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ON FUNCTIONAL MEASURES OF SKEWNESS

Abstract. We introduce a concept of functional measures of skewness which can be used in a wider context than some classical measures of asymmetry. The Hotelling and Solomons theorem is generalized.

1. Introduction. It was shown in [1] that the Pearson coefficient s of skewness:

$$s = (\text{mean} - \text{median})/(\text{standard deviation}) \tag{1.1}$$

necessarily lies between -1 and 1. A neat proof of that fact and an extension of the statement that the mean is within one standard deviation of any median can be found in [2]. Namely, it was proved that

$$|\mu - x_q| \le \sigma \max(\sqrt{(1-q)/q}, \sqrt{q/(1-q)}), \tag{1.2}$$

where μ denotes the mean and x_q the qth quantile of a random variable X. More details and references on this subject can be found in [3].

The goal of this note is to discuss measures of skewness of the type (1.1) for conditional distributions and to extend (1.1) to a class of random variables with infinite mean values. We are also interested in conditional versions of (1.2).

2. Measures of skewness of conditional distributions. We consider here only pairs (X, Y) of random variables with continuous strictly increasing marginal and conditional distribution functions.

For any given $p \in (0, 1)$, y_p stands for the pth quantile of F_Y. The qth quantiles of the distribution functions $P[X < x \mid Y > y_p]$, $P[X < x \mid Y < y_p]$

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are denoted by $x^{(1)}_{q|p}$ and $x^{(2)}_{q|p}$, respectively, i.e. we have

$$P[X < x^{(1)}_{q|p} \mid Y > y_p] \le q \le P[X \le x^{(1)}_{q|p} \mid Y > y_p],$$
$$P[X < x^{(2)}_{q|p} \mid Y < y_p] \le q \le P[X \le x^{(2)}_{q|p} \mid Y < y_p].$$

Moreover, we write

$$\mu^{(1)}_{X|Y}(p) := E[X \mid Y > y_p], \quad \mu^{(2)}_{X|Y}(p) := E[X \mid Y < y_p],$$
$$\sigma^{(k)}_{X|Y}(p) := \sqrt{\mathrm{Var}^{(k)}_{X|Y}(p)}, \quad k = 1, 2,$$
$$\mathrm{Var}^{(1)}_{X|Y}(p) := E[X^2 \mid Y > y_p] - (E[X \mid Y > y_p])^2,$$
$$\mathrm{Var}^{(2)}_{X|Y}(p) := E[X^2 \mid Y < y_p] - (E[X \mid Y < y_p])^2.$$

We introduce the following notions.

Definition 1. The quantities

$$s^{(k)}_{X|Y}(p) = (\mu^{(k)}_{X|Y}(p) - x^{(k)}_{1/2|p})/\sigma^{(k)}_{X|Y}(p), \quad p \in (0,1),\ k = 1, 2, \tag{2.1}$$

(if they exist) define the *functional measures of skewness* of conditional distribution functions for a pair (X, Y) of random variables.

We note that $s^{(1)}_{X|Y}(\cdot)$ defines a functional measure of skewness of the conditional distribution function of X under the condition that values of Y cross the pth quantile y_p. Similarly one can interpret $s^{(2)}_{X|Y}(\cdot)$. If X and Y are independent then (2.1) reduces to (1.1). Moreover, it is not difficult to see that the limit values (if they exist) of $s^{(k)}_{X|Y}(p)$ $k = 1, 2$, as $p \to 0$ and $p \to 1$, respectively, are $s^{(1)}_{X|Y}(0) = s$ and $s^{(2)}_{X|Y}(1) = s$.

Following the above idea we can introduce a concept of a functional measure of skewness which is a generalization of (1.1).

