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INITIAL-BOUNDARY VALUE PROBLEMS FOR IMPULSIVE PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Theorems on differential inequalities generated by an initial-boundary value problem for impulsive parabolic functional differential equations are considered. Comparison results implying uniqueness criteria are proved.

1. Introduction. The theory of impulsive ordinary differential equations made its start in [10] and it was an object of many investigations in the last three decades ([3], [4]). This theory is richer than the corresponding theory without impulses due to some new features and phenomena such as: "beating", "merging", "dying" of solutions, loss of autonomy, etc.

In the recent years the theory of impulsive partial differential equations began to emerge ([1], [5], [7], [8]). It gives greater possibilities for mathematical simulation of evolutional processes in theoretical physics, chemistry, population dynamics, biotechnology, etc., which are characterized by the fact that the system parameters are subject to short term perturbations in time. The authors believe that this new theory will undergo a rapid development in the coming years.

In the present paper impulsive parabolic functional differential inequalities are considered. It is shown that the impulsive ordinary functional differential inequalities find application in the proofs of theorems concerning the estimates of solutions and in the uniqueness theory for impulsive parabolic functional differential equations.

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We note that parabolic differential and functional differential inequalities with impulses are investigated in [2], [5], [6], [8].

2. Preliminaries. Let $E=[0,a)\times(-b,b)$, $a>0$, $b=(b_1,\ldots,b_n)\in\mathbb{R}^n_+$, $\mathbb{R}_+=[0,\infty)$ and $B=[-\tau_0,0]\times[-\tau,\tau]$, where $\tau_0\in\mathbb{R}_+$, $\tau=(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n_+$. We define $c=b+\tau$, $E_0=[-\tau_0,0]\times[-c,c]$, $\partial_0E=[0,a)\times([-c,c]\setminus(-b,b))$, $E^*=E\cup E_0\cup\partial_0E$. For $\tau_0>0$ we put $B^{(-)}=[-\tau_0,0)\times[-\tau,\tau]$.

Suppose that $0<x_1<\ldots<x_k<a$ are given numbers. We define

$$J_0=[-\tau_0,0],\quad J=[0,a),\quad J_{\mathrm{imp}}=\{x_1,\ldots,x_k\},$$
$$E_{\mathrm{imp}}=\{(x,y)\in E\colon x\in J_{\mathrm{imp}}\},$$
$$\partial_0E_{\mathrm{imp}}=\{(x,y)\in\partial_0E\colon x\in J_{\mathrm{imp}}\},$$
$$E^*_{\mathrm{imp}}=\{(x,y)\in E^*\colon x\in J_{\mathrm{imp}}\}.$$

Let $C_{\mathrm{imp}}[E^*,\mathbb{R}]$ be the class of all functions $z:E^*\to\mathbb{R}$ such that:

(i) the restriction of z to $E^*\setminus E^*_{\mathrm{imp}}$ is continuous,

(ii) for each $(x,y)\in E_{\mathrm{imp}}$, the limits

$$\lim_{\substack{(t,s)\to(x,y)\\ t<x}} z(t,s)=z(x^-,y),\tag{1}$$

$$\lim_{\substack{(t,s)\to(x,y)\\ t>x}} z(t,s)=z(x^+,y)\tag{2}$$

exist and $z(x,y)=z(x^+,y)$ for $(x,y)\in E_{\mathrm{imp}}$.

In the same way we define the set $C_{\mathrm{imp}}[\partial_0E,\mathbb{R}]$. If $z\in C_{\mathrm{imp}}[E^*,\mathbb{R}]$ and $(x,y)\in E_{\mathrm{imp}}$ then we write $\Delta z(x,y)=z(x,y)-z(x^-,y)$.

Suppose that $z:E^*\to\mathbb{R}$ and $(x,y)=(x,y_1,\ldots,y_n)\in\overline{E}$, the closure of E. We define a function $z_{(x,y)}:B\to\mathbb{R}$ as follows:

$$z_{(x,y)}(t,s)=z(x+t,y+s),\qquad (t,s)\in B.$$

Suppose that $\tau_0>0$. For the above z and (x,y) we also define $z_{(x^-,y)}:B^{(-)}\to\mathbb{R}$ by

$$z_{(x^-,y)}(t,s)=z(x+t,y+s),\qquad (t,s)\in B^{(-)}.$$

Assume that we have a sequence $\{t_1,\ldots,t_r\}$ such that $-\tau_0\le t_1<\ldots<t_r\le 0$. Let $\Gamma_i=(t_i,t_{i+1})\times[-\tau,\tau]$, $i=1,\ldots,r-1$ and

$$\Gamma_0=\begin{cases}\emptyset & \text{if } -\tau_0=t_1,\\ (-\tau_0,t_1)\times[-\tau,\tau] & \text{if } -\tau_0<t_1,\end{cases}$$
$$\Gamma_r=\begin{cases}\emptyset & \text{if } t_r=0,\\ (t_r,0)\times[-\tau,\tau] & \text{if } t_r<0.\end{cases}$$

Let $t_0 = -\tau_0$ if $t_1 > -\tau_0$ and $t_{r+1} = 0$ if $t_r < 0$. We denote by $C^*_{\mathrm{imp}}[B,\mathbb{R}]$ the class of all functions $w : B \to \mathbb{R}$ such that there exists a sequence $\{t_1,\dots,t_r\}$ (r and $t_1,\dots,t_r$ depend on w) satisfying:

(i) the functions $w_{|\Gamma_i}$, $i = 0,1,\dots,r$, are continuous,

(ii) for each i, $i = 1,\dots,r+1$ with $(t_i,s) \in B$, $t_i > -\tau_0$, the limit

$$\lim_{\substack{(t,y)\to(t_i,s)\\ t<t_i}} w(t,y) = w(t_i^-,s)$$

exists,

(iii) for each i, $i = 0,1,\dots,r$ with $(t_i,s) \in B$, $t_i < 0$, the limit

$$\lim_{\substack{(t,y)\to(t_i,s)\\ t>t_i}} w(t,y) = w(t_i^+,s)$$

exists,

(iv) for each $(t_i,s) \in B$, $i = 0,1,\dots,r-1$, and for $i = r$ if $t_r < 0$, we have $w(t_i,s) = w(t_i^+,s)$.

