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WAVELET TRANSFORM FOR TIME-FREQUENCY REPRESENTATION AND FILTRATION OF DISCRETE SIGNALS

Abstract. A method to analyse and filter real-valued discrete signals of finite duration $s(n)$, $n = 0, 1, \ldots, N-1$, where $N = 2^p$, $p > 0$, by means of time-frequency representation is presented. This is achieved by defining an invertible discrete transform representing a signal either in the time or in the time-frequency domain, which is based on decomposition of a signal with respect to a system of basic orthonormal discrete wavelet functions. Such discrete wavelet functions are defined using the Meyer generating wavelet spectrum and the classical discrete Fourier transform between the time and the frequency domains.

1. Introduction. In the sequel we assume that the considered discrete signals represent values of some complex-valued function of time f at $N = 2^p$, $p > 0$, equidistant time points, i.e. $s(n) = f(t_0 + n\Delta t)$, $n = 0, 1, \ldots, N-1$, $\Delta t > 0$. We use the symbol l_N^2 to denote the Hilbert space of complex-valued discrete signals $s(n)$, $n = 0, 1, \ldots, N-1$, with scalar product $\langle u, v\rangle = \sum_{n=0}^{N-1} u(n)\overline{v}(n)$. The *discrete Fourier transform* (DFT) of such a signal is defined by

$$(1)\quad \widetilde{s}(\nu) = \sum_{n=0}^{N-1} s(n)\exp(-i2\pi\nu n/N), \qquad \nu = -N/2+1, -N/2+2, \ldots, N/2,$$

and the inverse transform is defined as

$$(2)\qquad s(n) = \frac{1}{N}\sum_{\nu=-N/2+1}^{N/2} \widetilde{s}(\nu)\exp(i2\pi\nu n/N), \qquad n = 0, 1, \ldots, N-1.$$

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This transform is a good tool for analysis of a discrete signal as long as the frequency characteristics of its successive temporal components are similar. In that case it can be used to filter the stationary frequency components of the signal [5]. However, as one can plainly see from the formula (1), changing the signal value $s(k)$ at one point with index k influences the whole Fourier spectrum $\widetilde{s}(\nu)$, $\nu = -N/2+1, -N/2+2, \ldots, N/2$, so we cannot expect that the DFT will be an appropriate tool for analysis of transient signals whose frequency characteristics vary with time.

The aim of this work is to propose a new method for spectro-temporal analysis and filtering of finite duration signals having non-stationary frequency characteristics, which is based on the concept of the discrete wavelet transform of square-integrable functions of a real variable widely described in the wavelet literature [1], [2], [4]. It should be remarked that in the literature one can find at least two [2], [3], [8] definitions of a wavelet transform for discrete signals of finite duration, which, however, do not seem to be appropriate for time-frequency filtering of such signals. In [3] the dilation operator of discrete functions appears to be inconvenient if the analyzed signal is to be interpreted as a sampled continuously defined function of time. The second definition [2], [8] gives no formulae for discrete wavelet functions used, nor for their discrete Fourier transforms, so one has to examine the time and frequency domain localization of the wavelets used in the transform.

According to Meyer's results [6], [7] there exist wavelets g such that the system of functions $g_{jk}(t) = 2^{j/2} g(2^j t - k)$, $j,k = 0, \pm 1, \pm 2, \ldots,$ is an orthonormal base in the space $L^2(\mathbb{R})$, i.e. for $f \in L^2(\mathbb{R})$ we have the representation

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} g_{jk}, \quad \text{where} \quad c_{jk} = \int_{-\infty}^{\infty} f(t) \overline{g}_{jk}(t)\, dt,\ j,k = 0, \pm 1, \pm 2, \ldots$$

The Fourier transform of the orthonormal functions g_{jk}, $j,k = 0, \pm 1, \pm 2, \ldots,$ is given by the formula (see [1]) $\widehat{g}_{jk}(\omega) = 2^{-j/2} \exp(-i2\pi k 2^{-j}\omega)\widehat{g}(2^{-j}\omega)$, where $\widehat{g} = F[g]$ is the Fourier transform of the wavelet g. After applying the Parseval identity the formula for the coefficients c_{jk}, $j,k = 0, \pm 1, \pm 2, \ldots,$ takes the form (see [1])

$$c_{jk} = 2^{-j/2} F^{-1}[\widehat{f}(\omega)\overline{\widehat{g}}(2^{-j}\omega)](2^{-j}k). \tag{3}$$

If the wavelet g satisfies the condition $\widehat{g}(0) = 0$, then we easily obtain

$$f(t) = \widehat{f}(0) + F^{-1}\Big[\sum_{j=-\infty}^{\infty} 2^{-j/2} \sum_{k=-\infty}^{\infty} c_{jk} \exp(-i2\pi k 2^{-j}\omega)\widehat{g}(2^{-j}\omega)\Big](t). \tag{4}$$

Equations (3) and (4), respectively, define the direct and inverse discrete wavelet transform in the case of functions of a real variable. The Meyer wavelet g generating an orthonormal base is defined by its Fourier transform as follows (see [4], [6], [7]): $\widehat{g}(\omega) = \exp(-i\pi\omega)\theta(\omega)$, where $\theta : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies for $0 < \varepsilon \le 1/6$ the conditions:

$$\begin{aligned}
&\theta(t) = \theta(-t) && \text{for } t \in \mathbb{R},\\
&\theta(t) = 0 && \text{for } |t| \notin [1/2-\varepsilon, 1+2\varepsilon],\\
&\theta(t) = 1 && \text{for } |t| \in [1/2+\varepsilon, 1-2\varepsilon],\\
&\theta^2(t) + \theta^2(1-t) = 1 && \text{for } |t| \in [1/2-\varepsilon, 1/2+\varepsilon],\\
&\theta(2t) = \theta(1-t) && \text{for } |t| \in [1/2-\varepsilon, 1/2+\varepsilon].
\end{aligned} \tag{5}$$

It should be noted that in that case the orthonormal functions g_{jk}, $j,k = 0, \pm1, \pm2, \ldots$, are real-valued because their Fourier transforms $\widehat{g}_{jk}$ have hermitian symmetry.

