APPLICATIONES MATHEMATICAE
23,4 (1996), pp. 371–378

D. D. BAINOV (Sofia)
M. B. DIMITROVA (Sliven)

SUFFICIENT CONDITIONS FOR OSCILLATION AND NONOSCILLATION OF THE SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

Abstract. Effective sufficient conditions for oscillation and nonoscillation of solutions of some operator-differential equations with piecewise constant argument are found.

1. Introduction. In [1] sufficient conditions are obtained for oscillation of all solutions of the operator-differential equation with piecewise constant argument

$$y'(t) + q(t)y(t) + p(t)y([t]) = 0, \tag{1}$$

where $p, q \in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t) = \infty$.

Some mathematical models in biology [3] are described by means of equations of the form (1).

In [5] sufficient conditions are obtained for oscillation and nonoscillation of solutions of the equations

$$y'(t) + p(t)f(y([t])) = 0,$$
$$y'(t) + p(t)f(y([t])) = h(t).$$

In the present paper the operator-differential equations with piecewise constant argument

$$x'(t) + p(t)(\mathcal{A}x)([t]) = 0, \tag{2}$$
$$x'(t) + p(t)(\mathcal{A}x)([t]) = h(t) \tag{3}$$

1991 *Mathematics Subject Classification*: Primary 34K15; Secondary 34L30.
Key words and phrases: oscillation and nonoscillation, operator-differential equations, piecewise constant argument.

are investigated, where $\mathcal{A}$ is an operator with certain properties. Sufficient conditions for oscillation and nonoscillation of solutions of equations (2) and (3) are obtained. Some particular realizations of the operator $\mathcal{A}$ are considered.

2. Preliminaries. Consider the operator-differential equations

$$x'(t) + p(t)(\mathcal{A}x)([t]) = 0,$$
$$x'(t) + p(t)(\mathcal{A}x)([t]) = h(t),$$

where $\mathcal{A}$ is an operator and $p(\cdot)$ is locally integrable function in $\mathbb{R}$. Let t_0 be a fixed real number. Denote by $C([t_0,\infty);\mathbb{R})$ the set of all continuous functions $u : [t_0,\infty) \to \mathbb{R}$, and by $L_{\text{loc}}([t_0,\infty);\mathbb{R})$ the set of all functions $u : [t_0,\infty) \to \mathbb{R}$ which are Lebesgue integrable in each compact subinterval of $[t_0,\infty)$.

Definition 1. By a *solution* of equation (3) in the interval $[t_0,\infty)$ we mean any function $x(t)$ satisfying the following conditions:

1. $x \in C([t_0,\infty);\mathbb{R})$.
2. The derivative $x'(t)$ exists at any point $t \geq t_0$ with the possible exception of the integer values of t, at which the right-hand derivative exists.
3. The function $x(t)$ satisfies equation (3) in each finite interval $[n, n+1) \subset [t_0,\infty)$, where $n \geq t_0$ and n is an integer.

The set of all functions satisfying conditions 1 and 2 of Definition 1 will be denoted by $\mathcal{D}_{t_0}$.

Definition 2. A solution $x(t)$ of the equation (3) is said to be *regular* if $\sup\{x(t) : t \geq T\} > 0$ for $T \geq N_x$, where $N_x \geq t_0$ is an integer.

Definition 3. A regular solution $x(t)$ of the equation (3) is said to *oscillate* if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of points such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n) = 0$.

Otherwise the regular solution $x(t)$ is said to be *nonoscillating*.

Definition 4. A function $u : [t_0,\infty) \to \mathbb{R}$ is said to *eventually enjoy a property P* if there exists a point $t_{P,u} \geq t_0$ such that for $t \geq t_{P,u}$ it enjoys the property P.

We introduce the following conditions:

H1. $p \in L_{\text{loc}}([t_0,\infty);\mathbb{R})$, $\text{meas}\{s \geq t : p(s) \neq 0\} > 0$.

H2. $\mathcal{A} : \mathcal{D}_{t_0} \to L_{\text{loc}}([t_0,\infty);\mathbb{R})$.

H3. If $u \in \mathcal{D}_{t_0}$ and $u(t) \equiv 0$ eventually, then $(\mathcal{A}u)(t) \equiv 0$ eventually.

H4. If $u \in \mathcal{D}_{t_0}$ is eventually nonzero and of constant sign, then so is $\mathcal{A}u$, and they are of the same sign.

3. Main results

THEOREM 1. *Let the following conditions hold*:

1. *Conditions* H1–H4 *are satisfied.*
2. $p(t) \le 0$ *for* $t \in [t_0, \infty)$.

