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GROWTH AND ACCRETION OF MASS IN AN ASTROPHYSICAL MODEL, II

Abstract. Radially symmetric solutions of a nonlocal Fokker–Planck equation describing the evolution of self-attracting particles in a bounded container are studied. Conditions ensuring either global-in-time existence of solutions or their finite time blow up are given.

1. Introduction. In the second part of [1] we continue the study of radially symmetric solutions to the parabolic-elliptic system considered in [3–5], [10]:

$$u_t = \Delta u + \nabla \cdot (u \nabla \varphi), \tag{1}$$

$$\Delta \varphi = u. \tag{2}$$

Among physical interpretations of the system (1)–(2) we cite the evolution version of the Chandrasekhar equation from the theory of gravitating stars. For a discussion of other motivations leading to the nonlocal parabolic equation (1) of Fokker–Planck type we refer the reader to the introductions in [5], [10] and [11–12]. A related system of two parabolic equations modelling a biological phenomenon of chemotaxis has been studied in [7].

When the system (1)–(2) is considered in a bounded domain $\Omega \subset \mathbb{R}^n$, the nonlinear no-flux condition for the density of particles u,

$$\frac{\partial u}{\partial \nu} + u \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

is a natural one, since it guarantees the conservation of the total mass $M = \int_\Omega u(x,t)\,dx$.

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We assume for the gravitational potential φ generated by u either the Dirichlet condition

$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

or

$$\varphi = E_n * u, \tag{4.2}$$

with E_n being the fundamental solution of the n-dimensional Laplacian.

The solvability of the system (1)–(4) with the initial condition

$$u(x,0) = u_0(x) \geq 0 \tag{5}$$

has been studied in [5]. For instance, $u_0 \in L^p(\Omega)$ with some $p > n/2$ gives the existence of a local-in-time weak solution which becomes instantaneously regular: $u \in L^\infty_{\text{loc}}((0,T); L^\infty(\Omega))$ thanks to the parabolic character of (1). When Ω is star-shaped and u_0 is big enough (e.g. when either $M = \int_\Omega u_0(x)\,dx = |u_0|_1$ or the concentration of u_0 is sufficiently large), solutions cannot be global in time (see [5, Th. 2(v)], [3]); we say that *gravitational collapse* occurs. For the Cauchy problem (1)–(2), (4.2), (5) considered in the whole space $\mathbb{R}^n$, (nearly optimal) conditions for the existence of local-and global-in-time solutions are reported in the paper [2] (Theorems 1, 2). Finite time blow up of solutions is proved in [2, Proposition 1]; for the radially symmetric case see also [1, Theorem (i)] and [10, Th. 4].

In the case when Ω is the ball $B(0,R) \subset \mathbb{R}^n$ and u is radially symmetric the nonlocal problem (1)–(4) can be reformulated as the parabolic equation

$$Q_t = Q_{rr} - (n-1)r^{-1}Q_r + \sigma_n^{-1} r^{1-n} Q Q_r, \tag{6}$$

with the boundary conditions

$$Q(0,t) = 0, \qquad Q(R,t) = M. \tag{7}$$

Here $Q(r,t) = \int_{B(0,r)} u(x,t)\,dx$ is the integrated density, σ_n is the area of the unit sphere in $\mathbb{R}^n$, $\frac{\partial}{\partial r}Q(r,t) = \sigma_n r^{n-1} u(r,t)$ (see [4, (6)–(7)]). The initial condition

$$Q(r,0) = Q_0(r), \qquad 0 \leq r \leq R, \tag{8}$$

is a positive nondecreasing function, and the obvious compatibility condition is

$$Q_0(0) = 0, \qquad Q_0(R) = M.$$

Such a formulation allows us to consider singular densitites $u(r,t) = \sigma_n^{-1} r^{1-n} \frac{\partial}{\partial r} Q(r,t)$ (for instance: for $n \geq 3$ the Chandrasekhar stationary solution

$$\widetilde{u}(x) = 2(n-2)|x|^{-2} \tag{9}$$

or $u_0(x) = \text{const} \cdot \widetilde{u}(x)$, and for $n = 2$, a measure u_0), which are not necessarily smoothed out for $t > 0$ (see [4, Th. 2], [1, Theorem (i)], [2, Th. 2, Prop. 3]).