For the reasons which will become clear later let us now assume that for $0 < \varepsilon \le 1/6$ also a function $\theta' : \mathbb{R} \to \mathbb{R}$ is defined, which satisfies the conditions:

$$\begin{aligned}
&\theta'(t) = \theta'(-t) && \text{for } t \in \mathbb{R},\\
&\theta'(t) = 0 && \text{for } |t| \notin [1/2-\varepsilon, 1],\\
&\theta'(t) = \theta(t) && \text{for } |t| \in [1/2-\varepsilon, 1-2\varepsilon],\\
&\theta'(t) = 1 && \text{for } |t| \in [1-2\varepsilon, 1),\\
&\theta'(1) = \theta'(1/2) = 1/\sqrt{2}.
\end{aligned} \tag{6}$$

2. Definition and properties of the discrete wavelet functions. Since we are dealing with discrete signals of finite duration we abandon the idea of obtaining an orthonormal system of wavelets in the space l_N^2 by translations and dilations of a generating function. Instead, we define them in the discrete frequency domain, by discretizing modulated and dilated versions of a generating spectrum.

Definition 1. Let $N = 2^p$, $p > 0$, $0 < \varepsilon \le 1/6$ and let real functions θ and θ' satisfy the conditions (5) and (6). Then the discrete functions w_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, defined by their discrete Fourier transforms

$$\widetilde{w}_{jk}(\nu) = 2^{-j/2}\widehat{g}_j(2^{-j}\nu/N)\exp(-i2\pi k 2^{-j}\nu/N), \qquad \nu = -N/2+1, -N/2+2, \ldots, N/2,$$

where $\widehat{g}_j(t) = \widehat{g}(t) = \theta(t)\exp(-i\pi t)$ for $j = -p, -p+1, \ldots, -2$ and $\widehat{g}_{-1}(t) = \theta'(t)\exp(-i\pi t)$ for $j = -1$, will be referred to as the *discrete wavelet functions*.

Discrete wavelet spectra for $j = -p, -p+1, \ldots, -2$ are thus obtained by discretization of a continuous Meyer wavelet spectrum at frequency points $2^{-j}\nu/N$. It is important to note that while the original Meyer wavelets are non-periodic functions with unbounded supports our definition of discrete wavelets leads to periodic functions of an integer variable (with period N), due to the periodicity of the DFT. They strongly depart from discretizations of the corresponding dilations and translations of Meyer continuous wavelet for low values of j, i.e. close to $-p$ (for $j=-p$ the discrete wavelet $w_{-p,0}$ is simply a cosine function), and they approximate such discretizations better for j close to -2.

In Lemma 1 we prove that the discrete wavelet functions are real-valued and satisfy the condition $\sum_{n=0}^{N-1} w_{jk}(n)=0$ for $j=-p,-p+1,\ldots,-1$ and $k=0,1,\ldots,2^jN-1$.

LEMMA 1. *If* $N=2^p$, $p>0$ *and* $\beta:\mathbb{R}\to\mathbb{R}$ *is a symmetric real function with* $\beta(0)=0$, *then the discrete functions* $r_{jk}(n)$, $j=-p,-p+1,\ldots,-1$, $k=0,1,\ldots,2^jN-1$, $n=0,1,\ldots,N-1$, *with discrete Fourier transforms given by*

$$\widetilde{r}_{jk}(\nu)=\beta(\nu)\exp(-i\pi 2^{-j}\nu/N)\exp(-i2\pi k2^{-j}\nu/N),\qquad \nu=-N/2+1,-N/2+2,\ldots,N/2,$$

are real-valued and have zero mean values.

Proof. First we prove that the r_{jk} are real-valued. From the definition of $\widetilde{r}_{jk}$ and from (2) it follows that

$$r_{jk}(n)=\frac{1}{N}\sum_{\nu=-N/2+1}^{N/2}\beta(\nu)\exp(-i2\pi(k2^{-j}+2^{-j}/2)\nu/N)\exp(i2\pi n\nu/N)$$

and since β is symmetric, $\beta(0)=0$ and $\exp(i\pi(n-k2^{-j}-2^{-j}/2))=\pm1$ for $j=-p,-p+1,\ldots,-1$ and $k=0,1,\ldots,2^jN-1$, we easily obtain

$$r_{jk}(n)=\frac{1}{N}\sum_{\nu=1}^{N/2-1}2\beta(\nu)\cos(2\pi(n-k2^{-j}-2^{-j}/2)\nu/N)\pm\frac{1}{N}\beta(N/2).$$

It follows immediately that the values $r_{jk}(n)$, $n=0,1,\ldots,N-1$, are real numbers, because β is real-valued.

Now we prove the second property of r_{jk}. By (2) for fixed j,k we have

$$\begin{aligned}N\sum_{n=0}^{N-1}r_{jk}(n)&=\sum_{n=0}^{N-1}\sum_{\nu=-N/2+1}^{N/2}\widetilde{r}_{jk}(\nu)\exp(i2\pi n\nu/N)\\&=\sum_{\nu=-N/2+1}^{N/2}\widetilde{r}_{jk}(\nu)\sum_{n=0}^{N-1}\exp(i2\pi n\nu/N)\end{aligned}$$

and hence in view of the equality $(1-\exp(i2\pi\nu))/(1-\exp(i2\pi\nu/N))=0$

for $0 < |\nu| < N$, we get

$$\sum_{n=0}^{N-1} r_{jk}(n) = \frac{1}{N}\widetilde{r}_{jk}(0)N = \widetilde{r}_{jk}(0) = 0,$$

since the condition $\beta(0) = 0$ implies $\widetilde{r}_{jk}(0) = 0$ for $j = -p, -p+1, \ldots, -1$ and $k = 0, 1, \ldots, 2^j N - 1$, which proves the lemma. ■

In order to prove that the discrete wavelet functions $\widetilde{w}_{jk}$, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, form an orthonormal system in the space l_N^2 we need the following three lemmas.