Then all regular solutions of the equation (2) *are nonoscillating.*

Proof. Let $x(t)$ be a regular solution of (2) in $[N_x, \infty)$, where $N_x \ge t_0$ is an integer. Suppose that there exists an integer $n \ge N_x$ such that $x(n) = 0$. From (2) it follows that $x'(t) = -p(t)(\mathcal{A}x)(n)$ for $t \in [n, n+1)$. Then $x'(t) = 0$ for $t \in [n, n+1)$, i.e., $x(t) = \text{const}$ for $n \le t < n+1$. Hence if $x(n) = 0$ for any integer $n \ge N_x$, then by continuity of $x(t)$, $x(t) \equiv 0$ in $[n, \infty)$, which contradicts the requirement that $x(t)$ be a regular solution of (2). Hence there exists an integer $m \ge N_x$ such that $x(m) \ne 0$. Let $x(m) > 0$ (the case $x(m) < 0$ is analogous). Then

$$x'(t) = -p(t)(\mathcal{A}x)(m) \ge 0$$

for $t \in [m, m+1)$, and so $0 < x(m) \le x(t) \le x(m+1)$.

Analogously, we obtain $x(m+2) > 0$, etc. Hence $x(t) > 0$ for $t \ge m$. ■

THEOREM 2. *Let the following conditions hold*:

1. *Conditions* H1–H4 *are satisfied.*
2. $p(t) \ge 0$ *for* $t \ge t_0$ *and*

$$\limsup_{n\to\infty} \int_n^{n+1} p(t)\,dt < 1 \qquad \text{for } n \text{ integer}, \ n \ge t_0.$$

3. $(\mathcal{A}u)(t) \le u(t)$ *for any integer* $t \in [t_0, \infty)$ *and any* $u \in \mathcal{D}_{t_0}$.

Then all regular solutions of the equation (2) *are nonoscillating.*

Proof. Let $x(t)$ be a regular solution of (2) in $[N_x, \infty)$, where $N_x \ge t_0$ is an integer. There exists an integer $n_1 \ge N_x$ and a number ε, $0 < \varepsilon < 1$, such that for $n \ge n_1$,

$$\int_n^{n+1} p(t)\,dt < 1 - \varepsilon.$$

As in the proof of Theorem 1 we conclude that there exists an integer $n_2 \ge n_1$ such that $x(n_2) \ne 0$. Let $x(n_2) > 0$ (the case $x(n_2) < 0$ is analogous).

Integrate (2) from n_2 to t for $t \in [n_2, n_2+1)$ to obtain

$$x(t) = x(n_2) - \int_{n_2}^{t} p(s)(\mathcal{A}x)([s])\,ds$$

$$\geq x(n_2) - (\mathcal{A}x)(n_2) \int_{n_2}^{n_2+1} p(s)\,ds$$

$$\geq (\mathcal{A}x)(n_2)\Big[1 - \int_{n_2}^{n_2+1} p(s)\,ds\Big] > 0.$$

Repeating this process, we conclude that $x(t) > 0$ for $t \in [n_2 + 1, n_2 + 2)$, etc., i.e., $x(t) > 0$ for $t \geq n_2$. ∎

THEOREM 3. *Let the following conditions hold*:

1. *Conditions* H1–H4 *and condition* 3 *of Theorem* 2 *are satisfied.*
2. $p(t) \geq 0$ *for* $t \geq t_0$ *and*

$$\lim_{n\to\infty} \int_{n}^{n+1} p(t)\,dt = 0 \quad \textit{for } n \textit{ integer.}$$

Then each bounded solution of the equation (2) *is nonoscillating.*

The proof of Theorem 3 follows the scheme of the proof of Theorem 2.

THEOREM 4. *Let the following conditions hold*:

1. *Condition* H2 *is satisfied.*
2. $p \in C([t_0, \infty); \mathbb{R})$.
3. $h \in C([t_0, \infty); \mathbb{R})$.
4. $\lim_{t\to\infty} h(t)/p(t) = \infty$.
5. *If* $u \in \mathcal{D}_{t_0}$ *is eventually nonzero and bounded*, *then so is* $\mathcal{A}u$.

Then all bounded regular solutions of the equation (3) *are nonoscillating.*

Proof. Let $x(t)$ be a bounded regular solution of (3) in $[N_x, \infty)$, where $N_x \geq t_0$ is an integer, i.e., there exists a constant $M_1 > 0$ such that $|x(t)| \leq M_1$ for $t \geq N_x$. From condition 5 it follows that there exists a constant $M_2 > 0$ and a number $t_1 \geq N_x$ such that $|(\mathcal{A}x)(t)| \leq M_2$ for $t \geq t_1$. By condition 4, there exists $T \geq t_1$ such that $h(t) \geq M_2 p(t)$ for $t \geq T$.