Keeping in mind scaling properties of (6), we may assume without loss of generality that $R = 1$. Indeed, the function $R^{2-n}Q(Rr, R^2t)$, together with $Q(r,t)$, is a solution of (6). The problem (6)–(8) can be transformed, using a new independent variable $y = r^n$ (see [4, (12)], [1, (15)]) into

$$Q_t = n^2 y^{2-2/n} Q_{yy} + n\sigma_n^{-1} Q Q_y, \tag{10}$$

$$Q(0,t) = 0, \qquad Q(1,t) = M, \qquad Q(y,0) = Q_0(y). \tag{11}$$

As we have noted in [1, proof of Theorem], the parabolic comparison principle for solutions of (10)–(11) holds, and this can be proved using standard approximation techniques ([6], [9]; cf. [4, Th. 2]).

We will give in this note some results on global-in-time existence of solutions (Theorem 1) corresponding to *bounded* densities. We remark that Theorem 2 in [4] deals with Q which are classical solutions of (10)–(11) in $(0,1)\times(0,\infty)$, and may correspond to unbounded u, while here Q will be smooth in $[0,1]\times(0,\infty)$. Moreover, we will prove finite time blow up of solutions for certain Q_0 (Theorem 2), a phenomenon described by an unbounded growth of the derivative $\frac{\partial}{\partial y}Q$, i.e. quenching for (10). This gives an insight into the mechanism of formation of singularities leading to nonglobal existence of radial solutions proved in [4, Th. 3]. We will consider in the last section a related model which can be interpreted as a description of the evolution of a cloud of particles surrounding a fixed mass. The stationary version of this problem has been mentioned in [11, Lemma 4.5].

As in [1], the techniques used in this paper are based on the maximum principle, comparison of solutions to parabolic equations and variants of the Hopf lemma. The idea of constructing suitable super- and subsolutions was suggested by the form of stationary solutions in [4, Sec. 4], by (9) written as

$$\widetilde{Q}(r) = 2\sigma_n r^{n-2} \quad \text{or} \quad \widetilde{Q}(y) = 2\sigma_n y^{1-2/n},$$

and by the proof of Theorem (b) in [7].

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2. Global existence. Our main result in this section is the global-in-time existence of classical solutions under suitable assumptions on Q_0. Unlike the result in [4, Th. 2], here we suppose that Q_0 is differentiable and $\frac{d}{dy}Q_0$ is bounded, and we get as a part of the conclusion that $\frac{\partial}{\partial y}Q$ is bounded.

THEOREM 1. (i) *If* $n=2$, $Q_0(1)=M<8\pi$ *and* $\frac{d}{dy}Q_0(0)<\infty$, *then there exists a global-in-time solution* Q *of* (10)–(11) *which is classical in* $(0,1)\times(0,\infty)$ *and continuous up to the boundary.*

(ii) *If* $n=2$, $\frac{d}{dy}Q_0(y)\le AB(y+B)^{-2}$ *for some* $A<8\pi$, $B>0$ *such that* $B(8-A/\pi)\ge 16$, *and* $Q_0(y)\ge My^k$ *for some* $k\ge 1$, *then there exists a solution* Q *of* (10)–(11) *defined globally in time*, *with uniformly bounded space derivative* $\frac{\partial}{\partial y}Q$.

(iii) *If* $n\ge 3$ *and* $Q_0(y)\le Ay(y^{2/n}+B)^{-1}$ *for some* $A\le 2\sigma_n$ *and* $B>0$, *then there exists a solution* Q *of* (10)–(11) *which is classical in* $(0,1)\times(0,\infty)$.

(iv) *If* $n\ge 5$, $\frac{d}{dy}Q_0(y)\le A((1-2/n)y^{2/n}+B)(y^{2/n}+B)^{-2}$ *for some* $A\le 2((n-4)/(n-2))\sigma_n$, $B>0$, *and* $Q_0(y)\ge My^k$ *for some* $k\ge 1$, *then the conclusion of* (ii) *holds true.*

Remarks. (i)–(ii) For $n=2$ and $|u_0|_1=M<8\pi$ we proved in [5, Th. 2(iv)] that global weak solutions of (1)–(3), (4.1), (5) in arbitrary domains $\Omega\subset\mathbb{R}^2$ do exist. The proof is much more subtle than that below.