LEMMA 2. *For arbitrary complex-valued discrete signals* $u(n), v(n)$, $n = 0, 1, \ldots, N-1$,

$$\langle u, v\rangle = \sum_{n=0}^{N-1} u(n)\overline{v}(n) = \frac{1}{N}\sum_{\nu=-N/2+1}^{N/2} \widetilde{u}(\nu)\overline{\widetilde{v}}(\nu) = \frac{1}{N}\langle \widetilde{u}, \widetilde{v}\rangle. \tag{7}$$

P r o o f. Applying (2) we can write

$$u(n) = \frac{1}{N}\sum_{\nu=-N/2+1}^{N/2} \widetilde{u}(\nu)\exp(i2\pi n\nu/N),$$

$$v(n) = \frac{1}{N}\sum_{\mu=-N/2+1}^{N/2} \widetilde{v}(\mu)\exp(i2\pi n\mu/N)$$

for $n = 0, 1, \ldots, N-1$, and consequently

$$\sum_{n=0}^{N-1} u(n)\overline{v}(n) = \frac{1}{N^2}\sum_{\nu=-N/2+1}^{N/2}\sum_{\mu=-N/2+1}^{N/2} \widetilde{u}(\nu)\overline{\widetilde{v}}(\mu)\sum_{n=0}^{N-1}\exp(i2\pi n(\nu-\mu)/N).$$

In view of the equality

$$\sum_{n=0}^{N-1}\exp(i2\pi nk/N) = \frac{1-\exp(i2\pi k)}{1-\exp(i2\pi k/N)} = 0 \quad \text{for } k \neq 0, \pm N, \pm 2N, \ldots$$

we get

$$\sum_{n=0}^{N-1}\exp(i2\pi n(\nu-\mu)/N) = N\delta_{\nu\mu},$$

where $\delta_{\nu\mu}$ is the Kronecker delta, and in consequence

$$\sum_{n=0}^{N-1} u(n)\overline{v}(n) = \frac{1}{N}\sum_{\nu=-N/2+1}^{N/2} \widetilde{u}(\nu)\overline{\widetilde{v}}(\nu),$$

which establishes (7). ■

LEMMA 3. *If* $N = 2^p$, $p > 1$ *and for a given* $0 < \varepsilon \le 1/6$ *the real functions* θ *and* θ' *satisfy the conditions* (5) *and* (6), *then the functions* $F_j : [0, 2^{j+1}N] \to \mathbb{R}$ *defined by*

$$F_j(s) = \theta(2^{-j}s/N)\theta(2^{-j-1}s/N) \quad \textit{for } j = -p, -p+1, \ldots, -3,$$
$$F_{-2}(s) = \theta(2^2 s/N)\theta'(2s/N) \quad \textit{for } j = -2,$$

are symmetric with respect to the point $s_j = 2^j N$ *and* $F_j(2^j N + s) = 0$ *for* $2^{j+1}N\varepsilon \le |s| \le 2^j N$.

Proof. We prove that the function r defined for $0 \le s \le 2$ by $r(s) = \theta(s)\theta(s/2)$ is symmetric with respect to the point 1 and satisfies $r(1+s) = r(1-s) = 0$ for $2\varepsilon \le |s| \le 1$; in view of the equality $F_j(s) = r(2^{-j}s/N)$ this implies the assertion for $j = -p, -p+1, \ldots, -3$.

Since θ satisfies (5), necessarily $r(1+s) = r(1-s) = 0$ for $2\varepsilon \le |s| \le 1$, and for $0 \le |s| \le 2\varepsilon$ the same conditions yield

$$r(1+s) = \theta(2(1/2+s/2))\theta(1-(1/2-s/2)) = \theta(1/2-s/2)\theta(1-s) = r(1-s).$$

The proof in the case $j = -2$ is analogous because (5) and (6) imply that $r(s) = \theta(s)\theta(s/2) = \theta(s)\theta'(s/2)$ for $0 \le s \le 2$ since $0 < \varepsilon \le 1/6$. ∎

LEMMA 4. *If* $N = 2^p$, $p > 2$ *and integers* j *and* Δk *satisfy* $2-p \le j \le -1$ *and* $0 \le |\Delta k| < 2^j N$, *then for* $\nu_{0j} = 2^j N/2$ *and* $\nu'_{0j} = 2^j N$ *we have*

$$\sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \cos(2\pi\Delta k 2^{-j}\nu/N) = 0 \quad \textit{for } \Delta k = 2l+1,$$
$$1 + \sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \cos(2\pi\Delta k 2^{-j}\nu/N) = 2^{j-1}N\delta_{0,\Delta k} \quad \textit{for } \Delta k = 2l.$$

Proof. In order to prove the equality for $\Delta k = 2l+1$ observe that

$$\begin{aligned}
&\sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N)\\
&\quad = \exp(i2\pi\Delta k 2^{-j}(\nu_{0j}+1)/N) \sum_{\nu=0}^{\nu'_{0j}-\nu_{0j}-2} \exp(i2\pi\Delta k 2^{-j}\nu/N)\\
&\quad = \exp(i2\pi\Delta k 2^{-j}\nu_{0j}/N)\exp(i2\pi\Delta k 2^{-j}/N)\\
&\qquad \times \frac{1 - \exp(i2\pi\Delta k 2^{-j}(\nu'_{0j}-\nu_{0j}-1)/N)}{1 - \exp(i2\pi\Delta k 2^{-j}/N)}
\end{aligned}$$

so taking into account the equalities $2^{-j}\nu_{0j}/N = 1/2$, $2^{-j}(\nu'_{0j} - \nu_{0j})/N =$