Put

$$m^{(1)}_X(p) := \mathrm{median}(P[X < x \mid X > x_p]),$$
$$m^{(2)}_X(p) := \mathrm{median}(P[X < x \mid X < x_p]),$$
$$\mu^{(1)}_X(p) := E[X \mid X > x_p], \quad \mu^{(2)}_X(p) := E[X \mid X < x_p],$$
$$\sigma^{(k)}_X(p) := \sqrt{\mathrm{Var}^{(k)}_X(p)}, \quad k = 1, 2,$$
$$\mathrm{Var}^{(1)}_X(p) := E[X^2 \mid X > x_p] - E^2[X \mid X > x_p],$$
$$\mathrm{Var}^{(2)}_X(p) := E[X^2 \mid X < x_p] - E^2[X \mid X < x_p].$$

DEFINITION 2. The quantities

$$s_X^{(k)}(p) = (\mu_X^{(k)}(p) - m_X^{(k)}(p))/\sigma_X^{(k)}(p), \qquad p \in (0,1),\ k = 1,2, \tag{2.2}$$

(if they exist) are called the *functional measures of skewness* of a random variable X (or of its probability distribution function).

DEFINITION 3. The *measures of skewness* $s_X^{(k)}$ of any probability distribution function are defined by

$$s_X^{(1)} = \lim_{p\to 0} s_X^{(1)}(p), \qquad s_X^{(2)} = \lim_{p\to 1} s_X^{(2)}(p), \tag{2.3}$$

provided that at least one of the above limits exists.

One can see that in the case when $EX^2 < \infty$, we have $s_X^{(k)} = s$, $k = 1,2$, with s defined by (1.1).

The following examples present applications of the introduced measures of skewness.

EXAMPLE 1. Let $F(x) = 1 - 1/x^3$, $x \ge 1$, and 0 otherwise. Then

$$EX = 3/2, \quad \sigma^2 X = 3/4, \quad x_p = 1/\sqrt[3]{1-p},$$
$$m_X^{(1)}(p) = \sqrt[3]{2/(1-p)}, \qquad m_X^{(2)}(p) = \sqrt[3]{2/(2-p)},$$
$$\mu_X^{(1)}(p) = 3\sqrt[3]{1/(1-p)}/2, \qquad \mu_X^{(2)}(p) = 3(1 - \sqrt[3]{(1-p)^2})/(2p),$$
$$(\sigma_X^{(1)}(p))^2 = 3\sqrt[3]{1/(1-p)^2}/4,$$
$$(\sigma_X^{(2)}(p))^2 = 3(4p - 3 - (p+3)\sqrt[3]{1-p} + 6\sqrt[3]{(1-p)^2})/(4p^2).$$

Hence the coefficient of skewness (1.1) is $s = \sqrt{3} - 2\sqrt[3]{2}/\sqrt{3}$, while the functional coefficients are

$$s_X^{(1)}(p) = \sqrt{3} - 2\sqrt[3]{2}/\sqrt{3},$$
$$s_X^{(2)}(p) = \frac{\sqrt{3}(1 - \sqrt[3]{(1-p)^2}) - 2p\sqrt[3]{2/(2-p)}/\sqrt{3}}{\sqrt{4p - 3 - (p+3)\sqrt[3]{1-p} + 6\sqrt[3]{(1-p)^2}}}.$$

Moreover, $\lim_{p\to 0} s_X^{(1)}(p) = \lim_{p\to 1} s_X^{(2)}(p) = \sqrt{3} - 2\sqrt[3]{2}/\sqrt{3} = s$.

EXAMPLE 2. Let $F(x) = 1 - 1/x$, $x \ge 1$. We see that $EX = \infty$ and the classical measure of skewness (1.1) is undefined. Moreover,

$$x_p = 1/(1-p), \quad m_X^{(1)}(p) = 2/(1-p), \quad m_X^{(2)}(p) = 2/(2-p),$$
$$\mu_X^{(1)}(p) = \infty, \quad \mu_X^{(2)}(p) = -p^{-1}\ln(1-p), \quad (\sigma_X^{(2)})^2 = 1/(1-p) - p^{-2}\ln^2(1-p).$$

Hence $s_X^{(1)}(p)$ is undefined but

$$s_X^{(2)}(p) = \frac{-p^{-1}\ln(1-p) - 2/(2-p)}{\sqrt{1/(1-p) - p^{-2}\ln^2(1-p)}},$$

and

$$s = \lim_{p\to 1} s_X^{(2)}(p) = 0.$$

Now we give examples elucidating the quantities (2.1) (the conditional measures of skewness).