Let $C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}] = \{w_{|B^{(-)}} : w \in C^*_{\mathrm{imp}}[B,\mathbb{R}]\}$ in the case $\tau_0 > 0$. Elements of the sets $C^*_{\mathrm{imp}}[B,\mathbb{R}]$ and $C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}]$ will be denoted by the same symbols. It is easy to see that if $z \in C_{\mathrm{imp}}[E^*,\mathbb{R}]$ and $(x,y) \in \overline{E}$, then $z_{(x,y)} \in C^*_{\mathrm{imp}}[B,\mathbb{R}]$ and $z_{(x^-,y)} \in C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}]$ in the case $\tau_0 > 0$.

For $w \in C^*_{\mathrm{imp}}[B,\mathbb{R}]$ we define $\|w\|_B = \sup\{|w(t,s)| : (t,s) \in B\}$. We denote by $\|\cdot\|_{B^{(-)}}$ the supremum norm in the space $C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}]$.

Let $M[n]$ be the class of all matrices $\gamma = [\gamma_{ij}]_{1\le i,j\le n}$, where $\gamma_{ij} \in \mathbb{R}$ and $\gamma_{ij} = \gamma_{ji}$.

Suppose that

$$\Omega = (E \setminus E_{\mathrm{imp}}) \times \mathbb{R} \times C^*_{\mathrm{imp}}[B,\mathbb{R}] \times \mathbb{R}^n \times M[n],$$
$$\Omega_{\mathrm{imp}} = (E_{\mathrm{imp}} \cup \partial_0 E_{\mathrm{imp}}) \times \mathbb{R} \times C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}]$$

and $f : \Omega \to \mathbb{R}$, $g : \Omega_{\mathrm{imp}} \to \mathbb{R}$ and $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$, where $\varphi_{|\partial_0 E} \in C_{\mathrm{imp}}[\partial_0 E,\mathbb{R}]$, are given functions.

A function $z \in C_{\mathrm{imp}}[E^*,\mathbb{R}]$ will be called a function of class $C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$ if z has continuous derivatives $D_x z(x,y)$, $D_y z(x,y)$ and $D_{yy} z(x,y)$ for $(x,y) \in E \setminus E_{\mathrm{imp}}$, where

$$D_y z = (D_{y_1} z,\dots,D_{y_n} z), \qquad D_{yy} z = [D_{y_i y_j} z]_{1\le i,j\le n}.$$

A function $f : \Omega \to \mathbb{R}$ is said to be *parabolic with respect to* $z \in C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$ in $E \setminus E_{\mathrm{imp}}$ if for $(x,y) \in E \setminus E_{\mathrm{imp}}$ and for any $\gamma, s \in M[n]$ such that

$$\sum_{i,j=1}^{n} (\gamma_{ij} - s_{ij})\lambda_i\lambda_j \le 0, \qquad \lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n,$$

we have

$$f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),\gamma) \le f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),s).$$

We consider the initial-boundary value problem:

$$D_x z(x,y) = f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),D_{yy} z(x,y)), \qquad (x,y)\in E\setminus E_{\mathrm{imp}}, \tag{3}$$

$$z(x,y)=\varphi(x,y), \quad (x,y)\in E_0\cup\partial_0 E, \tag{4}$$

$$\Delta z(x,y) = g(x,y,z(x^-,y),z_{(x^-,y)}), \quad (x,y)\in E_{\mathrm{imp}}\cup\partial_0 E_{\mathrm{imp}}. \tag{5}$$

For $f:\Omega\to\mathbb{R}$, $g:\Omega_{\mathrm{imp}}\to\mathbb{R}$ and $z\in C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$ we write

$$F[z](x,y) = D_x z(x,y) - f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),D_{yy} z(x,y)), \qquad (x,y)\in E\setminus E_{\mathrm{imp}},$$

and

$$G[z](x,y) = \Delta z(x,y) - g(x,y,z(x^-,y),z_{(x^-,y)}), \quad (x,y)\in E_{\mathrm{imp}}.$$

3. Main results

3.1. *Impulsive parabolic functional differential inequalities*. We introduce

ASSUMPTION H1. Suppose that:

1. the function $f:\Omega\to\mathbb{R}$ of the variables (x,y,p,w,q,s) is non-decreasing with respect to the functional argument w,
2. the function $g:\Omega_{\mathrm{imp}}\to\mathbb{R}$ of (x,y,p,w) is non-decreasing with respect to w and for each $(x,y)\in E_{\mathrm{imp}}$ and $w\in C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}]$ the function $\delta(p)=p+g(x,y,p,w)$, $p\in\mathbb{R}$, is non-decreasing on $\mathbb{R}$.

THEOREM 1. *Suppose that*:

1. *Assumption* H1 *holds*,
2. $u,v\in C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$ *satisfy the initial-boundary inequality*

$$u(x,y)<v(x,y), \quad (x,y)\in E_0\cup\partial_0 E, \tag{6}$$

3. *the functional differential inequality*

$$F[u](x,y) < F[v](x,y), \quad (x,y)\in E\setminus E_{\mathrm{imp}}, \tag{7}$$

and the inequality for impulses

$$G[u](x,y) < G[v](x,y), \quad (x,y)\in E_{\mathrm{imp}}, \tag{8}$$

are satisfied,

4. f *is parabolic with respect to* u *in* $E\setminus E_{\mathrm{imp}}$.

