilon)) \\ &\le \int_t^{T+\varepsilon} b(x) e^{c|x(r)-x|}\,dr + b(x) e^{c|x(T+\varepsilon)-x|} \\ &\le b(x)\Big[\int_t^{T+\varepsilon} e^{cK|r-t|}\,dr + e^{cK|T+\varepsilon-t|}\Big] \le Cb(x). \end{aligned}$$

Taking the infimum over $\mathcal{U}_\varepsilon(t)$ yields the lemma. ■

Lemma 6.2. *There exist $\varepsilon_1 > 0$ and $C > 0$ such that for all $\varepsilon < 1$ and $|x - y|, |s - t| < \varepsilon_1$,*

$$|J_\varepsilon^*(t,x) - J_\varepsilon^*(s,y)| \le C[|x-y| + |s-t|]b(x).$$

Proof. Assume $t < s$ and let $\mathbf{u} \in \mathcal{U}_\varepsilon(t)$ be an arbitrary control function. Put

$$\begin{aligned} \dot{x}_1(r) &= f(r, x_1(r), \mathbf{u}(r)), \quad t < r \le T + \varepsilon, \quad \text{with } x_1(t) = x, \\ \dot{x}_2(r) &= f(r, x_2(r), \mathbf{u}(r)), \quad s < r \le T + \varepsilon, \quad \text{with } x_2(s) = y. \end{aligned}$$

Then

$$
\begin{aligned}
(6.4)\qquad |J_\varepsilon(t,x;\mathbf{u}) - J_\varepsilon(s,y;\mathbf{u})| &\le \Big|\int_t^s l_0(r,x_1(r),\mathbf{u}(r))\,dr\Big| \\
&\quad + \Big|\int_s^{T+\varepsilon} [l_0(r,x_1(r),\mathbf{u}(r)) - l_0(r,x_2(r),\mathbf{u}(r))]\,dr\Big| \\
&\quad + |L_0(x_1(T+\varepsilon)) - L_0(x_2(T+\varepsilon))| \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Using (6.2), (6.3) and (H2), we have

$$
\begin{aligned}
(6.5)\qquad I_1 &\le \int_t^s b(x_1(r))\,dr \le \int_t^s b(x)e^{c|x_1(r)-x|}\,dr \\
&\le b(x)\int_t^s e^{cK|r-t|}\,dr \le b(x)e^{cK(T+2)}(s-t).
\end{aligned}
$$

We now majorize I_2. From (6.3), $|x_1(s)-x| \le K|s-t|$; hence

$$
(6.6)\qquad |x_1(s)-y| \le |x-y| + K|s-t|.
$$

Consequently, by (H1),

$$
\begin{aligned}
(6.7)\qquad |x_1(r)-x_2(r)| &= |x_1(s)-y| + \int_s^r |f(z,x_1(z),\mathbf{u}(z)) - f(z,x_2(z),\mathbf{u}(z))|\,dz \\
&\le |x-y| + K|t-s| + \int_s^r c|x_1(z)-x_2(z)|\,dz.
\end{aligned}
$$

Thus, Gronwall's inequality implies

$$
(6.8)\qquad |x_1(r)-x_2(r)| \le [|x-y| + K|t-s|]e^{c|r-s|}.
$$

Taking $\varepsilon_1 < 1$ such that $(K+1)\varepsilon_1 e^{c(T+2)} < \varepsilon_0$ we have (see condition (H3))

$$
\begin{aligned}
(6.9)\qquad I_2 &\le \int_s^{T+\varepsilon} |l_0(r,x_1(r),\mathbf{u}(r)) - l_0(r,x_2(r),\mathbf{u}(r))|\,dr \\
&\le \int_s^{T+\varepsilon} b(x_1(r))[|x-y| + K|t-s|]e^{c(T+\varepsilon-s)}\,dr \\
&\le b(x)[|x-y| + K|t-s|]e^{c(T+2)} \int_s^{T+\varepsilon} e^{c|x_1(r)-x|}\,dr
\end{aligned}
$$

$$\leq b(x)[|x-y|+K|t-s|]e^{c(T+2)}\int\limits_{-\varepsilon}^{T+\varepsilon} e^{cK|r-t|}\,dr$$
$$= C[|x-y|+|t-s|]b(x).$$

Similarly, using (6.8), (6.2), (6.3) and (H3), we may majorize I_3 as follows:

$$\begin{aligned} I_3 &= |L_0(x_1(T+\varepsilon)) - L_0(x_2(T+\varepsilon))| \\ &\leq cb(x_1(T+\varepsilon))|x_2(T+\varepsilon) - x_1(T+\varepsilon)| \\ &\leq cb(x)e^{c|x_1(T+\varepsilon)-x|}(|x-y|+K|t-s|)e^{c(T+\varepsilon-s)} \\ &\leq cb(x)e^{cK|T+\varepsilon-t|}(|x-y|+|t-s|)(K+1)e^{c(T+\varepsilon-s)} \\ &= Cb(x)(|x-y|+|t-s|). \end{aligned} \tag{6.10}$$