EXAMPLE 3. Let $F(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+xy)}$, $x, y \ge 0$, and 0 otherwise. Then

$$y_p = -\ln(1-p), \quad x_{q|p}^{(1)} = (\ln(1-q))/(\ln(1-p)-1),$$
$$\mu_{X|Y}^{(1)}(p) = 1/(1-\ln(1-p)), \quad (\sigma_{X|Y}^{(1)}(p))^2 = 1/(1-\ln(1-p))^2,$$

which gives $s_{X|Y}^{(1)}(p) = 1 - \ln 2$, $p \in (0,1)$, proving that the functional measure of skewness of $P[X < x \mid Y > y_p]$ is a constant function.

The quantity $s_{X|Y}^{(2)}(p)$ can be determined only by an approximation.

EXAMPLE 4. Let $F(x,y) = 1 - 1/x - 1/y + 1/(x^y y)$, $x, y \ge 1$, and 0 otherwise. Then $EX = \infty$. Moreover,

$$y_p = 1/(1-p), \quad x_{q|p}^{(1)} = (1-q)^{p-1}, \quad \mu_{X|Y}^{(1)}(p) = 1/p,$$
$$(\sigma_{X|Y}^{(1)}(p))^2 = \begin{cases} \infty, & 0 < p \le 1/2, \\ (1-p)^2/(p^2(2p-1)), & 1/2 < p < 1. \end{cases}$$

Hence we get

$$s_{X|Y}^{(1)}(p) = \begin{cases} 0, & 0 < p \le 1/2, \\ \dfrac{1 - p2^{1-p}}{1-p}\sqrt{2p-1}, & 1/2 < p < 1. \end{cases}$$

The characteristic $s_{X|Y}^{(2)}(p)$ can be given by an approximation.

3. Properties of functional measures of skewness. The following generalization of the estimate derived in [2] (cf. (1.2)) gives bounds for functional measures of skewness.

THEOREM. *Under the notations of Section 2 we have*:

$$\text{(3.1)} \quad \begin{aligned} &\text{(i)} \quad |\mu_{X|Y}^{(k)}(p) - x_{q|p}^{(k)}| \le \sigma_{X|Y}^{(k)}(p) M(q), \quad p \in (0,1),\ k = 1,2, \\ &\text{(ii)} \quad |\mu_X^{(k)}(p) - \widetilde{x}_{q|p}^{(k)}| \le \sigma_X^{(k)}(p) M(q), \quad p \in (0,1),\ k = 1,2, \end{aligned}$$

where $M(q) = \max\{\sqrt{q/(1-q)}, \sqrt{(1-q)/q}\}$, *and* $\widetilde{x}_{q|p}^{(1)}$ *and* $\widetilde{x}_{q|p}^{(2)}$ *denote the* q*th quantiles of* $P[X < x \mid X > x_p]$ *and* $P[X < x \mid X < x_p]$, *respectively.*

Proof. We need only prove (i) with $k = 1$ as the remaining cases can be shown similarly.

Note that the distribution function $P[X < x \mid Y > y_p]$ can be written as a mixture of distribution functions as follows:

$$P[X < x \mid Y > y_p] = qP_1[X < x \mid Y > y_p] + (1-q)P_2[X < x \mid Y > y_p], \tag{3.2}$$

where

$$P_1[X < x \mid Y > y_p] = \begin{cases} \frac{1}{q}P[X < x \mid Y > y_p], & x \le x^{(1)}_{q|p}, \\ 1, & x > x^{(1)}_{q|p}, \end{cases} \tag{3.3}$$

and

$$P_2[X < x \mid Y > y_p] = \begin{cases} 0, & x \le x^{(1)}_{q|p}, \\ \frac{1}{1-q}P[X < x \mid Y > y_p] - \frac{q}{1-q}, & x > x^{(1)}_{q|p}. \end{cases} \tag{3.4}$$

From (3.2) we have

$$\mu^{(1)}_{X|Y}(p) = q\mu_1(p) + (1-q)\mu_2(p), \tag{3.5}$$

where

$$\mu_i(p) = \int x\, dP_i[X < x \mid Y > y_p], \qquad i = 1, 2.$$

Moreover, (3.3) and (3.4) imply

$$\mu_1(p) \le x^{(1)}_{q|p} \tag{3.6}$$

and

$$\mu_2(p) \ge x^{(1)}_{q|p}, \tag{3.7}$$

respectively.