Then

$$u(x,y)<v(x,y) \quad \text{on } E^*. \tag{9}$$

Proof. If (9) is false then the set $Z = \{x \in [0, a) : \text{there exists } y \in (-b, b) \text{ such that } u(x, y) \geq v(x, y)\}$ is non-empty. Defining $\widetilde{x} = \inf Z$ it follows from (6) that $\widetilde{x} > 0$ and there exists $\widetilde{y} \in (-b, b)$ such that

$$\begin{aligned} &u(x, y) < v(x, y), \qquad (x, y) \in E^* \cap ([-\tau_0, \widetilde{x}) \times \mathbb{R}^n), \\ &u(\widetilde{x}, \widetilde{y}) = v(\widetilde{x}, \widetilde{y}). \end{aligned} \tag{10}$$

There are two cases to be distinguished:

Case 1: $(\widetilde{x}, \widetilde{y}) \in E \setminus E_{\text{imp}}$. Then

$$D_x(u - v)(\widetilde{x}, \widetilde{y}) \geq 0, \qquad D_y(u - v)(\widetilde{x}, \widetilde{y}) = 0$$

and

$$\sum_{i,j=1}^{n} D_{y_i y_j}(u - v)(\widetilde{x}, \widetilde{y})\lambda_i \lambda_j \leq 0,$$

for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, which leads to a contradiction with (7).

Case 2: $(\widetilde{x}, \widetilde{y}) \in E_{\text{imp}}$. Then there exists i, $1 \leq i \leq k$, such that $\widetilde{x} = x_i$. From (10) we have

$$u(\widetilde{x}^-, \widetilde{y}) \leq v(\widetilde{x}^-, \widetilde{y}). \tag{11}$$

It follows from (8) and (11) that

$$\begin{aligned} u(\widetilde{x}, \widetilde{y}) - v(\widetilde{x}, \widetilde{y}) &< u(\widetilde{x}^-, \widetilde{y}) + g(\widetilde{x}, \widetilde{y}, u(\widetilde{x}^-, \widetilde{y}), u_{(\tilde{x}^-, \tilde{y})}) \\ &\quad - v(\widetilde{x}^-, \widetilde{y}) - g(\widetilde{x}, \widetilde{y}, v(\widetilde{x}^-, \widetilde{y}), v_{(\tilde{x}^-, \tilde{y})}) \leq 0, \end{aligned}$$

which contradicts (10).

Hence Z is empty and the statement (9) follows. ■

Remark 1. In Theorem 1 we can assume instead of (7), (8) that

$$\begin{aligned} &F[u](x, y) < F[v](x, y) \qquad \text{for } (x, y) \in T \setminus E_{\text{imp}}, \\ &G[u](x, y) < G[v](x, y) \qquad \text{for } (x, y) \in T \cap E_{\text{imp}}, \end{aligned}$$

where

$$T = \{(x, y) \in E : u(t, s) < v(t, s) \text{ for } (t, s) \in E,\ t \in [0, x),\ u(x, y) = v(x, y)\}.$$

Now we consider weak impulsive parabolic functional differential inequalities.

Assumption H2. Suppose that:

1. $\sigma : ([0, a] \setminus J_{\text{imp}}) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\sigma(x, 0) = 0$ for $x \in [0, a] \setminus J_{\text{imp}}$,

2. $\sigma_0 : J_{\mathrm{imp}} \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $\sigma_0(x,0) = 0$ for $x \in J_{\mathrm{imp}}$ and the right-hand maximal solution of the problem

$$\begin{aligned} &\alpha'(x) = \sigma(x, \alpha(x)), \qquad x \in J \setminus J_{\mathrm{imp}}, \\ &\alpha(0) = 0, \\ &\Delta\alpha(x) = \sigma_0(x, \alpha(x^-)), \qquad x \in J_{\mathrm{imp}}, \end{aligned}$$

is $\alpha(x) = 0$, $x \in J$,

3. $f : \Omega \to \mathbb{R}$ satisfies the inequality

$$\begin{aligned} f(x,y,p,w,q,s) &- f(x,y,\overline{p},\overline{w},q,s) \\ &\geq -\sigma(x, \max\{\overline{p} - p, \|\bar{w} - w\|_B\}) \quad \text{on } \Omega, \end{aligned}$$

where $p \leq \overline{p}$ and $w \leq \overline{w}$,

4. for $(x,y,p,w) \in E_{\mathrm{imp}} \times \mathbb{R} \times C^*_{\mathrm{imp}}[B^{(-)}, \mathbb{R}]$ we have

$$g(x,y,p,w) - g(x,y,\overline{p},\overline{w}) \geq -\sigma_0(x, \max\{\overline{p} - p, \|\overline{w} - w\|_{B^{(-)}}\}),$$

where $p \leq \overline{p}$, $w \leq \overline{w}$.

THEOREM 2. *Suppose that*:

1. *Assumptions* H1 *and* H2 *hold*,
2. $u, v \in C^{(1,2)}_{\mathrm{imp}}[E^*, \mathbb{R}]$ *and*

$$u(x,y) \leq v(x,y) \qquad \text{on } E_0 \cup \partial_0 E, \tag{12}$$

3. *the functional differential inequality*

$$F[u](x,y) \leq F[v](x,y), \qquad (x,y) \in E \setminus E_{\mathrm{imp}}, \tag{13}$$

and the inequality for impulses

$$G[u](x,y) \leq G[v](x,y), \qquad (x,y) \in E_{\mathrm{imp}}, \tag{14}$$

are satisfied,

4. f *is parabolic with respect to* u *in* $E \setminus E_{\mathrm{imp}}$.

Then $u(x,y) \leq v(x,y)$ *on* E^*.

Proof. Suppose that $a_0 \in (x_k, a)$. We prove that

$$u(x,y) \leq v(x,y) \tag{15}$$

for $(x,y) \in ([-\tau_0, a_0) \times \mathbb{R}^n) \cap E^*$.