Combining (6.4), (6.5), (6.9) and (6.10) and taking the supremum over all control functions $\mathbf{u}(\cdot)$, we complete the proof of the lemma, since

$$|J_\varepsilon^*(t,x) - J_\varepsilon^*(s,y)| \leq \sup_{\mathcal{U}_\varepsilon(t)} |J_\varepsilon(t,x;\mathbf{u}) - J_\varepsilon(s,y;\mathbf{u})|. \ \blacksquare$$

Remark 6.3. From Lemma 6.2 it follows that J_ε^* is differentiable for almost all $(t,x) \in \Sigma_\varepsilon$, and $|D_i J_\varepsilon^*(t,x)| \leq Cb(x)$, $i = 0, 1, \ldots, n$.

LEMMA 6.4. *There exists $C > 0$ such that for all $(t,x) \in \Sigma$ and all sufficiently small $\varepsilon > 0$,*

$$|J_\varepsilon^*(t,x) - J^*(t,x)| \leq C\varepsilon b(x). \tag{6.11}$$

Proof. Let $0 \leq t \leq T$ and let $\mathbf{u}(\cdot)$ be any control function in $\mathcal{U}_\varepsilon(t)$. Let

$$\dot{x}(r) = f(r, x(r), \mathbf{u}(r)), \quad t < r \leq T+\varepsilon,$$
$$x(t) = x.$$

Then (H3) and the inequalities (6.2) and (6.3) show that for $\varepsilon < 1$,

$$\begin{aligned} &|J_\varepsilon(t,x;\mathbf{u}) - J(t,x;\mathbf{u})| \\ &\quad\leq \int\limits_T^{T+\varepsilon} l_0(r, x(r), \mathbf{u}(r))\,dr + |L_0(x(T+\varepsilon)) - L_0(x(T))| \\ &\quad\leq \int\limits_T^{T+\varepsilon} b(x)e^{c|x(r)-x|}\,dr + cb(x(T))|x(T+\varepsilon) - x(T)| \\ &\quad\leq b(x)e^{cK(T+\varepsilon)}\varepsilon + cb(x)e^{c|x(T)-x|}K\varepsilon \\ &\quad\leq b(x)e^{cK(T+\varepsilon)}\varepsilon + cb(x)e^{cK|T-t|}K\varepsilon \leq C\varepsilon b(x). \end{aligned} \tag{6.12}$$

Finally, as in the proof of Lemma 6.2, taking the supremum over all $\mathbf{u}(\cdot)$, we get (6.11). $\blacksquare$

LEMMA 6.5. *There exist $\varepsilon_2 > 0$ and $C > 0$ such that for any $\varepsilon < \varepsilon_2$, any initial condition (t,x), and any sufficiently small $0 < h < \varepsilon$ and $u \in U$,*

$$J_\varepsilon^*(t,x) \le l_0(t,x,u)h + J_\varepsilon^*(t+h, x+f(t,x,u)h) + C\varepsilon h b(x). \tag{6.13}$$

Proof. Let $u \in U$ be fixed and let

$$\dot{x}(r) = f(r,x(r),u), \qquad t < r \le T+\varepsilon,$$
$$x(t) = x.$$

The dynamic programming principle [4, p. 9] implies

$$\begin{aligned} J_\varepsilon^*(t,x) &\le \int_t^{t+h} l_0(r,x(r),u)\,dr + J_\varepsilon^*(t+h, x(t+h)) \\ &=: I_1 + I_2. \end{aligned} \tag{6.14}$$

Using (H3) and (6.3), we get

$$\begin{aligned} |I_1 - l_0(t,x,u)h| &\le \int_t^{t+h} |l_0(r,x(r),u) - l_0(r,x,u)|\,dr \\ &\quad + \int_t^{t+h} |l_0(r,x,u) - l_0(t,x,u)|\,dr \\ &\le \int_t^{t+h} c|x(r)-x|b(x)\,dr + \int_t^{t+h} cb(x)|r-t|\,dr \\ &\le cb(x)\int_t^{t+h} K|r-t|\,dr + cb(x)\varepsilon h/2 \\ &\le c(K+1)\varepsilon h b(x)/2. \end{aligned} \tag{6.15}$$