Now by (3.5)–(3.7) we conclude that

$$\mu^{(1)}_{X|Y}(p) - x^{(1)}_{q|p} \le (1-q)(\mu_2(p) - \mu_1(p))$$

and

$$x^{(1)}_{q|p} - \mu^{(1)}_{X|Y}(p) \le q(\mu_2(p) - \mu_1(p)).$$

Hence

$$(\mu^{(1)}_{X|Y}(p) - x^{(1)}_{q|p})^2 \le \max\{q^2, (1-q)^2\}(\mu_2(p) - \mu_1(p))^2. \tag{3.8}$$

Then using (3.2) and (3.5) we see that

$$\begin{aligned}(\sigma^{(1)}_{X|Y}(p))^2 &= q \int (x - q\mu_1(p) - (1-q)\mu_2(p))^2\, dP_1[X < x \mid Y > y_p] \\ &\quad + (1-q) \int (x - q\mu_1(p) - (1-q)\mu_2(p))^2\, dP_2[X<x \mid Y>y_p] \\ &= q \int (x - \mu_1(p))^2\, dP_1[X < x \mid Y > y_p]\end{aligned}$$

$$\begin{aligned}
&+ q \int (x - \mu_2(p))^2 \, dP_2[X < x \mid Y > y_p] \\
&+ q(1-q)^2(\mu_2(p) - \mu_1(p))^2 + q^2(1-q)(\mu_2(p) - \mu_1(p))^2 \\
&\ge q(1-q)(\mu_2(p) - \mu_1(p))^2.
\end{aligned}$$

Hence after using (3.8) we get

$$(\sigma^{(1)}_{X|Y}(p))^2 \ge \frac{q(1-q)}{\max\{q^2, (1-q)^2\}} (\mu^{(1)}_{X|Y}(p) - x^{(1)}_{q|p})^2.$$

Corollary. *The limits of functional measures of skewness are as follows*:

(i) $$|\mu^{(k)}_{X|Y}(p) - x^{(k)}_{1/2|q}| \le \sigma^{(k)}_{X|Y}(p), \qquad p \in (0,1),\ k = 1, 2,$$

(ii) $$|\mu^{(k)}_{X}(p) - m^{(k)}_{X}(p)| \le \sigma^{(k)}_{X}(p), \qquad p \in (0,1),\ k = 1, 2.$$

4. Examples. We now give examples of functional measures of skewness using conditional distribution functions of order statistics.

Example. Let U and V be independent random variables with a common strictly monotone distribution function. We consider two cases:

(i) $X = U$, $Y = \max(U, V)$,
(ii) $X = U$, $Y = \min(U, V)$,

and put $F_X = F$.

In the case (i) we have $y_p = x_{\sqrt{p}}$ and

$$P[X < x \mid Y > y_p] = \begin{cases} P[X < x]/(1 + \sqrt{p}), & x \le x_{\sqrt{p}}, \\ (P[X < x] - p)/(1 - p), & x > x_{\sqrt{p}}, \end{cases}$$

$$P[X < x \mid Y < y_p] = \begin{cases} P[X < x]/\sqrt{p}, & x \le x_{\sqrt{p}}, \\ 1, & x > x_{\sqrt{p}}, \end{cases}$$

$$x^{(1)}_{q|p} = \begin{cases} x_{q(1+\sqrt{p})}, & q < \sqrt{p}/(1 + \sqrt{p}), \\ x_{q(1-p)+p}, & q \ge \sqrt{p}/(1 + \sqrt{p}), \end{cases}$$

$$x^{(2)}_{q|p} = x_{q\sqrt{p}}, \quad x^{(1)}_{1/2|p} = x_{(1+p)/2}, \quad x^{(2)}_{1/2|p} = x_{\sqrt{p}/2},$$

$$\mu^{(1)}_{X|Y}(p) = (1 + \sqrt{p})^{-1} EX\mathrm{I}[X < x_{\sqrt{p}}] + (1 - p)^{-1} EX\mathrm{I}[X > x_{\sqrt{p}}],$$

$$E[X^2 \mid Y > y_p] = (1 + \sqrt{p})^{-1} EX^2\mathrm{I}[X < x_{\sqrt{p}}] + (1 - p)^{-1} EX^2\mathrm{I}[X > x_{\sqrt{p}}],$$