Consider the problem

$$\begin{aligned} &\alpha'(x) = \sigma(x, \alpha(x)) + \varepsilon_0, \qquad x \in J \setminus J_{\mathrm{imp}}, \\ &\alpha(0) = \varepsilon_1, \\ &\Delta\alpha(x) = \sigma_0(x, \alpha(x^-)) + \varepsilon_2, \qquad x \in J_{\mathrm{imp}}. \end{aligned} \tag{16}$$

There exists $\widetilde{\varepsilon} > 0$ such that for $0 < \varepsilon_i < \widetilde{\varepsilon}$, $i = 0,1,2$, there exists a solution $\omega(\cdot;\varepsilon)$, $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$, of (16) and this solution is defined on $[0, a_0)$.

Let

$$\widetilde{v}(x,y)=\begin{cases} v(x,y)+\varepsilon_0, & (x,y)\in E_0,\\ v(x,y)+\omega(x;\varepsilon), & (x,y)\in([0,a_0)\times\mathbb{R}^n)\cap(E\cup\partial_0 E).\end{cases}$$

We prove that

$$u(x,y)<\widetilde{v}(x,y)\quad \text{on } ([0,a_0)\times\mathbb{R}^n)\cap(E\cup\partial_0 E). \tag{17}$$

We have

$$\begin{aligned}
&F[u](x,y)-F[\widetilde{v}](x,y)\\
&\quad=F[u](x,y)-D_x v(x,y)-\omega'(x;\varepsilon)\\
&\qquad+f(x,y,\widetilde{v}(x,y),\widetilde{v}_{(x,y)},D_y v(x,y),D_{yy}v(x,y))\\
&\quad\le F[u](x,y)-D_x v(x,y)-\omega'(x;\varepsilon)\\
&\qquad+f(x,y,v(x,y),v_{(x,y)},D_y v(x,y),D_{yy}v(x,y))+\sigma(x,\omega(x;\varepsilon))\\
&\quad=F[u](x,y)-F[v](x,y)-\varepsilon_0<0,\qquad (x,y)\in(E\setminus E_{\mathrm{imp}})\cap([0,a_0)\times\mathbb{R}^n).
\end{aligned}$$

For $(x,y)\in E_{\mathrm{imp}}$ we have

$$\begin{aligned}
&G[u](x,y)-G[\widetilde{v}](x,y)\\
&\qquad=G[u](x,y)-\Delta\widetilde{v}(x,y)+g(x,y,\widetilde{v}(x^-,y),\widetilde{v}_{(x^-,y)})\\
&\qquad\le G[u](x,y)-\Delta v(x,y)-\Delta\omega(x;\varepsilon)+g(x,y,v(x^-,y),v_{(x^-,y)})\\
&\qquad\quad+\sigma_0(x,\omega(x^-;\varepsilon))\\
&\qquad=G[u](x,y)-G[v](x,y)-\varepsilon_2<0.
\end{aligned}$$

Since $u(x,y)<\widetilde{v}(x,y)$ on $(E_0\cup\partial_0E)\cap([0,a_0)\times\mathbb{R}^n)$, from Theorem 1 we have assertion (17). Since $\lim_{\varepsilon\to 0}\omega(x;\varepsilon)=0$ uniformly with respect to x on $[0,a_0)$, we obtain (15). The constant $a_0\in(x_k,a)$ is arbitrary and therefore the proof is complete. ■

Assumption H3. Suppose that:

1. $\widetilde{\sigma}:([0,a]\setminus J_{\mathrm{imp}})\times\mathbb{R}_-\to\mathbb{R}_+$, $\mathbb{R}_-=(-\infty,0]$, is continuous, $\widetilde{\sigma}(x,0)=0$ for $x\in[0,a]\setminus J_{\mathrm{imp}}$ and for $p\le\overline{p}$ we have

$$f(x,y,p,w,q,s)-f(x,y,\overline{p},w,q,s)\le\widetilde{\sigma}(x,p-\overline{p})\quad\text{on } \Omega,$$

2. $\widetilde{\sigma}_0:J_{\mathrm{imp}}\times\mathbb{R}_-\to\mathbb{R}_+$ is continuous, $\widetilde{\sigma}_0(x,0)=0$ for $x\in J_{\mathrm{imp}}$ and for $p\le\overline{p}$ we have

$$g(x,y,p,w)-g(x,y,\overline{p},w)\le\widetilde{\sigma}_0(x,p-\overline{p})\quad\text{on } E_{\mathrm{imp}}\times\mathbb{R}\times C^*_{\mathrm{imp}}[B^{(-)},\mathbb{R}],$$

3. the left-hand minimal solution of the problem

$$\begin{aligned}
&\alpha'(x) = \widetilde{\sigma}(x, \alpha(x)), \qquad x \in J \setminus J_{\mathrm{imp}},\\
&\Delta\alpha(x) = \widetilde{\sigma}_0(x, \alpha(x^-)), \qquad x \in J_{\mathrm{imp}},\\
&\lim_{x \to a^-} \alpha(x) = 0
\end{aligned}$$

is $\alpha(x) = 0$, $x \in J$.

THEOREM 3. *Suppose that*:

1. *Assumptions* H1 *and* H3 *hold*,
2. $u, v \in C^{(1,2)}_{\mathrm{imp}}[E^*, \mathbb{R}]$ *satisfy the initial-boundary inequality* (6), *and the functional differential inequality* (13) *holds on* $E \setminus E_{\mathrm{imp}}$,
3. *estimate* (14) *is satisfied*,
4. f *is parabolic with respect to* v *in* $E \setminus E_{\mathrm{imp}}$.