$1/2$ and $\exp(i\pi(2l+1)) = -1$ we obtain

$$\sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N) = \frac{\exp(i2\pi\Delta k 2^{-j}/N)+1}{\exp(i2\pi\Delta k 2^{-j}/N)-1}$$
$$= \frac{\exp(i\pi\Delta k 2^{-j}/N)+\exp(-i\pi\Delta k 2^{-j}/N)}{\exp(i\pi\Delta k 2^{-j}/N)-\exp(-i\pi\Delta k 2^{-j}/N)}$$
$$= \frac{2\cos(\pi\Delta k 2^{-j}/N)}{2i\sin(\pi\Delta k 2^{-j}/N)} = -i\cot(\pi\Delta k 2^{-j}/N).$$

The last equality implies that the real part of the computed sum equals zero, which proves the assertion for $\Delta k = 2l+1$.

Now assume that $\Delta k = 2l$, $l \neq 0$. Since $2^{-j}\nu_{0j}/N = 1/2$ and $\exp(i2\pi l) = 1$, we have

$$1 + \sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N)$$
$$= 1 + \exp(i2\pi\Delta k 2^{-j}\nu_{0j}/N) \sum_{\nu=1}^{\nu'_{0j}-\nu_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N)$$
$$= \sum_{\nu=0}^{\nu'_{0j}-\nu_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N)$$
$$= \frac{1-\exp(i2\pi\Delta k 2^{-j}(\nu'_{0j}-\nu_{0j})/N)}{1-\exp(i2\pi\Delta k 2^{-j}/N)} = 0$$

because $2^{-j}(\nu'_{0j}-\nu_{0j})/N = 1/2$. For $\Delta k = 0$ we have

$$1 + \sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \exp(i2\pi\Delta k 2^{-j}\nu/N) = \nu'_{0j}-\nu_{0j} = N2^j/2,$$

which completes the proof. ∎

In Theorem 1 we prove that the discrete wavelet functions form an orthonormal system in l^2_N.

THEOREM 1. *Let* $N = 2^p$, $p > 0$. *Then the discrete wavelet functions* w_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, *with discrete Fourier transforms given in Definition* 1, *form an orthonormal system in the Hilbert space* l^2_N, *i.e.*

$$\langle w_{jk}, w_{j'k'}\rangle = \sum_{n=0}^{N-1} w_{jk}(n)\overline{w}_{j'k'}(n) = \delta_{jj'}\delta_{kk'}.$$

Proof. Since the w_{jk} are real-valued, by Lemma 2 we have

$$\langle w_{jk}, w_{j'k'}\rangle = \sum_{n=0}^{N-1} w_{jk}(n) w_{j'k'}(n) = \frac{1}{N}\sum_{\nu=-N/2+1}^{N/2} \widetilde{w}_{jk}(\nu)\overline{\widetilde{w}}_{j'k'}(\nu)$$

and in view of Definition 1 we obtain for $j, j' = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$ and $k' = 0, 1, \ldots, 2^{j'} N - 1$,

$$\text{(8)}\quad \langle w_{jk}, w_{j'k'}\rangle = \frac{2^{-(j+j')/2}}{N} \sum_{\nu=-N/2+1}^{N/2} \theta_j(2^{-j}\nu/N)\theta_{j'}(2^{-j'}\nu/N)\Phi_{jj'kk'}(\nu),$$

where $\theta_j = \theta$ for $j = -p, -p+1, \ldots, -2$, $\theta_{-1} = \theta'$ and

$$\Phi_{jj'kk'}(\nu) = \exp(-i\pi 2^{-j}\nu/N)\exp(i\pi 2^{-j'}\nu/N) \times \exp(-i2\pi k 2^{-j}\nu/N)\exp(i2\pi k' 2^{-j'}\nu/N).$$

First we show that $\langle w_{jk}, w_{j'k'}\rangle = 0$ for $j \neq j'$. Observe that (5) and (6) imply that the symmetric functions θ and θ' are for $0 < \varepsilon \leq 1/6$ different from zero only for $1/3 < |t| < 4/3$. We can of course assume that $j' = j + \Delta j$, where $\Delta j > 0$. The values of $\theta_j(2^{-j}\nu/N)$ are different from zero only for $2^j N/3 < |\nu| < 2^j 4N/3$, and the values of $\theta_{j+\Delta j}(2^{-j-\Delta j}\nu/N)$ only for $2^{\Delta j}2^j N/3 < |\nu| < 2^{\Delta j}2^j 4N/3$; consequently, for $\Delta j \geq 2$ we have $\theta_j(2^{-j}\nu/N)\theta_{j+\Delta j}(2^{-j-\Delta j}\nu/N) = 0$ for $\nu = -N/2 + 1, -N/2+2, \ldots, N/2$ and this clearly forces $\langle w_{jk}, w_{j+\Delta j,k'}\rangle = 0$ for $\Delta j \geq 2$.