By virtue of (H3), the inequality (6.15) is valid for h such that $|x(r)-x| < \varepsilon_0$ for all $t \le r \le t+h$. This requirement is satisfied by choosing $h \le \varepsilon \le \varepsilon_0/K$. On the other hand, using Lemma 6.2, (H1) and (H3), we get

$$\begin{aligned} &|I_2 - J_\varepsilon^*(t+h, x+f(t,x,u)h)| \\ &\quad \le Cb(x)|x(t+h) - x - f(t,x,u)h| \\ &\quad \le Cb(x)\int_t^{t+h} |f(r,x(r),u) - f(t,x,u)|\,dr \\ &\quad \le Cb(x)\Big[\int_t^{t+h} |f(r,x(r),u) - f(r,x,u)|\,dr \\ &\qquad + \int_t^{t+h} |f(r,x,u) - f(t,x,u)|\,dr\Big] \end{aligned} \tag{6.16}$$

$$\le Cb(x)\Big[\int_t^{t+h} c|x(r)-x|\,dr + \varepsilon h\Big]$$

$$= Cb(x)(cKh^2/2+\varepsilon h) \le Cb(x)\varepsilon h(cK/2+1).$$

In (6.16), h is chosen such that $|f(r,x,u)-f(t,x,u)|<\varepsilon$ for $r\in[t,t+h]$. The inequalities (6.14)–(6.16) yield (6.13). ∎

Remark 6.6. From Remark 6.3 it follows that subtracting $J^*_\varepsilon(t,x)$ from both sides of (6.13), dividing by h and letting $h\to 0$, we get

$$0 \le l_0(t,x,u) + AJ^*_\varepsilon(t,x,u) + C\varepsilon b(x) \tag{6.17}$$

for almost all $(t,x)\in\Sigma_\varepsilon$, and all $u\in U$.

We shall now use J^*_ε to construct a smooth approximation of J^*. Let $\varrho_\varepsilon(t,x)$ be an infinitely differentiable nonnegative function such that $\varrho_\varepsilon(t,x)=0$ if $|t|+|x|>\varepsilon$ and

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \varrho_\varepsilon(t,x)\,dx\,dt = 1.$$

For $(t,x)\in\Sigma$ define the convolution

$$\widetilde{J}_\varepsilon(t,x) := \varrho_\varepsilon * J^*_\varepsilon(t,x) = \int_{t-\varepsilon}^{t+\varepsilon}\int_{B_\varepsilon(x)} \varrho_\varepsilon(t-s,x-y)J^*_\varepsilon(s,y)\,dy\,ds \tag{6.18}$$

$$= \int_{-\varepsilon}^{+\varepsilon}\int_{B_\varepsilon(0)} \varrho_\varepsilon(s,y)J^*_\varepsilon(t-s,x-y)\,dy\,ds,$$

where $B_\varepsilon(x)$ is the ball in $\mathbb{R}^n$ with radius ε and center x.

Lemma 6.7. *$\widetilde{J}_\varepsilon$ belongs to $C^1_b(\Sigma)$.*

Proof. Continuous differentiability of $\widetilde{J}_\varepsilon$ is obvious from its definition. On the other hand, applying Lemmas 6.1 and 6.2 to J^*_ε, we see that

$$\widetilde{J}_\varepsilon(t,x) \le J^*_\varepsilon(t,x) + \sup_{|s-t|,|x-y|<\varepsilon} |J^*_\varepsilon(s,y)-J^*_\varepsilon(t,x)| \tag{6.19}$$

$$\le Cb(x) + 2C\varepsilon b(x) = (1+2\varepsilon)Cb(x).$$

Let ε_1 be as in Lemma 6.2. From (6.18) we see that for each $\varepsilon<\varepsilon_1$ and each $(t,x),(s,y)$ subject to $|t-s|,|x-y|<\varepsilon$,

$$|\widetilde{J}_\varepsilon(t,x)-\widetilde{J}_\varepsilon(s,y)| \tag{6.20}$$

$$= \Big|\int_{-\varepsilon}^{\varepsilon}\int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z)[J^*_\varepsilon(t-r,x-z)-J^*_\varepsilon(s-r,y-z)]\,dz\,dr\Big|$$

$$\le \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z) C(|s-t|+|x-y|) b(x)\, dz\, dr$$

$$= C(|s-t|+|x-y|) b(x).$$

Inequality (6.20) shows that

$$|D_i \widetilde{J}_\varepsilon| \le C b(x). \tag{6.21}$$

Combining (6.21) and (6.19), we get the statement of the lemma. ■

LEMMA 6.8. $\|\widetilde{J}_\varepsilon - J^*\|_b \to 0$ *as* $\varepsilon \to 0$.