(iii) We stress the fact that solutions of (10)–(11) can be obtained under the sole assumption $Q_0(y)\le 2\sigma_n y^{1-2/n}$, but we are unable to show any estimate for its derivative. An approximating sequence has been constructed in [4, Th. 2], but even if Q_0 is regular, no bound for $\frac{\partial}{\partial y}Q$ near $y=0$ is available when we work with the family of supersolutions to (10)–(11) in the proof below.

(iv) We recall that for $n\ge 5$, $Q_0\in C^{1-2/n}(0,1)$, $\frac{d}{dy}Q_0(1)<\infty$ and $\frac{d}{dy}Q_0(y)\le 2((n-4)/n)\sigma_n y^{-2/n}$ (hence $Q_0(y)\le 2((n-4)/(n-2))\sigma_n y^{1-2/n}$), we considered in [4, Th. 2] global solutions of (10)–(11) which are classical in $(0,1)\times(0,\infty)$, i.e. in the interior of their domain of definition. These solutions converge to the unique stationary solution as t tends to $+\infty$ (see [4, Th. 2] and [1, Prop. 2]).

The convergence of solutions as $t\to+\infty$ in Theorem 1(ii), (iv) can be obtained by using a similar argument involving Lyapunov functions introduced in [4–5].

Proof of Theorem 1. Let us begin with some calculations which will be useful also in the proof of Theorem 2. Define a nonlinear differential operator

$$\mathcal{L}q=n^2y^{2-2/n}q_{yy}+n\sigma_n^{-1}qq_y-q_t \tag{12}$$

and consider a function

$$q(y,t)=\frac{Ay}{y^{2/n}+B(t)}, \tag{13}$$

where $A > 0$ is a constant and $B(t) \ge 0$ is a (monotone) function of t. With these definitions we have

$$\begin{aligned}
q_t(y,t) &= -AyB'(t)(y^{2/n}+B(t))^{-2}, \qquad ' = d/dt,\\
q_y(y,t) &= A((1-2/n)y^{2/n}+B(t))(y^{2/n}+B(t))^{-2},\\
q_{yy}(y,t) &= -\frac{2}{n}Ay^{2/n-1}((1-2/n)y^{2/n}\\
&\quad + (1+2/n)B(t))(y^{2/n}+B(t))^{-3},\\
q_{yyy}(y,t) &= \frac{2}{n}Ay^{2/n-2}((1-2/n)(1+2/n)(y^{4/n}+B^2(t))\\
&\quad + 2(1+8/n^2)y^{2/n}B(t))(y^{2/n}+B(t))^{-4},\\
q_{yt}(y,t) &= -AB'(t)((1-4/n)y^{2/n}+B(t))(y^{2/n}+B(t))^{-3},
\end{aligned} \tag{14}$$

and consequently we obtain

$$\begin{aligned}
\mathcal{L}q = A(y^{2/n}+B(t))^{-3}\{&y^{2/n}(n-2)(A\sigma_n^{-1}-2)\\
&+ B(t)(nA\sigma_n^{-1}-2(n+2)) + B'(t)(y^{2/n}+B(t))\}.
\end{aligned} \tag{15}$$

If Q solves (10), then the derivative $P = \frac{\partial}{\partial y}Q$ satisfies the equation $\mathcal{L}_1 P = 0$, where

$$\mathcal{L}_1 p = n^2 y^{2-2/n} p_{yy} + 2n(n-1)y^{1-2/n}p_y + n\sigma_n^{-1}p^2 + n\sigma_n^{-1}Qp_y - p_t. \tag{16}$$

Thus we will also need for $p = \frac{\partial}{\partial y}q$ the identity

$$\begin{aligned}
\mathcal{L}_1 p = A(y^{2/n}+B(t))^{-4}\{&y^{4/n}((1-2/n)((n-2)A\sigma_n^{-1}\\
&- 2(n-4)) + (1-4/n)B'(t)) + 2y^{2/n}B(t)((n-2)A\sigma_n^{-1}\\
&- 2(n-4)(1+2/n) + (1-4/n)B'(t))\\
&+ B^2(t)(nA\sigma_n^{-1} - 2(n+2) + B'(t))\} + n\sigma_n^{-1}Qp_y.
\end{aligned} \tag{17}$$