Now consider the case $j' = j + 1$. The function $F_j(t) = \theta_j(2^{-j}t/N) \times \theta_{j+1}(2^{-j-1}t/N)$ is symmetric and real-valued and the function $\operatorname{Im}\Phi_{j,j+1,k,k'}(t) = \sin(\pi(k' - 2k - 1/2)2^{-j}t/N)$ is antisymmetric. The equalities $\operatorname{Im}\Phi_{j,j+1,k,k'}(0) = 0$, $\operatorname{Im}\Phi_{j,j+1,k,k'}(N/2) = \sin(\pi(k'-2k-1/2)2^{-j-1}) = 0$ for $j = -p, -p+1, \ldots, -2$, $k = 0, 1, \ldots, 2^j N - 1$ and $k' = 0, 1, \ldots, 2^{j+1}N - 1$ now clearly imply that the imaginary part of the considered scalar product $\langle w_{jk}, w_{j+1,k'}\rangle$ is zero. Furthermore, since the function $\operatorname{Re}\Phi_{j,j+1,k,k'}(t) = \cos(\pi(k' - 2k - 1/2)2^{-j}t/N)$ is symmetric and $F_j(N/2) = \theta_j(2^{-j}/2)\theta_{j+1} \times (2^{-j-1}/2) = 0$ and $F_j(0) = 0$ for $j = -p, -p+1, \ldots, -2$, from the last formula for $\langle w_{jk}, w_{j+1,k'}\rangle$ we easily obtain

$$\text{(9)}\qquad \langle w_{jk}, w_{j+1,k'}\rangle = \frac{2^{-(2j+1)/2}}{N}\sum_{\nu=1}^{N/2-1} 2F_j(\nu)\operatorname{Re}\Phi_{j,j+1,k,k'}(\nu)$$

for $j = -p, -p+1, \ldots, -2$, $k = 0, 1, \ldots, 2^j N - 1$ and $k' = 0, 1 \ldots, 2^{j+1}N - 1$.

By Lemma 3 the real function $F_j(2^jN+s)$ defined for $0 \le |s| \le 2^jN$, $j = -p, -p+1, \ldots, -2$, is symmetric and equal to zero for $2^jN/3 \le |s| \le 2^jN$. The function $\operatorname{Re}\Phi_{j,j+1,k,k'}(2^jN+s)$ is antisymmetric for the same values of s and, what is more, for $j = -p, -p+1, \ldots, -2$, $k = 0, 1, \ldots, 2^jN-1$ and $k' = 0, 1, \ldots, 2^{j+1}N-1$ we have $\operatorname{Re}\Phi_{j,j+1,k,k'}(2^jN) = \cos(\pi(k'-2k-1/2)) = 0$. The inequalities $2^jN + 2^jN/3 = 2^j4N/3 \le N/3 < N/2$ and $2^jN - 2^jN/3 = 2^j2N/3 \ge 2/3 > 0$ for $j = -p, -p+1, \ldots, -2$, where $N = 2^p$, now yield

$$\langle w_{jk}, w_{j+1,k'}\rangle = \frac{2^{-(2j+1)/2}}{N} \sum_{-2^jN/3<\mu<2^jN/3} 2F_j(2^jN+\mu)\operatorname{Re}\Phi_{j,j+1,k,k'}(2^jN+\mu) = 0.$$

We still have to consider the case of $j = -p, -p+1, \ldots, -1$ and $k, k' = 0, 1, \ldots, 2^jN-1$. From (8) we obtain

$$\langle w_{jk}, w_{jk'}\rangle = \frac{2^{-j}}{N} \sum_{\nu=-N/2+1}^{N/2} \theta_j^2(2^{-j}\nu/N)\Phi_{jjkk'}(\nu),$$

where $\Phi_{jjkk'}(\nu) = \exp(i2\pi\Delta k 2^{-j}\nu/N)$, $\Delta k = k'-k$, $\theta_j = \theta$ for $j = -p, -p+1, \ldots, -2$ and $\theta_{-1} = \theta'$. Since $\theta_j^2(2^{-j}t/N)$ is symmetric and real-valued and $\sin(\pi\Delta k 2^{-j}) = 0$ and $\theta_j(0) = 0$ for $j = -p, -p+1, \ldots, -1$ it follows that

$$\langle w_{jk}, w_{jk'}\rangle = \frac{2^{-j+1}}{N} \sum_{\nu=1}^{N/2-1} \theta_j^2(2^{-j}\nu/N)\cos(2\pi\Delta k 2^{-j}\nu/N) + \frac{2^{-j}}{N}\theta_j^2(2^{-j}/2)\cos(\pi\Delta k 2^{-j})$$

for $j = -p, -p+1, \ldots, -1$ and $k, k' = 0, 1, \ldots, 2^jN-1$.

Remembering that $N = 2^p$ we see that for $j = -p$ the index k can only have one value $k = 2^{-p}N-1 = 0$ and then necessarily $\Delta k = 0$. Further, since $\theta(t) = \theta'(t) = 0$ for $t > 4/3$ and $\theta(1) = \theta(1/2) = \theta'(1/2) = \theta'(1) = 1/\sqrt{2}$, we easily obtain $\langle w_{-p,0}, w_{-p,0}\rangle = 2\theta^2(1) = 1$.

In the case $j = -p+1$ we have $0 \le k \le 2^{-p+1}N-1 = 1$ and consequently $\Delta k = 0, 1$, and it follows from the same formula for $\langle w_{jk}, w_{jk'}\rangle$ that

$$\langle w_{-p+1,k}, w_{-p+1,k'}\rangle = \theta^2(1/2)\cos(2\pi\Delta k/2) + \theta^2(1)\cos(2\pi\Delta k) = \delta_{0,\Delta k}.$$