Suppose that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of zeros of $x(t)$ such that $\lim_{n\to\infty} t_n = \infty$. Denote by t_k, t_{k+1} two consecutive zeros of $x(t)$ such that $T \leq t_k \leq t_{k+1}$.

Integrate (3) from t_k to t_{k+1} and obtain

$$0 = \int_{t_k}^{t_{k+1}} [h(s) - p(s)(\mathcal{A}x)([s])]\,ds \geq \int_{t_k}^{t_{k+1}} [h(s) - M_2 p(s)]\,ds > 0. \quad ∎$$

THEOREM 5. *Let the following conditions hold*:

1. *Conditions* H1, H2 *and* H4 *are satisfied.*
2. $\limsup_{n\to\infty} \int_n^{n+1} p(t)\,dt = \infty$.

3. *If* $u \in \mathcal{D}_{t_0}$, *then* $\lim_{n\to\infty} u(n)/(\mathcal{A}u)(n) < \infty$.

Then all regular solutions of the equation (2) *oscillate.*

Proof. Suppose that $x(t)$ is a nonoscillating solution of (2). Without loss of generality we can assume that $x(t) > 0$ in $[N_x, \infty)$, $N_x \ge t_0$, N_x is an integer. From H4 it follows that there exists an integer $N_{\mathcal{A}x} \ge N_x$ such that $(\mathcal{A}x)(t) > 0$ for $t \ge N_{\mathcal{A}x}$. Let N be an integer, $N \ge N_{\mathcal{A}x}$. Integrate (2) from N to $N+1$ and obtain

$$x(N+1) - x(N) = -\int_N^{N+1} p(t)(\mathcal{A}x)([t])\,dt = -(\mathcal{A}x)(N)\int_N^{N+1} p(t)\,dt.$$

But $-x(N) < x(N+1) - x(N)$. Hence $x(N) > (\mathcal{A}x)(N)\int_N^{N+1} p(t)\,dt$, i.e.,

$$\limsup_{N\to\infty} \int_N^{N+1} p(t)\,dt = \lim_{N\to\infty} \frac{x(N)}{(\mathcal{A}x)(N)} < \infty,$$

which contradicts condition 2. ∎

Theorem 6. *Let the following conditions hold*:

1. *Conditions* H1, H2 *and* H4 *are satisfied.*
2. $p(t) \ge 0$ *for* $t \ge t_0$.
3. $h \in L_{\text{loc}}([t_0, \infty); \mathbb{R})$ *and*

$$\liminf_{t\to\infty} \int_{t_0}^{t} h(s)\,ds = -\infty, \qquad \limsup_{t\to\infty} \int_{t_0}^{t} h(s)\,ds = \infty.$$

Then all regular solutions of the equation (3) *oscillate.*

Proof. Suppose that $x(t)$ is a nonoscillating solution of (3). Assume that $x(t) > 0$ for $t \ge N$, where $N \ge t_0$ is an integer. Integrate (3) from N to t ($t > N$) and obtain

$$x(t) = x(N) + \int_N^t h(s)\,ds - \int_N^t p(s)(\mathcal{A}x)([s])\,ds \le x(N) + \int_N^t h(s)\,ds.$$

Hence $\liminf_{t\to\infty} x(t) < 0$, which contradicts the assumption that $x(t)$ is eventually positive. ∎

4. Some particular realizations of the operator $\mathcal{A}$

Corollary 1. *Let the following conditions hold*:

1. $(\mathcal{A}x)(t) = \max_{s\in M(t)} x(s)$, *where* $M(t) = [p_1(t), q_1(t)]$ *is a compact subset of* $[t_0, \infty)$ *for* $t \ge t_0$ *and* $p_1(t) < q_1(t)$ *for* $t \ge t_0$, $p_1, q_1 \in C([t_0, \infty); \mathbb{R})$, $\lim_{t\to\infty} p_1(t) = \infty$.
2. *Condition* H1 *and condition* 2 *of Theorem* 1 *are satisfied.*

Then all regular solutions $x(t)$ *of the equation*

$$x'(t) + p(t) \max_{s \in M([t])} x(s) = 0$$

are nonoscillating.

Proof. It is immediately verified that condition 1 implies H3 and H4. Condition H2 follows from Lemma 1 of [2]. Thus Corollary 1 follows from Theorem 1. ∎

COROLLARY 2. *Let the following conditions hold*:

1. $(\mathcal{A}x)(t) = \min_{s \in M(t)} x(s)$, *where* $M(t)$ *is as in condition* 1 *of Corollary* 1.
2. *Condition* H1 *and condition* 2 *of Theorem* 1 *are satisfied.*

Then all regular solutions $x(t)$ *of the equation*

$$x'(t) + p(t) \min_{s \in M([t])} x(s) = 0 \tag{4}$$

are nonoscillating.