The idea of the proof is the following: we approximate (10) by uniformly parabolic equations for $Q = Q_\varepsilon$, $\varepsilon > 0$,

$$Q_t = n^2(y+\varepsilon)^{2-2/n}Q_{yy} + n\sigma_n^{-1}QQ_y$$

in $[0,1]\times[0,T]$, $T > 0$ arbitrary. The functions $q(y,t)$ of the form (13) with suitably chosen A, B, and y replaced by $y+\varepsilon$ are supersolutions of the above regularized equation, and the comparison principle holds for these equations. Finally, we pass to the limit $\varepsilon \to 0$ on $[\delta,1]\times[0,T]$ for each $\delta > 0$. Similarly we work with the equation for the derivative $\frac{\partial}{\partial y}Q$ of a solution Q to (10)–(11). Since this construction is standard in the theory of parabolic equations ([6], [9]; for similar considerations see the proof of Theorem 2 in [4]), we restrict ourselves to the verification that (13) are supersolutions to (10)–(11), and that the derivatives $p = \frac{\partial}{\partial y}q$ of suitable q's satisfy $\mathcal{L}_1 p \le 0$ in (16).

(i) Now if $n = 2$ and $A \le 4\sigma_2 = 8\pi$, then (15) implies that $\mathcal{L}q \le 0$, so q is a supersolution of the equation (10) whenever $B'(t) \le 0$, e.g. when $B(t) \equiv B > 0$. On the other hand, if $\frac{d}{dy}Q_0(0) < \infty$, $M = Q_0(1) < 8\pi$, then $Q_0(y) \le \min(Ky, M)$ for some $K > 0$, hence $Q_0(y) \le Ay(y+B)^{-1} \equiv q(y)$, e.g. for $B = (8\pi - M)/(2K)$, $A = \sup_{y\in[0,1]} Q_0(y)(1 + B/y) < 8\pi$.

(ii) Observe that under the assumption on $\frac{d}{dy}Q_0$ we have $Q_0(y) \le Ay(y+B)^{-1} \equiv q(y)$, so (i) applies. If $M \le 8\pi/12$, then there exist many Q_0's satisfying all the hypotheses in (ii). In fact, $Q_0(y) = My^k$ with $1 \le k \le 8\pi/(12M)$ is such an initial condition since $kMy^{k-1} \le kM \le AB(1+B)^{-2} = q_y(1) \le q_y(y)$ for $A/\pi = 8 - 16/B$, $B = 5$ and each $y \in [0,1]$.

Concerning the equation for the derivative of the solution $Q(y,t)$ to (10)–(11), it is easy to see from (17) that $p = \frac{d}{dy}q$ is a supersolution of this equation if $B(8 - A/\pi) \ge 16$ (note that $Qp_y \le 0$).

Next we see that q, being a supersolution of the problem (10)–(11) such that $q(0) = Q(0,t)$, satisfies $\frac{\partial}{\partial y}Q(0,t) \le \frac{d}{dy}q(0) = p(0)$. Similarly, $\underline{Q}(y) = My^k$ is a subsolution of (10)–(11), $\underline{Q}(1) = Q(1,t)$, so $\frac{\partial}{\partial y}Q(1,t) \le \frac{d}{dy}\underline{Q}(1) = kM \le p(1)$. This means that p is a supersolution of the equation $\mathcal{L}_1 Q_y = 0$ with the above initial and boundary conditions.

(iii) If $n \ge 3$, $A \le 2\sigma_n < 2(n+2)\sigma_n/n$ and $B'(t) \le 0$, then (15) implies that q is a supersolution of (10)–(11) provided that $Ay(y^{2/n} + B(0))^{-1} \ge Q_0(y)$, $y \in [0,1]$. Moreover, even if $A < 2(n+2)\sigma_n/n$, then taking $B(t) \equiv B$ large enough we can assure that q is a supersolution of the problem (10)–(11) (note that $y \le 1$) for suitable Q_0's.