Then

$$u(x, y) < v(x, y) \qquad \text{on } E^*. \tag{18}$$

Proof. First we prove (18) for $(x, y) \in ([0, a - \varepsilon) \times \mathbb{R}^n) \cap E$, where $a - x_k > \varepsilon > 0$. Let $0 < p_0 < \min\{v(x, y) - u(x, y) : (x, y) \in E_0 \cup \partial_0 E\}$. For $\delta > 0$ denote by $\omega(\cdot; \delta)$ the right-hand minimal solution of the problem

$$\begin{aligned}
&\alpha'(x) = -\widetilde{\sigma}(x, -\alpha(x)) - \delta, \qquad x \in J \setminus J_{\mathrm{imp}},\\
&\alpha(0) = p_0,\\
&\Delta\alpha(x) = -\widetilde{\sigma}_0(x, -\alpha(x^-)) - \delta, \qquad x \in J_{\mathrm{imp}}.
\end{aligned} \tag{19}$$

If $p_0 > 0$ is fixed then to every $\varepsilon > 0$ corresponds $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the solution $\omega(\cdot; \delta)$ of (19) exists and is positive on $[0, a - \varepsilon)$. Suppose that $\delta > 0$ is a constant such that $\omega(\cdot; \delta)$ satisfies the above conditions. Let

$$\widetilde{u}(x, y) = \begin{cases} u(x, y) + p_0, & (x, y) \in E_0,\\ u(x, y) + \omega(x; \delta), & (x, y) \in (E \cup \partial_0 E) \cap ([0, a - \varepsilon) \times \mathbb{R}^n). \end{cases}$$

We will prove that

$$\widetilde{u}(x, y) < v(x, y) \qquad \text{on } E \cap ([0, a - \varepsilon) \times \mathbb{R}^n). \tag{20}$$

It follows from H1 and H3 that

$$\begin{aligned}
F[\widetilde{u}](x, y) - F[v](x, y) &\le D_x u(x, y) + \omega'(x; \delta)\\
&\qquad - f(x, y, u(x, y), u_{(x,y)}, D_y u(x, y), D_{yy} u(x, y))\\
&\qquad + \widetilde{\sigma}(x, -\omega(x; \delta)) - F[v](x, y)\\
&= F[u](x, y) - F[v](x, y) - \delta < 0
\end{aligned}$$

for $(x, y) \in (E \setminus E_{\mathrm{imp}}) \cap ([0, a - \varepsilon) \times \mathbb{R}^n)$.

Now we prove that

$$G[\widetilde{u}](x, y) < G[v](x, y), \qquad (x, y) \in E_{\mathrm{imp}}. \tag{21}$$

It follows from H3, (14) and (19) that

$$\begin{aligned} G[\widetilde{u}](x,y) - G[v](x,y) &\le \Delta u(x,y) + \Delta\omega(x;\delta) - g(x,y,u(x^-,y),u_{(x^-,y)}) \\ &\quad + \widetilde{\sigma}_0(x,-\omega(x^-;\delta)) - G[v](x,y) \\ &= G[u](x,y) - G[v](x,y) - \delta < 0, \qquad (x,y) \in E_{\mathrm{imp}}, \end{aligned}$$

which completes the proof of (21). Since $\widetilde{u}(x,y) < v(x,y)$ for $(x,y) \in (E_0 \cup \partial_0 E) \cap ([0,a-\varepsilon) \times \mathbb{R}^n)$, we have estimate (20) from Theorem 1. It follows from (20) that $u(x,y) < v(x,y)$ on $([0,a-\varepsilon)\times\mathbb{R}^n)\cap E$. Since $\varepsilon > 0$ is arbitrary, inequality (18) holds on E^*. ∎

3.2. *Comparison theorems for parabolic functional differential inequalities*. In this section we prove estimates of functions satisfying impulsive parabolic functional differential inequalities by means of solutions of impulsive ordinary functional differential equations.

Let $C_{\mathrm{imp}}[J_0 \cup J, \mathbb{R}]$ be the class of all functions $\alpha : J_0 \cup J \to \mathbb{R}$ such that:

(i) the restriction of α to $J_0 \cup J \setminus J_{\mathrm{imp}}$ is continuous,

(ii) for each $x \in J_{\mathrm{imp}}$ the limits

$$\lim_{\substack{t\to x\\ t<x}} \alpha(t) = \alpha(x^-), \qquad \lim_{\substack{t\to x\\ t>x}} \alpha(t) = \alpha(x^+)$$

exist and $\alpha(x) = \alpha(x^+)$ for $x \in J_{\mathrm{imp}}$.

Suppose that we have a sequence $\{t_1,\dots,t_r\}$ such that $-\tau_0 \le t_1 < t_2 < \ldots < t_r \le 0$. For $t_1 > -\tau_0$ we also define $t_0 = -\tau_0$ and for $t_r < 0$ we put $t_{r+1} = 0$. Let $J^{(i)} = (t_i, t_{i+1})$, $i = 0,1,\dots,r$.

We denote by $C^*_{\mathrm{imp}}[J_0,\mathbb{R}]$ the class of all functions $\eta : J_0 \to \mathbb{R}$ such that there exists a sequence $\{t_0,t_1,\dots,t_r,t_{r+1}\}$ depending on η such that:

(i) the functions $\eta_{|J^{(i)}}$, $i = 0,1,\dots,r$, are continuous,

(ii) for each i, $i = 2,\dots,r+1$, and for $t_1 > -\tau_0$, the limit

$$\lim_{\substack{t\to t_i\\ t<t_i}} \eta(t) = \eta(t_i^-)$$

exists,

(iii) for each i, $i = 0,1,\dots,r-1$, and for $t_r < 0$, the limit

$$\lim_{\substack{t\to t_i\\ t>t_i}} \eta(t) = \eta(t_i^+)$$

exists and $\eta(t_i) = \eta(t_i^+)$.