Proof. It is immediately verified that condition 1 implies H3, H4 and condition 3 of Theorem 2. Condition H2 follows from Lemma 1 of [2]. Thus Corollary 2 follows from Theorem 2. ∎

COROLLARY 3. *Let the following conditions hold*:

1. *Condition* 1 *of Corollary* 2 *is satisfied.*
2. *Condition* H1, *condition* 3 *of Theorem* 2 *and condition* 2 *of Theorem* 3 *are satisfied.*

Then each bounded solution of the equation (4) *is nonoscillating.*

Proof. Apply Corollary 2 and Theorem 3. ∎

COROLLARY 4. *Let the following conditions hold*:

1. $(\mathcal{A}x)(t) = \int_{t-a}^{t} k(t,s)x(s)\,ds$, *where* a *is a positive constant and* $k \in C([t_0 + a)^2; (0, \infty))$.
2. *Condition* 2 *of Corollary* 1 *is satisfied.*

Then all regular solutions $x(t)$ *of the equation*

$$x'(t) + p(t) \int_{[t]-a}^{[t]} k([t], s)x(s)\,ds = 0$$

are nonoscillating.

Proof. This follows from Theorem 1. ∎

Example 1. Consider the differential equation

$$x'(t) - \frac{1}{a} e^{t-[t]} \int_{[t]-a}^{[t]} e^{[t]-s} x(s)\, ds = 0, \tag{5}$$

where $a = \text{const} > 0$ and $t \geq t_0 > a + 2$. Here the functions

$$p(t) = -\frac{1}{a} e^{t-[t]}, \qquad (\mathcal{A}x)(t) = \int_{t-a}^{t} e^{t-s} x(s)\, ds$$

satisfy the conditions of Corollary 4. Thus all solutions of the equation (5) are nonoscillating.

Corollary 5. *Let the following conditions hold*:

1. $(\mathcal{A}x)(t) = f(x(g(t)))$, *where* $g \in C([t_1, \infty); \mathbb{R})$ *and* $t_1 \geq t_0$ *is such that* $g(t) \geq t_0$ *for* $t \geq t_1$, $\lim_{t\to\infty} g(t) = \infty$, $f \in C(\mathbb{R}; \mathbb{R})$, $uf(u) > 0$, $f(0) = 0$.
2. *Condition* 2 *of Corollary* 1 *is satisfied.*

Then all regular solutions $x(t)$ *of the equation*

$$x'(t) + p(t) f(x(g([t]))) = 0$$

are nonoscillating.

Proof. This follows from Theorem 1. ■

Example 2. Consider the differential equation

$$x'(t) - e^{t-3[t]} x^3([t]) = 0, \qquad t \geq t_0 > 0. \tag{6}$$

Here the functions $f(u) = u^3$, $p(t) = -e^{t-3[t]}$, and $(\mathcal{A}x)(t) = x(t)$ satisfy the conditions of Corollary 5. Thus all solutions of the equation (6) are nonoscillating.

Corollary 6. *Let the following conditions hold*:

1. *Condition* 1 *of Corollary* 4 *holds.*
2. *Conditions* 2 *and* 3 *of Theorem* 6 *hold.*

Then all solutions of the equation

$$x'(t) + p(t) \int_{[t]-a}^{[t]} k([t], s) x(s)\, ds = h(t)$$

are nonoscillating.

Proof. This follows from Theorem 6 since it is immediately verified that the corresponding operator $\mathcal{A}$ satisfies conditions H2 and H4. ■

Acknowledgements. The present investigation is supported by the Bulgarian Ministry of Education, Science and Technologies under Grant MM-422.

References

[1] A. R. Aftabizadeh and J. Wiener, *Oscillatory properties of first order linear functional differential equations*, Appl. Anal. 20 (1985), 165–187.

[2] V. G. Angelov and D. D. Bainov, *On the functional differential equations with "maximums"*, ibid. 16 (1983), 187–194.

[3] S. Busenberg and K. L. Cooke, *Models of vertically transmitted diseases with sequential continuous dynamics*, in: *Nonlinear Phenomena in Mathematical Science*, V. Lakshmikantham (ed.), Academic Press, New York, 1982, 179–187.

[4] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. 99 (1984), 265–297.

[5] B. G. Zhang and N. Parhi, *Oscillatory and nonoscillatory properties of first order differential equations with piecewise constant deviating arguments*, ibid. 139 (1989), 23–35.

D. D. Bainov
Medical University of Sofia
P.O. Box 45
Sofia 1504, Bulgaria

M. B. Dimitrova
Technical University
Sliven, Bulgaria

Received on 8.11.1994