From now on we can assume that $p > 2$ because we have already considered the cases $p = 1, 2$. For $j = -p+2, -p+3, \ldots, -2$ and $k, k' = 0, 1, \ldots, 2^jN-1$, we have $\theta_j = \theta$ and we can write

$$\langle w_{jk}, w_{jk'} \rangle = \frac{2^{-j+1}}{N}(S_1 + S_2 + S_3),$$

where

$$S_1 = \sum_{\nu=\nu_{1j}}^{\nu_{2j}} \theta^2(2^{-j}\nu/N)\cos(2\pi\Delta k 2^{-j}\nu/N),$$

$$S_2 = \sum_{\nu=\nu_{2j}+1}^{\nu_{3j}-1} \theta^2(2^{-j}\nu/N)\cos(2\pi\Delta k 2^{-j}\nu/N),$$

$$S_3 = \sum_{\nu=\nu_{3j}}^{\nu_{4j}} \theta^2(2^{-j}\nu/N)\cos(2\pi\Delta k 2^{-j}\nu/N),$$

and the natural numbers $\nu_{1j}, \nu_{2j}, \nu_{3j}, \nu_{4j}$ satisfy

$$0 \le 2^{-j}(\nu_{1j} - 1)/N < 1/2 - \varepsilon \le 2^{-j}\nu_{1j}/N,$$
$$2^{-j}\nu_{2j}/N \le 1/2 + \varepsilon < 2^{-j}(\nu_{2j} + 1)/N,$$
$$2^{-j}(\nu_{3j} - 1)/N < 1 - 2\varepsilon \le 2^{-j}\nu_{3j}/N,$$
$$2^{-j}\nu_{4j}/N \le 1 + 2\varepsilon < 2^{-j}(\nu_{4j} + 1)/N.$$

Observe that $\nu_{4j} < N/2$ since $(1+2\varepsilon)2^j N < 2^j 4N/3 \le N/3$ for $0 < \varepsilon \le 1/6$ and $j = -p+2, -p+3, \ldots, -2$, and then we can also choose natural numbers ν_{0j} and ν'_{0j} such that $2^{-j}\nu_{0j}/N = 1/2$ and $2^{-j}\nu'_{0j}/N = 1$. Now, since θ satisfies (5) we have $\theta^2(t) + \theta^2(1-t) = 1$, $\theta(2t) = \theta(1-t)$ for $1/2 - \varepsilon \le t \le 1/2 + \varepsilon$ and $\theta(t) = 1$ for $1/2 + \varepsilon \le t \le 1 - 2\varepsilon$, and we can write

$$S_1 = 1/2 + \sum_{\nu=\nu_{0j}+1}^{\nu_{2j}} \cos(2\pi\Delta k 2^{-j}\nu/N) \qquad \text{for } \Delta k = 2l,$$

$$S_1 = -1/2 + \sum_{\nu=\nu_{0j}+1}^{\nu_{2j}} \cos(2\pi\Delta k 2^{-j}\nu/N) \qquad \text{for } \Delta k = 2l+1,$$

$$S_2 = \sum_{\nu=\nu_{2j}+1}^{\nu_{3j}-1} \cos(2\pi\Delta k 2^{-j}\nu/N),$$

$$S_3 = 1/2 + \sum_{\nu=\nu_{3j}}^{\nu'_{0j}-1} \cos(2\pi\Delta k 2^{-j}\nu/N).$$

Summing up we obtain

$$S_1 + S_2 + S_3 = 1 + \sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \cos(2\pi\Delta k 2^{-j}\nu/N) \qquad \text{for } \Delta k = 2l,$$

$$S_1 + S_2 + S_3 = \sum_{\nu=\nu_{0j}+1}^{\nu'_{0j}-1} \cos(2\pi \Delta k 2^{-j}\nu/N) \quad \text{for } \Delta k = 2l+1$$

and further since $\Delta k = k' - k$ satisfies $0 \le |\Delta k| < 2^j N$ we have, by Lemma 4,

$$\langle w_{jk}, w_{jk'} \rangle = \frac{2^{-j+1}}{N}(S_1 + S_2 + S_3) = \delta_{0,\Delta k}$$

for $j = -p+2, -p+3, \ldots, -2$ and $k, k' = 0, 1, \ldots, 2^j N - 1$.

In the case $j = -1$ we have $\theta_{-1} = \theta'$ and similarly to the previous case we write

$$\langle w_{-1,k}, w_{-1,k'} \rangle = \frac{4}{N}(S'_1 + S'_2)$$

for $k, k' = 0, 1, \ldots, N/2 - 1$, where

$$S'_1 = \sum_{\nu=\nu'_1}^{\nu'_2} \theta'^2(2\nu/N) \cos(2\pi \Delta k 2\nu/N),$$

$$S'_2 = \sum_{\nu=\nu'_2+1}^{\nu'_3} \theta'^2(2\nu/N) \cos(2\pi \Delta k 2\nu/N),$$

and the natural numbers ν'_1, ν'_2, ν'_3 satisfy

$$0 \le 2(\nu'_1 - 1)/N < 1/2 - \varepsilon \le 2\nu'_1/N,$$
$$2\nu'_2/N \le 1/2 + \varepsilon < 2(\nu'_2 + 1)/N, \qquad 2\nu'_3/N = 1.$$

We can also choose natural numbers ν_0 and ν'_0 such that $2\nu_0/N = 1/2$ and $2\nu'_0/N = 1$. The conditions (5) and (6) imply that $\theta'^2(t) + \theta'^2(1-t) = 1$ for $1/2 - \varepsilon \le t \le 1/2 + \varepsilon$, $\theta'(t) = 1$ for $1/2 + \varepsilon \le t < 1$ and also $\theta'^2(1) = 1/2$ so the sums S'_1, S'_2 can be rewritten in the form

$$S'_1 = 1/2 + \sum_{\nu=\nu_0+1}^{\nu'_2} \cos(2\pi \Delta k 2\nu/N) \qquad \text{for } \Delta k = 2l,$$

$$S'_1 = -1/2 + \sum_{\nu=\nu_0+1}^{\nu'_2} \cos(2\pi \Delta k 2\nu/N) \quad \text{for } \Delta k = 2l+1,$$