$$\mu^{(2)}_{X|Y}(p) = p^{-1/2} EX\mathrm{I}[X < x_{\sqrt{p}}],$$

$$E[X^2 \mid Y < y_p] = p^{-1/2} EX^2\mathrm{I}[X < x_{\sqrt{p}}].$$

From this one gets

$$\begin{aligned}
s^{(1)}_{X|Y}(p) = {} & \{(1+\sqrt{p})^{-1} EX\mathrm{I}[X < x_{\sqrt{p}}] \\
& + (1-p)^{-1} EX\mathrm{I}[X < x_{\sqrt{p}}] - x_{(1+p)/2}\} \\
& \times \{(1+\sqrt{p})^{-1} EX^2\mathrm{I}[X < x_{\sqrt{p}}] + (1-p)^{-1} EX^2\mathrm{I}[X < x_{\sqrt{p}}] \\
& - ((1+\sqrt{p})^{-1} EX\mathrm{I}[X < x_{\sqrt{p}}] + (1-p)^{-1} EX\mathrm{I}[X > x_{\sqrt{p}}])^2\}^{-1/2},
\end{aligned}$$

$$s^{(2)}_{X|Y}(p) = \frac{p^{-1/2} EX\mathrm{I}[X < x_{\sqrt{p}}] - x_{\sqrt{p}/2}}{\{p^{-1/2} EX^2\mathrm{I}[X < x_{\sqrt{p}}] - p^{-1} E^2 X\mathrm{I}[X < x_{\sqrt{p}}]\}^{1/2}}.$$

Now we give the values of $s^{(1)}_{X|Y}(p)$ and $s^{(2)}_{X|Y}(p)$ for exponential, uniform, and Pareto distributions.

(a) Let $F(x) = 1 - e^{-x}$, $x \geq 0$, and zero otherwise. Then

$$s^{(1)}_{X|Y}(p) = \frac{(1+\sqrt{p})(1-\ln 2) + \ln(1-p)^{1+\sqrt{p}}(1-\sqrt{p})^{-\sqrt{p}}}{\sqrt{(1+\sqrt{p})^2 + \sqrt{p}\ln^2(1-\sqrt{p})}},$$

$$s^{(2)}_{X|Y}(p) = \frac{\sqrt{p}(1-\ln 2) + \ln(2-\sqrt{p})^{\sqrt{p}}(1-\sqrt{p})^{1-\sqrt{p}}}{\sqrt{p - (1-\sqrt{p})\ln^2(1-\sqrt{p})}},$$

$$\lim_{p\to 0} s^{(1)}_{X|Y} = 1 - \ln 2, \qquad \lim_{p\to 1} s^{(2)}_{X|Y} = 1 - \ln 2.$$

(b) Let $F(x) = x$, $x \in [0,1]$, and zero otherwise. Then

$$s^{(1)}_{X|Y}(p) = -\frac{\sqrt{3}p\sqrt{p}}{\sqrt{1-p+2\sqrt{p}(1+p)+p^2}}, \quad \lim_{p\to 0} s^{(1)}_{X|Y}(p) = 0, \quad s^{(2)}_{X|Y}(p) \equiv 0.$$

(c) Let $F(x) = 1 - 1/x$, $x \geq 1$, and zero otherwise. Then $s^{(1)}_{X|Y}(p)$ is undefined and

$$s^{(2)}_{X|Y}(p) = -\frac{\sqrt{1-\sqrt{p}}\,[2\sqrt{p} + (2-\sqrt{p})\ln(1-\sqrt{p})]}{(2-\sqrt{p})\sqrt{p-(1-\sqrt{p})\ln^2(1-\sqrt{p})}}, \qquad \lim_{p\to 1} s^{(2)}_{X|Y}(p) = 0.$$