For $\tau_0 > 0$ we put $J_0^{(-)} = [-\tau_0, 0)$ and $C^*_{\mathrm{imp}}[J_0^{(-)},\mathbb{R}] = \{\eta_{|J_0^{(-)}} : \eta \in C^*_{\mathrm{imp}}[J_0,\mathbb{R}]\}$. We will denote the elements of $C^*_{\mathrm{imp}}[J_0,\mathbb{R}]$ and $C^*_{\mathrm{imp}}[J_0^{(-)},\mathbb{R}]$ by the same symbols. We denote by $\|\cdot\|_0$ the supremum norm in the space $C^*_{\mathrm{imp}}[J_0,\mathbb{R}]$ and in the space $C^*_{\mathrm{imp}}[J_0^{(-)},\mathbb{R}]$. For $z \in C_{\mathrm{imp}}[E^*,\mathbb{R}]$ we define

$Tz : J_0 \cup J \to \mathbb{R}_+$ by

$$(Tz)(x) = \max\{|z(x,y)| : y \in [-c,c]\}, \qquad x \in [-\tau_0, a).$$

If $\alpha : J_0 \cup J \to \mathbb{R}$ and $x \in J$ then we define $\alpha_{(x)} : J_0 \to \mathbb{R}$ by $\alpha_{(x)}(t) = \alpha(x+t)$, $t \in J_0$. For the above α and x we define $\alpha_{(x^-)} : J_0^{(-)} \to \mathbb{R}$ by $\alpha_{(x^-)}(t) = \alpha(x+t)$ for $t \in J_0^{(-)}$. For $w \in C^*_{\mathrm{imp}}[B, \mathbb{R}]$ we define $T^* w : J_0 \to \mathbb{R}_+$ by

$$(T^* w)(t) = \max\{|w(t,s)| : \ s \in [-\tau, \tau]\}.$$

Lemma 1. *If $z \in C_{\mathrm{imp}}[E^*, \mathbb{R}]$ then $Tz \in C_{\mathrm{imp}}[J_0 \cup J, \mathbb{R}_+]$. If $w \in C^*_{\mathrm{imp}}[B, \mathbb{R}]$ then $T^* w \in C^*_{\mathrm{imp}}[J_0, \mathbb{R}_+]$.*

We omit the proof.

Assumption H4. Suppose that:

1. the functions $\sigma : ([0,a) \setminus J_{\mathrm{imp}}) \times \mathbb{R}_+ \times C^*_{\mathrm{imp}}[J_0, \mathbb{R}_+] \to \mathbb{R}_+$ and $\widetilde{\sigma} : J_{\mathrm{imp}} \times \mathbb{R}_+ \times C^*_{\mathrm{imp}}[J_0^{(-)}, \mathbb{R}_+] \to \mathbb{R}_+$ are continuous and non-decreasing with respect to the functional argument,
2. for each $(x, \eta) \in J \times C^*_{\mathrm{imp}}[J_0^{(-)}, \mathbb{R}_+]$ the function $\gamma(p) = p + \widetilde{\sigma}(x, p, \eta)$, $p \in \mathbb{R}_+$, is non-decreasing on $\mathbb{R}_+$.

Lemma 2. *Suppose that*:

1. *Assumption* H4 *holds and $\psi \in C_{\mathrm{imp}}[J_0 \cup J, \mathbb{R}]$,*
2. *$\widetilde{\eta} \in C(J_0, \mathbb{R}_+)$ and $\omega(\cdot; \widetilde{\eta}) : [-\tau_0, a) \to \mathbb{R}_+$ is the maximal solution of the problem*

$$\begin{aligned} &\alpha'(x) = \sigma(x, \alpha(x), \alpha_{(x)}), \qquad x \in J \setminus J_{\mathrm{imp}}, \\ &\alpha(x) = \widetilde{\eta}(x), \qquad x \in J_0, \\ &\Delta\alpha(x) = \widetilde{\sigma}(x, \alpha(x^-), \alpha_{(x^-)}), \qquad x \in J_{\mathrm{imp}}, \end{aligned} \tag{22}$$

3. *the function ψ satisfies*

$$\begin{aligned} \psi(x) &\le \widetilde{\eta}(x), \qquad x \in J_0, \\ \Delta\psi(x) &\le \widetilde{\sigma}(x, \psi(x^-), \psi_{(x^-)}), \qquad x \in J_{\mathrm{imp}}, \end{aligned}$$

4. *for $x \in P_+ = \{x > 0, \ x \in J \setminus J_{\mathrm{imp}} : \psi(x) > \omega(x; \widetilde{\eta})\}$ we have*

$$D_- \psi(x) \le \sigma(x, \psi(x), \psi_{(x)}),$$

where D_- is the left-hand lower Dini derivative.

Then $\psi(x) \le \omega(x; \widetilde{\eta})$ for $x \in [-\tau_0, a)$.

We omit the proof.

THEOREM 4. *Suppose that*:

1. *Assumption* H4 *holds*, $f \in C(\Omega, \mathbb{R})$ *and for each* $(x, y, p, w) \in (E \setminus E_{\mathrm{imp}}) \times \mathbb{R} \times C^*_{\mathrm{imp}}[B, \mathbb{R}]$ *we have*

$$f(x, y, p, w, 0, 0) \operatorname{sign} p \le \sigma(x, |p|, T^* w), \tag{23}$$

where $\operatorname{sign} p$ *denotes* 1 *if* $p \ge 0$ *and* -1 *if* $p < 0$,

2. $u \in C^{(1,2)}_{\mathrm{imp}}[E^*, \mathbb{R}]$ *and*

$$D_x u(x, y) = f(x, y, u(x, y), u_{(x,y)}, D_y u(x, y), D_{yy} u(x, y)), \tag{24}$$

for $(x, y) \in E \setminus E_{\mathrm{imp}}$,

3. $\widetilde{\eta} \in C(J_0, \mathbb{R}_+)$ *and*

$$|u(x, y)| \le \widetilde{\eta}(x), \qquad (x, y) \in E_0, \tag{25}$$

4. $\omega(\cdot; \widetilde{\eta}) : [-\tau_0, a) \to \mathbb{R}_+$ *is the maximal solution of the problem* (22),

5. *the boundary estimate*

$$|u(x, y)| \le \omega(x; \widetilde{\eta}), \qquad (x, y) \in \partial_0 E \tag{26}$$

and the impulsive estimate

$$\begin{aligned} &|u(x, y)| \\ &\le |u(x^-, y)| + \widetilde{\sigma}(x, |u(x^-, y)|, (T^* u)_{(x^-)}), \qquad (x, y) \in E_{\mathrm{imp}} \cup \partial_0 E_{\mathrm{imp}}, \end{aligned} \tag{27}$$

are satisfied,

6. f *is parabolic with respect to* u *in* $E \setminus E_{\mathrm{imp}}$.