$$S'_2 = 1/2 + \sum_{\nu=\nu'_2+1}^{\nu'_0-1} \cos(2\pi \Delta k 2\nu/N).$$

Summing up we obtain

$$S_1' + S_2' = 1 + \sum_{\nu=\nu_0+1}^{\nu_0'-1} \cos(2\pi\Delta k 2\nu/N) \quad \text{for } \Delta k = 2l,$$

$$S_1' + S_2' = \sum_{\nu=\nu_0+1}^{\nu_0'-1} \cos(2\pi\Delta k 2\nu/N) \quad \text{for } \Delta k = 2l+1$$

and further since $\Delta k = k' - k$ satisfies $0 \le |\Delta k| < N/2$ we have, by Lemma 4,

$$\langle w_{-1,k}, w_{-1,k'} \rangle = \frac{4}{N}(S_1' + S_2') = \delta_{0,\Delta k}$$

for $k, k' = 0, 1, \ldots, N/2 - 1$. This completes the proof of Theorem 1. ∎

The last part of the above proof clarifies why we have to use θ' instead of θ in the definition of the discrete wavelet functions for $j = -1$.

The next lemma concerns the representation of discrete signals from the space l_N^2 with the use of the orthonormal system of discrete wavelet functions.

LEMMA 5. *Any complex-valued discrete signal $s(n)$, $n = 0, 1, \ldots, N-1$, where $N = 2^p$, $p > 0$, can be represented in the form*

$$s(n) = s_{00} + \sum_{j=-p}^{-1} \sum_{k=0}^{2^j N - 1} s_{jk} w_{jk}(n), \qquad n = 0, 1, \ldots, N-1, \tag{10}$$

where $w_{jk}(n)$, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, are the discrete wavelet functions and the coefficients s_{jk} are uniquely determined by the formulae

$$s_{00} = \frac{1}{N} \sum_{n=0}^{N-1} s(n),$$

$$s_{jk} = \sum_{n=0}^{N-1} s(n) \overline{w}_{jk}(n) = \sum_{n=0}^{N-1} s(n) w_{jk}(n),$$

for $j = -p, -p+1, \ldots, -1$ and $k = 0, 1, \ldots, 2^j N - 1$. Moreover, for arbitrary complex-valued signals $u, v \in l_N^2$,

$$\langle u, v \rangle = \sum_{n=0}^{N-1} u(n) \overline{v}(n) = N u_{00} \overline{v}_{00} + \sum_{j=-p}^{-1} \sum_{k=0}^{2^j N - 1} u_{jk} \overline{v}_{jk}. \tag{11}$$

Proof. If we add to the set of discrete wavelet functions w_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, the constant function $w_{00}(n) = 1/\sqrt{N}$, $n = 0, 1, \ldots, N-1$, we obtain an orthonormal system in l_N^2, since according to Theorem 1 the discrete wavelet functions form an orthonormal system in that space and having zero mean values they are also orthogonal to the constant function w_{00}, which was proved in Lemma 1. This extended orthonormal system consists of $N/2 + N/4 + \ldots + N/N + 1 = N$ linearly independent elements in the N-dimensional space l_N^2 and since the discrete wavelet functions w_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, are real-valued, this immediately implies the validity of the decomposition (10) together with the formulae for the coefficients s_{jk}. The formula (11) follows from (10) and from orthonormality of the extended system. ∎

The coefficients s_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, occurring in (10) can be obtained without computing the values of the discrete wavelet functions since Lemma 2 yields a formula analogous to (3):

$$(12)\quad s_{jk} = \langle s, w_{jk} \rangle = \frac{2^{-j/2}}{N} \sum_{\nu=-N/2+1}^{N/2} \widetilde{s}(\nu) \overline{\widetilde{g}}_j (2^{-j}\nu/N) \exp(i2\pi k 2^{-j}\nu/N).$$

If the coefficients s_{00}, s_{jk}, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, are known, then the values of the signal $s(n)$, $n = 0, 1, \ldots, N-1$, can be obtained using the DFT according to the formula

$$(13)\qquad s(n) = s_{00} + \frac{1}{N} \sum_{\nu=-N/2+1}^{N/2} \exp(i2\pi n\nu/N) \sum_{j=-p}^{-1} \sum_{k=0}^{2^j N-1} s_{jk} \widetilde{w}_{jk}(\nu),$$

which follows from (10) and from the definition of discrete wavelet functions. The above formula is analogous to the formula (4) for functions of a real variable.

The formulae (12) and (13) define the direct and inverse discrete wavelet transform (DWT) of finite duration discrete signals $s(n)$, $n = 0, 1, \ldots, N-1$, where $N = 2^p$, $p > 0$.

3. Time-frequency filtration of discrete signals. Let us now make two remarks about the discrete wavelet functions defined in the previous section. If we define translation operators T_m, $m = 0, 1, \ldots, N-1$, for a discrete signal $s(n)$, $n = 0, 1, \ldots, N-1$, as in [3], i.e. $(T_m s)(n) = s(n - m \pmod N)$, $n = 0, 1, \ldots, N-1$, then as one can easily verify [3] for $v = T_m s$ we have $\widetilde{v}(\nu) = \exp(-i2\pi m\nu/N)\widetilde{s}(\nu)$ for $\nu = -N/2+1, -N/2+2, \ldots, N/2$. Thus for fixed j the discrete wavelet functions w_{jk}, $k = 0, 1, \ldots, 2^j N - 1$, can be treated as translations T_{m_k}, $m_k = k2^{-j}$, of the discrete functions w_{j0}, which become more and more localized in time as j grows. The DWT is thus sensitive to time-shift of the signal being analyzed. A translated version of a signal leads to a different time distribution of the wavelet coefficients in the spectro-temporal plane. Moreover, as already remarked in the proof of Theorem 1, the supports of $\widetilde{w}_{jk}$ and $\widetilde{w}_{j'k'}$ are disjoint if $|j - j'| \geq 2$. These supports become more and more localized in the frequency domain as j decreases. In Lemma 6 we prove that the supports of $\widetilde{w}_{jk}$, $j = -p, -p+1, \ldots, -1$, $k = 0, 1, \ldots, 2^j N - 1$, cover the discrete frequency spectrum $\nu = -N/2+1, -N/2+2, \ldots, N/2$.