In the case (ii) we have $y_p = x_{1-\sqrt{1-p}}$ and

$$P[X < x \mid Y > y_p] = \begin{cases} 0, & x \leq x_{1-\sqrt{1-p}}, \\ \dfrac{P[X<x] - (1-\sqrt{1-p})}{\sqrt{1-p}}, & x > x_{1-\sqrt{1-p}}, \end{cases}$$

$$P[X < x \mid Y < y_p] = \begin{cases} P[X<x]/p, & x \leq x_{1-\sqrt{1-p}}, \\ \dfrac{1-\sqrt{1-p}}{p}(P[X<x] + \sqrt{1-p}), & x > x_{1-\sqrt{1-p}}, \end{cases}$$

$$x^{(1)}_{q|p} = x_{1-(1-q)\sqrt{1-p}}, \qquad x^{(1)}_{1/2|p} = x_{1-\sqrt{1-p}/2},$$

$$x^{(2)}_{q|p} = \begin{cases} x_{qp}, & q < 1/(1+\sqrt{1-p}), \\ x_{q-(1-q)\sqrt{1-p}}, & q \geq 1/(1+\sqrt{1-p}), \end{cases} \qquad x^{(2)}_{1/2|p} = x_{p/2},$$

$$\mu^{(1)}_{X|Y}(p) = (1-p)^{-1/2} EX\mathrm{I}[X > x_{1-\sqrt{1-p}}],$$

$$E[X^2 \mid Y > y_p] = (1-p)^{-1/2} EX^2\mathrm{I}[X > x_{1-\sqrt{1-p}}],$$

$$\mu^{(2)}_{X|Y}(p) = p^{-1} EX\mathrm{I}[X < x_{1-\sqrt{1-p}}] + (1-\sqrt{1-p})p^{-1} EX\mathrm{I}[X > x_{1-\sqrt{1-p}}],$$

$$E[X^2 \mid Y < y_p] = p^{-1} EX^2\mathrm{I}[X < x_{1-\sqrt{1-p}}] + (1-\sqrt{1-p})p^{-1} EX^2\mathrm{I}[X > x_{1-\sqrt{1-p}}].$$

From this one gets

$$s^{(1)}_{X|Y}(p) = \frac{(1-p)^{-1/2} EX\mathrm{I}[X > x_{1-\sqrt{1-p}}] - x_{1-\sqrt{1-p}/2}}{\sqrt{(1-p)^{-1/2} EX^2\mathrm{I}[X > x_{1-\sqrt{1-p}}] - (1-p)^{-1} EX^2\mathrm{I}[X \geq x_{1-\sqrt{1-p}}]}},$$

$$\begin{aligned} s^{(2)}_{X|Y}(p) &= \{p^{-1} EX\mathrm{I}[X < x_{1-\sqrt{1-p}}] \\ &\quad + (1-\sqrt{1-p})p^{-1} EX\mathrm{I}[X > x_{1-\sqrt{1-p}}] - x_{p/2}\} \\ &\quad \times \{p^{-1} EX^2\mathrm{I}[X < x_{1-\sqrt{1-p}}] \\ &\quad + (1-\sqrt{1-p})p^{-1} EX^2\mathrm{I}[X > x_{1-\sqrt{1-p}}] - (p^{-1} EX\mathrm{I}[X < x_{1-\sqrt{1-p}}] \\ &\quad + (1-\sqrt{1-p})p^{-1} EX\mathrm{I}[X > x_{1-\sqrt{1-p}}])^2\}^{-1/2}. \end{aligned}$$

Now we see that in the case (a),

$$s^{(1)}_{X|Y}(p) = 1 - \ln 2,$$

$$s^{(2)}_{X|Y}(p) = \frac{p(1-\ln 2) + \ln(2-p)^p (1-p)^{(1-p)/2}}{\sqrt{p^2 - (1-p)\ln^2\sqrt{1-p}}}, \qquad \lim_{p\to 1} s^{(2)}_{X|Y}(p) = 1 - \ln 2;$$

in the case (b) we have

$$s^{(1)}_{X|Y}(p) \equiv 0,$$

$$s^{(2)}_{X|Y}(p) = \frac{\sqrt{3}(1-p)^{3/2}}{\sqrt{1+2\sqrt{1-p}(2-p) - p(1-p)}}, \qquad \lim_{p\to 1} s^{(2)}_{X|Y}(p) = 0,$$

while in the case (c) both quantities $s^{(1)}_{X|Y}(p)$ and $s^{(2)}_{X|Y}(p)$ are undefined.

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