Then

$$|u(x, y)| \le \omega(x; \widetilde{\eta}) \quad \textit{for } (x, y) \in E^*. \tag{28}$$

Proof. We prove that the function $\psi = Tu$ satisfies all conditions of Lemma 2. It follows from (25) and (27) that condition 3 of Lemma 2 holds. Suppose that $x \in P_+$. There exists $y \in [-c, c]$ such that $\psi(x) = |u(x, y)|$. It follows from (26) that $y \in (-b, b)$. There are two possibilities: either

$$\psi(x) = u(x, y) \tag{29a}$$

or

$$\psi(x) = -u(x, y). \tag{29b}$$

Suppose that (29b) holds. Then $D_y u(x, y) = 0$,

$$\sum_{i,j=1}^{n} D_{y_i y_j} u(x, y) \lambda_i \lambda_j \ge 0, \qquad \lambda = (\lambda_1, \ldots \lambda_n) \in \mathbb{R}^n,$$

and

$$\begin{aligned} D_- \psi(x) &\le -D_x u(x, y) \\ &= -f(x, y, u(x, y), u_{(x,y)}, D_y u(x, y), D_{yy} u(x, y)) \end{aligned}$$

$$\begin{aligned}&\le -f(x,y,u(x,y),u_{(x,y)},0,0)\\&= f(x,y,u(x,y),u_{(x,y)},0,0)\operatorname{sign}u(x,y)\\&\le \sigma(x,|u(x,y)|,(Tu)_{(x)}).\end{aligned}$$

Thus ψ satisfies condition 4 of Lemma 2. The case when (29a) holds is analogous. Thus all conditions of Lemma 2 are satisfied and (28) follows. ■

Let us consider two problems: the problem (3)–(5) and the problem

(30) $$D_x z(x,y) = \widetilde{f}(x,y,z(x,y),z_{(x,y)},D_y z(x,y),D_{yy}z(x,y)), \qquad (x,y)\in E\setminus E_{\mathrm{imp}},$$

(31) $$z(x,y)=\widetilde{\varphi}(x,y),\qquad (x,y)\in E_0\cup\partial_0 E,$$

(32) $$\Delta z(x,y)=\widetilde{g}(x,y,z(x^-,y),z_{(x^-,y)}),\qquad (x,y)\in E_{\mathrm{imp}}\cup\partial_0 E_{\mathrm{imp}},$$

where $\widetilde{f}:\Omega\to\mathbb{R}$ and $\widetilde{g}:\Omega_{\mathrm{imp}}\to\mathbb{R}$ and $\widetilde{\varphi}:E_0\cup\partial_0E\to\mathbb{R}$, where $\widetilde{\varphi}_{|\partial_0 E}\in C_{\mathrm{imp}}[\partial_0E,\mathbb{R}]$, are given functions.

We prove an estimate of the difference between solutions of (3)–(5) and (30)–(32).

THEOREM 5. *Suppose that*:

1. *Assumption* H4 *holds*,
2. $f,\widetilde{f}\in C(\Omega,\mathbb{R})$, $g,\widetilde{g}\in C(\Omega_{\mathrm{imp}},\mathbb{R})$ *satisfy the inequalities*

(33) $$\begin{aligned}&(f(x,y,p,w,q,s)-\widetilde{f}(x,y,\overline{p},\overline{w},q,s))\operatorname{sign}(p-\overline{p})\\&\qquad\qquad\le\sigma(x,|p-\overline{p}|,T^*(w-\overline{w}))\qquad \text{on } \Omega,\\&|g(x,y,p,w)-\widetilde{g}(x,y,\overline{p},\overline{w})|\le\widetilde{\sigma}(x,|p-\overline{p}|,T^*(w-\overline{w}))\qquad\text{on } \Omega_{\mathrm{imp}},\end{aligned}$$

3. $\varphi,\widetilde{\varphi}:E_0\cup\partial_0E\to\mathbb{R}$, $\widetilde{\eta}\in C(J_0,\mathbb{R}_+)$, $\varphi_{|E_0}$, $\widetilde{\varphi}_{|E_0}\in C(E_0,\mathbb{R})$, $\varphi_{|\partial_0E}$, $\widetilde{\varphi}_{|\partial_0E}\in C_{\mathrm{imp}}[\partial_0E,\mathbb{R}]$, *and*

$$|\varphi(x,y)-\widetilde{\varphi}(x,y)|\le\widetilde{\eta}(x),\qquad (x,y)\in E_0,$$

4. *the maximal solution* $\omega(\cdot;\widetilde{\eta})$ *of* (22) *is defined on* $[-\tau_0,a)$ *and* $u,\widetilde{u}\in C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$ *are solutions of* (3)–(5) *and* (28)–(30), *respectively*,
5. $|\varphi(x,y)-\widetilde{\varphi}(x,y)|\le\omega(x;\widetilde{\eta})$ *on* ∂_0E,
6. f *is parabolic with respect to* u *in* $E\setminus E_{\mathrm{imp}}$.

Then $|u(x,y)-\widetilde{u}(x,y)|\le\omega(x;\widetilde{\eta})$ *for* $(x,y)\in E^*$.