(iv) First we note that the condition on $\frac{d}{dy}Q_0$ implies

$$Q_0(y) \le Ay(y^{2/n} + B)^{-1} \equiv q(y),$$

hence (iii) applies to such initial data. If $M < 2((n-4)/n)\sigma_n$, then there exist many Q_0's satisfying all the assumptions in (iv), for instance $Q_0(y) = My^k$ with $1 \le k < 2((n-4)/n)\sigma_n M^{-1}$. Indeed, we have $kMy^{k-1} \le kM \le A(1 - 2/n + B)(1+B)^{-2} = q_y(1) \le q_y(y)$ for some $A \le 2((n-4)/(n-2))\sigma_n$, suitably small $B > 0$ and each $y \in [0,1]$.

Applying the operator $\mathcal{L}_1$ in (16) to $p = \frac{d}{dy}q$ we deduce from (17) that if $n \ge 5$ and

$$A\sigma_n^{-1} \le 2\frac{n-4}{n-2} < 2\frac{n-4}{n-2}\cdot\frac{n+2}{n} < 2 < 2\frac{n+2}{n},$$

then p is a supersolution of the equation $\mathcal{L}_1 Q_y = 0$. Note that if $A\sigma_n^{-1} < 2(n+2)/n$ and $B > 0$ is sufficiently large, then p is also a supersolution of $\mathcal{L}_1 Q_y = 0$ even if $n = 3$ or $n = 4$ (remember that $y \le 1$). However, in this case conditions for $p(y) \ge \frac{d}{dy}Q_0(y)$ are not as simple to express explicitly as before.

Clearly, $\underline{Q}(y) = My^k$ is a subsolution of the problem (10)–(11), $\underline{Q}(1) = Q(1,t)$, and the inequality $\underline{Q}(y) \le Q(y,t)$ implies $\frac{\partial}{\partial y}Q(1,t) \le \frac{d}{dy}\underline{Q}(1) = kM \le p(1)$. Therefore the function p is a supersolution of the problem $\mathcal{L}_1 Q_y = 0$ with the initial condition $\frac{d}{dy}Q_0$ and the boundary conditions $\frac{\partial}{\partial y}Q(0,t) \le \frac{d}{dy}q(0) = p(0)$, $\frac{\partial}{\partial y}Q(1,t) \le p(1)$.

The remaining part of the proof is a standard consequence of the comparison principle for parabolic equations (cf. [6], [9]).

3. Blow up of solutions. As was already mentioned in the introduction, large initial data u_0 lead to nonglobal solutions, i.e. ones that cease to exist after a finite time. The proofs of explosion of solutions published in [2, Prop. 1], [5, Th. 2(v)], [4, Th. 3] and [3] are based on the so-called virial method, that is, moment functionals of u are considered. These proofs are indirect; they do not explain *how* the solutions blow up. In Theorem 2 below we will use another technique to prove the gravitational collapse: we construct some blowing up subsolutions for suitably large initial data Q_0. They show that the phenomenon of quenching is possible for the problem (10)–(11), i.e. $\frac{\partial}{\partial y}Q(y,t)$ becomes unbounded (together with $u(y,t)$) when $t \to T^-$. On the other hand, Theorems (ii), (iii) in [1] show that if the blow up is accompanied by concentration of u near $y = 0$, then this concentration must be sufficiently large. Together with Theorem 2 below this gives bounds for threshold values of the concentration of initial data leading to global existence versus those with finite time explosion.

THEOREM 2. (i) *If* $n = 2$ *and* $Q_0(1) = M > 8\pi$, *then the solution* Q *of* (10)–(11) *cannot be defined globally in time.*

(ii) *If* $n \ge 3$ *and* $Q_0(y) \ge Ay(y^{2/n} + b)^{-1} + \varepsilon y$ *for some* $b, \varepsilon > 0$ *and* $A > 2(n+2)\sigma_n/n$, *then* Q *solving the problem* (10)–(11) *cannot be continued for all* $t \ge 0$.

Remarks. (i) The sufficient condition for the blow up of solutions in the two-dimensional situation coincides with that in [4, Th. 3].

(ii) Sufficient conditions for the blow up in higher-dimensional radial situation are more subtle than that ($Q_0(1) = M > 2n\sigma_n$) given in [4, Th. 3], and incomparable with that in [3, Th. 1]. In particular, for $n \ge 3$ solutions with $2(n+2)\sigma_n/n < M \le 2n\sigma_n$ can blow up provided that their initial distributions are highly concentrated near $y = 0$.