Proof. In view of Lemma 6.4, it is sufficient to show that

$$\|J_\varepsilon^* - \widetilde{J}_\varepsilon\|_b \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.22}$$

From (6.18),

$$\begin{aligned} (6.23)\quad & |\widetilde{J}_\varepsilon(t,x) - J_\varepsilon^*(t,x)| \\ & \le \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z) |J_\varepsilon^*(t-r, x-z) - J_\varepsilon^*(t,x)|\, dz\, dr \\ & \le \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z) \sup_{|t-s|,|x-y|<\varepsilon} |J_\varepsilon^*(t-r, x-z) - J_\varepsilon^*(t,x)|\, dz\, dr \\ & = \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z) \sup_{|t-s|,|x-y|<\varepsilon} |J_\varepsilon^*(s,y) - J_\varepsilon^*(t,x)|\, dz\, dr \\ & \le \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} \varrho_\varepsilon(r,z) [\sup_{|t-s|,|x-y|<\varepsilon} C(|t-s|+|x-y|) b(x)]\, dz\, dr \\ & \le 2C\varepsilon b(x), \end{aligned}$$

where the last inequality in (6.23) follows from Lemma 6.2. ■

To conclude this section, we shall use the previous lemmas to prove Theorem 5.2.

Proof of Theorem 5.2. Let $L_\varepsilon(x) := \widetilde{J}_\varepsilon(x,T)$. Then, from Lemma 6.8 and the equality $J^*(x,T) = L_0(x)$, we have

$$\|\widetilde{J}_\varepsilon - J^*\|_b \to 0 \quad \text{and} \quad \|L_\varepsilon - L_0\|_b \to 0 \quad \text{as } \varepsilon \to 0, \tag{6.24}$$

which proves (5.1)–(5.2). Now, from (6.17) it follows that

$$0 \le l_0 * \varrho_\varepsilon + (AJ_\varepsilon^*) * \varrho_\varepsilon + C\varepsilon(b * \varrho_\varepsilon) \quad \text{on } S. \tag{6.25}$$

Thus, to complete the proof of the theorem it suffices to show that, as $\varepsilon \to 0$,

(i) $\|(AJ_\varepsilon^*) * \varrho_\varepsilon - A\widetilde{J}_\varepsilon\|_b \to 0$,
(ii) $\|l_0 * \varrho_\varepsilon - l_0\|_b \to 0$, and
(iii) $\|b * \varrho_\varepsilon\|_b < \infty$.

Fix $\varepsilon < \varepsilon_2$ (where ε_2 is the same as in Lemma 6.5) and $(t,x,u) \in S$. Then (i) follows from

$$\begin{aligned}
(6.26)\quad &\frac{1}{b(x)}|(AJ_\varepsilon^*) * \varrho_\varepsilon(t,x,u) - A\widetilde{J}_\varepsilon(t,x,u)| \\
&= \frac{1}{b(x)}\Big|\sum_{i=1}^n \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} f_i(t-r,x-z,u) D_i J_\varepsilon^*(t-r,x-z)\varrho_\varepsilon(r,z)\,dz\,dr \\
&\quad - \sum_{i=1}^n f_i(t,x,u) D_i \widetilde{J}_\varepsilon(t,x)\Big| \\
&= \frac{1}{b(x)}\sum_{i=1}^n \Big|\int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} [f_i(t-r,x-z,u) \\
&\quad - f_i(t,x,u)] D_i J_\varepsilon^*(t-r,x-z)\varrho_\varepsilon(r,z)\,dz\,dr\Big| \\
&\le \sum_{i=1}^n \delta(f_i)\|D_i J_\varepsilon^*\|_b,
\end{aligned}$$

where $\delta(f_i)$ denotes the modulus of continuity of f_i.

We now prove (ii) using (H3):

$$\begin{aligned}
(6.27)\quad &\frac{1}{b(x)}|l_0 * \varrho_\varepsilon(t,x,u) - l_0(t,x,u)| \\
&\le \frac{1}{b(x)} \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} |l_0(t-r,x-z,u) - l_0(t,x,u)|\varrho_\varepsilon(r,z)\,dz\,dr \\
&\le \frac{1}{b(x)} \int_{-\varepsilon}^{\varepsilon} \int_{B_\varepsilon(0)} cb(x)[|r| + |z|]\varrho_\varepsilon(r,z)\,dz\,dr \le 2c\varepsilon.
\end{aligned}$$

Finally, to prove (iii) we use (6.2):

$$(6.28)\quad \frac{1}{b(x)} \int_{B_\varepsilon(0)} b(x-z)\varrho_\varepsilon(z)\,dz \le \frac{1}{b(x)}\, b(x) \int_{B_\varepsilon(0)} \varrho_\varepsilon(z) e^{c|z|}\,dz = \text{const}.$$

Combining (6.25)–(6.28), we get the statement of the theorem. ∎

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