Proof. We prove that the function $\psi=T(u-\widetilde{u})$ satisfies all conditions of Lemma 2. It is easy to see that condition 3 of Lemma 2 holds. Suppose that $x\in P_+$. There exists $y\in[-c,c]$ such that $\psi(x)=|u(x,y)-\widetilde{u}(x,y)|$. From condition 5 of the theorem it follows that $y\in(-b,b)$. There are two possibilities: either

(34a) $$\psi(x)=u(x,y)-\widetilde{u}(x,y)$$

or

$$\psi(x) = -(u(x,y) - \widetilde{u}(x,y)). \tag{34b}$$

Suppose that (34a) holds. Then $D_y(u-\widetilde{u})(x,y)=0$,

$$\sum_{i,j=1}^{n} D_{y_iy_j}(u-\widetilde{u})(x,y)\lambda_i\lambda_j \le 0, \qquad \lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n,$$

and

$$\begin{aligned}D_-\psi(x) &\le D_x u(x,y) - D_x\widetilde{u}(x,y)\\ &= [f(x,y,u(x,y),u_{(x,y)},D_yu(x,y),D_{yy}u(x,y))\\ &\quad - f(x,y,u(x,y),u_{(x,y)},D_yu(x,y),D_{yy}\widetilde{u}(x,y))]\\ &\quad + [f(x,y,u(x,y),u_{(x,y)},D_yu(x,y),D_{yy}\widetilde{u}(x,y))\\ &\quad - \widetilde{f}(x,y,\widetilde{u}(x,y),\widetilde{u}_{(x,y)},D_yu(x,y),D_{yy}\widetilde{u}(x,y))].\end{aligned}$$

The first difference in brackets is non-positive by the parabolicity of f with respect to u. Since $u(x,y)\ge\widetilde{u}(x,y)$ by (34a), in view of condition 2 we get

$$D_-\psi(x)\le\sigma(x,\psi(x),(T\psi)_{(x)}), \qquad x\in P_+.$$

The case when (34b) holds is analogous. Thus all conditions of Lemma 2 are satisfied and the statement of the theorem follows. ■

THEOREM 6. *Suppose that*:

1. *Assumption* H4 *holds*,
2. $f\in C(\Omega,\mathbb{R})$, $g\in C(\Omega_{\mathrm{imp}},\mathbb{R})$ *and*

$$\begin{aligned}&(f(x,y,p,w,q,s)-f(x,y,\bar{p},\overline{w},q,s))\,\mathrm{sign}(p-\overline{p})\\ &\qquad\qquad \le\sigma(x,|p-\overline{p}|,T^*(w-\overline{w})) \qquad \text{on } \Omega,\\ &|g(x,y,p,w)-g(x,y,\bar{p},\overline{w})|\le\widetilde{\sigma}(x,|p-\overline{p}|,T^*(w-\overline{w})) \qquad \text{on } \Omega_{\mathrm{imp}},\end{aligned}$$

3. $\sigma(x,0,\theta)=0$ *for* $x\in J\setminus J_{\mathrm{imp}}$ *and* $\widetilde{\sigma}(x,0,\theta)=0$ *for* $x\in J_{\mathrm{imp}}$, *where* $\theta(t)=0$ *for* $t\in J_0$,
4. *the maximal solution of the problem*

$$\begin{aligned}&\alpha'(x)=\sigma(x,\alpha(x),\alpha_{(x)}), \qquad x\in J\setminus J_{\mathrm{imp}},\\ &\alpha(x)=0, \qquad x\in J_0,\\ &\Delta\alpha(x)=\widetilde{\sigma}(x,\alpha(x^-),\alpha_{(x^-)}), \qquad x\in J_{\mathrm{imp}},\end{aligned}$$

is $\alpha(x)=0$, $x\in J_0\cup J$.

Then the problem (3)–(5) *admits at most one solution in* $C^{(1,2)}_{\mathrm{imp}}[E^*,\mathbb{R}]$.

Proof. Put $\widetilde{f}=f$ and $\widetilde{g}=g$ and apply Theorem 5. ■

Remark 2. Suppose that $\varrho:([0,a)\setminus J_{\mathrm{imp}})\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ and $\widetilde{\varrho}:J_{\mathrm{imp}}\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ are given functions and $\sigma:([0,a)\setminus J_{\mathrm{imp}})\times\mathbb{R}_+\times$

$C^*_{\mathrm{imp}}[J_0, \mathbb{R}_+] \to \mathbb{R}_+$ and $\widetilde{\sigma} : J_{\mathrm{imp}} \times \mathbb{R}_+ \times C^*_{\mathrm{imp}}[J_0^{(-)}, \mathbb{R}_+] \to \mathbb{R}_+$ are defined by

$$\sigma(x, p, \eta) = \varrho(x, p, \sup\{\eta(t) : t \in J_0\}),$$
$$\widetilde{\sigma}(x, p, \eta) = \widetilde{\varrho}(x, p, \sup\{\eta(t) : t \in J_0^{(-)}\}).$$

Then:

1. Inequality (27) is equivalent to

$$|u(x, y)| \le |u(x^-, y)| + \widetilde{\varrho}(x, |u(x^-, y)|, \|u_{(x^-, y)}\|_0), \qquad (x, y) \in E_{\mathrm{imp}} \cup \partial_0 E_{\mathrm{imp}}.$$

2. Estimates (23) and (33) are equivalent to

$$f(x, y, p, w, 0, 0) \operatorname{sign} p \le \varrho(x, |p|, \|w\|_0),$$
$$(f(x, y, p, w, q, s) - \widetilde{f}(x, y, \overline{p}, \overline{w}, q, s)) \operatorname{sign}(p - \overline{p}) \le \varrho(x, |p - \overline{p}|, \|w - \overline{w}\|_0),$$
$$|g(x, y, p, w) - \widetilde{g}(x, y, \bar{p}, \overline{w})| \le \widetilde{\varrho}(x, |p - \overline{p}|, \|w - \overline{w}\|_0).$$

3. If we assume that $\widetilde{\eta} \in C(J_0, \mathbb{R}_+)$ is non-decreasing on J_0 then the problem (22) is equivalent to

$$\alpha'(x) = \varrho(x, \alpha(x), \alpha(x)), \qquad x \in J \setminus J_{\mathrm{imp}},$$
$$\alpha(0) = \widetilde{\eta}(0),$$
$$\Delta\alpha(x) = \widetilde{\varrho}(x, \alpha(x^-), \alpha(x^-)), \qquad x \in J_{\mathrm{imp}}.$$

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