Proof of Theorem 2. The idea of the proof is similar to that of Theorem 1. We consider approximating equations and use systematically the comparison principle. Of course, here we will construct subsolutions that have unbounded space derivatives.

(i) Consider $\underline{Q}(y,t) = q(y,t) + \varepsilon y$ with q defined in (13), $B(t) = (b-ct)^2$ and $0 \le t \le T = b/c$, where the parameters b, c, ε will be determined later. Using the formulas (14), (15) we arrive at

$$\begin{aligned}\mathcal{L}\underline{Q} &\ge Ay(y+B(t))^{-3}\{-2c(b-ct)(y+B(t)) \\ &\quad + (A/\pi - 8)B(t) + \varepsilon\pi^{-1}(y+B(t))^2\} \\ &\ge Ay(y+B(t))^{-3}\{-cB(t) - cy^2 - 2cB^{3/2}(t) \\ &\quad + (A/\pi - 8)B(t) + \varepsilon\pi^{-1}(y^2 + B^2(t))\}.\end{aligned}$$

Therefore, for each $b > 0$, $A > 8\pi$, $\varepsilon > 0$ there exists $c > 0$ such that $\mathcal{L}\underline{Q} \ge 0$ for all $t \in [0,T]$, $y \in [0,1]$. Clearly, if $Q_0(1) = M > 8\pi$ and $\frac{d}{dy}Q_0(0) > 0$, then there exist $A > 8\pi$, b large enough and $\varepsilon > 0$ such that $\underline{Q}(y,0) = Ay(y+b^2)^{-1} + \varepsilon y \le Q_0(y)$ for all $y \in [0,1]$ and $\underline{Q}(1,t) \le M$ for all $t \in [0,T]$. Finally, shifting the initial data Q_0 to any time level $t_0 > 0$ we can assume, without loss of generality, that $\frac{d}{dy}Q_0(0) > 0$. This is a consequence of the Hopf lemma mentioned in [10, Remark 4] which gives $\frac{\partial}{\partial y}Q(0,t_0) > 0$. Consequently, we get for each $\delta > 0$,

$$\lim_{t\to T^-} \sup_{y<\delta} \underline{Q}(y,t) \ge A,$$

and this shows that Q cannot be defined for $t = T$ as a smooth function. Indeed, the derivative $\frac{\partial}{\partial y}Q(0,T)$ should be infinite by the comparison with the subsolution $\underline{Q}$: $Q \ge \underline{Q}$. Since $\frac{\partial}{\partial y}Q = n^{-1}\sigma_n u$, this means that $u(0,T)$ would become infinite together with $\frac{\partial}{\partial y}Q(0,T)$.

Incidentally, note that if a finite time blow up is accompanied by the concentration of mass near the origin, this mass is at least 4π ([1, Theorem (ii)]).

(ii) We construct a subsolution of the form $\underline{Q} = q(y,t) + \varepsilon y$ as in (i). Now the action of the operator $\mathcal{L}$ on $\underline{Q}$ gives

$$\begin{aligned}\mathcal{L}\underline{Q} &\ge Ay(y^{2/n}+B(t))^{-3}\{-2c(b-ct)(y^{2/n}+B(t)) \\ &\quad + (n-2)(A\sigma_n^{-1} - 2)y^{2/n} + (nA\sigma_n^{-1} - 2(n+2))B(t) \\ &\quad + \varepsilon n\sigma_n^{-1}(y^{2/n}+B(t))^2\} \\ &\ge Ay(y^{2/n}+B(t))^{-3}\{-cB(t) - cy^{4/n} - 2cB^{3/2}(t) \\ &\quad + (n-2)(A\sigma_n^{-1} - 2)y^{2/n} + (nA\sigma_n^{-1} - 2(n+2))B(t) \\ &\quad + \varepsilon n\sigma_n^{-1}y^{4/n} + \varepsilon n\sigma_n^{-1}B^2(t)\}.\end{aligned}$$