LEMMA 6. *For each ν satisfying $-N/2+1 \le \nu \le N/2$, where $N = 2^p$, $p > 0$, there exists an integer j such that $-p \le j \le -1$ and $\widehat{g}_j(2^{-j}\nu/N) \ne 0$.*

Proof. According to Definition 1, $\widehat{g}_j(t) = \theta(t)\exp(-i\pi t)$ for $j = -p, -p+1, \ldots, -2$ and $\widehat{g}_{-1}(t) = \theta'(t)\exp(-i\pi t)$. From (5), (6) and $\varepsilon > 0$ it follows that $\theta(t) > 0$ and $\theta'(t) > 0$ for $1/2 \le |t| \le 1$. Consequently, $\widehat{g}_j(2^{-j}\nu/N)$, $j = -p, -p+1, \ldots, -1$, is different from zero for $2^j N/2 \le |\nu| \le 2^j N$. Since $\bigcup_{j=-p}^{-1}[2^j N/2, 2^j N] = [1/2, N/2]$, this proves the lemma. ■

The wavelet coefficients s_{jk} of a discrete signal s are computed as scalar products in the time domain: $s_{jk} = \langle s, w_{jk}\rangle$, or equivalently as scaled scalar products in the frequency domain: $\langle \widetilde{s}, \widetilde{w}_{jk}\rangle = N\langle s, w_{jk}\rangle$. Thus in view of the above indicated properties of w_{jk} those coefficients contain information simultaneously about the time and frequency components of the analyzed signal. The information about the signal contained in its N time samples is transformed into N wavelet coefficients, which describe the signal in the time-frequency domain. This suggests that it is possible to use the above defined DWT to perform time-frequency filtering of discrete signals. This can be done by multiplying the wavelet coefficients s_{jk} of the signal by real coefficients a_{jk} and then applying the inverse DWT to the set of products $s_{jk}a_{jk}$ to obtain a filtered signal s'. The coefficients a_{jk} can be set to zero for w_{jk} concentrated outside a specific time-frequency region, and left without change for functions concentrated within this region. The real coefficients a_{jk} with values 0 or 1 define the time-frequency transfer function of the filter, analogous to standard transfer functions of digital filters applied in the frequency domain [5].

It should be remarked that this filtering technique is defined by means of multiplication in the time-frequency domain and has no equivalent in the time domain like convolution of signals in the case of filtering based on the DFT. However, the same precautions as in the case of filtering using the DFT should be taken when using the DWT. For instance, in order to avoid edge effects, signals should be tapered by a time window [5], before transforming them by the DWT. Although the described DWT is defined for complex-valued discrete signals it is in general appropriate only for analysis of real-valued signals. This can be seen if we examine the effect of transforming the signal which is the sum of prograde and retrograde oscillations $s(n) = a\exp(i2\pi pn) + b\exp(-i2\pi pn)$, $n = 0, 1, \ldots, N-1$, where $a \ne b$, $p > 0$ are real numbers. Analyzing the wavelet coefficients s_{jk} of such a signal we can detect the oscillation $s(n) = (a+b)\cos(2\pi pn) + i(a-b)\sin(2\pi pn)$, $n =$

$0, 1, \ldots, N-1$, but because of hermitian symmetry of the discrete functions $\widetilde{w}_{jk}$ the DWT does not reveal that there are two frequency components in the signal with different amplitudes and periods, which would be detected by the DFT.

3. Conclusions. In order to perform the presented DWT a particular function θ satisfying the conditions (5) must be chosen. It is enough to define a real function β on the interval $[-\varepsilon, \varepsilon]$ such that

$$\beta(t) = \begin{cases} \gamma(t) & \text{for } 0 \le t \le \varepsilon, \\ \sqrt{1-\gamma^2(-t))} & \text{for } -\varepsilon \le t \le 0, \end{cases}$$

where γ can be a polynomial, trigonometric or exponential function satisfying $\gamma(0) = 1/\sqrt{2}$ to assure continuity of β. The function θ may be constructed by symmetry, translation and dilation of β. For instance, $\gamma(t) = 1 - \exp(\alpha\varepsilon^2/(t-\varepsilon)^2)$ with $\alpha = \ln(1 - 1/\sqrt{2})$ can be used. The choice of γ slightly influences the form of the discrete wavelets.

The discrete values of θ involved in (12) and (13) can be tabulated. It suffices to compute the values needed by the highest frequency discrete wavelet spectrum ($j = -1$) with step $2/N$. Since $\theta(-t) = \theta(t)$ and $\theta(t) = 0$ for $t \notin [1/2 - \varepsilon, 1 + 2\varepsilon]$, only the $[(1/2 + 3\varepsilon)N/2]$ values $\theta(2m/N)$ for $(1/2-\varepsilon)N/2 < m < (1+2\varepsilon)N/2$ must be computed. Then for a given j the discrete values $\theta(2^{-j}\nu/N)$, $\nu = 0, 1, \ldots, N/2$, involved in (12) and (13) can be picked in a vector storing the values $\theta(2m/N)$, $m = 0, 1, \ldots, [(1+2\varepsilon)N/2]$, with step 2^{-j-1}. It is worth remarking that since the number of signal values must satisfy the condition $N = 2^p$, $p > 0$, it is possible to implement fast DWT computer programs basing on formulae (12), (13) and widely applied FFT algorithms [9].

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