Hence, for each $A > 2(n+2)\sigma_n/n > 2\sigma_n$, $b > 0$, $\varepsilon > 0$, there exists $c > 0$ such that $\mathcal{L}\underline{Q} \ge 0$. Therefore, if $\underline{Q}(y,0) = Ay(y^{2/n}+b^2)^{-1} + \varepsilon y \le Q_0(y)$, then $\underline{Q}$ is a subsolution of the problem (10)–(11). The asymptotic behavior of $\underline{Q}$ as $t \to T^-$, $T = b/c$, shows that $Q(y,t) \ge Ay^{1-2/n}$ as $t \to T^-$, so

$\frac{\partial}{\partial y}Q(0,T)$ becomes infinite. This contradicts the global-in-time existence of the classical solution Q. As before, the density u blows up in L^∞ when $t \to T^-$.

The high concentration assumption on Q_0 in (ii) is slightly more difficult to interpret than (i) for $n = 2$. Nevertheless, if $Q_0(y) \geq (2(n+2)\sigma_n/n + \delta)y^{1-2/n}$ for $y \in [y_0, 1]$ with some $y_0 > 0$, $\delta > 0$, and $Q_0(y) \geq (2(n+2)\sigma_n/n + 2\delta)y$ for $y \in [0, y_0]$, then Q blows up in a finite time. Such a Q_0 corresponds to the density $u_0(x) \approx (n-2)(2(n+2)+\delta)n^{-1}|x|^{-2}$, $|x| \geq |x_0| > 0$, but $u_0(x)$ can be chosen bounded near $x = 0$. This is a weaker condition ensuring blow up than that in [4, Th. 3]. Of course, a sufficient condition for the blow up ($Q_0(r) > 2\sigma_n r^{n-2}$ for each $r > 0$) for the problem in the whole space in [1, Theorem (i)] is more transparent.

4. A modified problem. Here we apply methods developed in [4] and in the preceding sections to analyze solutions to a related problem mentioned in [11, Lemma 4.5]. Consider a star of mass $M^* > 0$ fixed at an interior point x_0 of a domain $\Omega \subset \mathbb{R}^n$ and surrounded by a cloud of particles. The evolution of the density u of particles is now described by a modification of the system (1)–(2) where (2) is replaced by the equation $\Delta\varphi = u + M^*\delta_{x_0}$ with δ_{x_0} the Dirac measure at x_0.

Assuming that Ω is the ball $B(0,R)$, $x_0 = 0$ and u is radially symmetric, we obtain

$$Q_t = Q_{rr} - (n-1)r^{-1}Q_r + \sigma_n^{-1}r^{1-n}(Q + M^*)Q_r \tag{18}$$

as the counterpart of (6) for this modified problem, with the initial and boundary conditions (7)–(8). Likewise, (10) (after the rescaling of M^* to $R^{2-n}M^*$) becomes

$$Q_t = n^2 y^{2-2/n}Q_{yy} + n\sigma_n^{-1}(Q + M^*)Q_y \tag{19}$$

with the conditions (11) as before.

The author of [11] has observed in Lemma 4.5 that for $n = 2$ stationary solutions of (18) exist if and only if $M + 2M^* < 8\pi$. Indeed, in our notation we get

$$Q(y) = M(1+c)y^\gamma(y^\gamma + c)^{-1}$$

with $M^* \in (0, 4\pi)$, $\gamma = 1 - M^*/(4\pi) \in (0,1)$, $c = (8\pi - 2M^*)/M - 1 > 0$ and

$$M + 2M^* < 8\pi, \tag{20}$$

since the stationary version of (19) is an integrable equation; for related calculations see [4, Sec. 4]. Note that these steady states have unbounded derivatives $\frac{d}{dy}Q(y)$ when $y \to 0$. We prove below that (20) is a crucial condition in order that a counterpart of Theorem 1(i) hold true. Moreover, local-in-time solutions to (19), (11) cannot be continued to global ones when $M+2M^* > 8\pi$. Thus, the critical total mass for the gravitational collapse in (19) is $8\pi - M^*$, so it is *strictly less than* 8π as was for the original problem (10)–(11).

THEOREM 3. *Let* $n = 2$ *in the problem* (19) *with the conditions* (11).

(i) *If* $M + 2M^* < 8\pi$, $\frac{d}{dy}Q_0(0) < \infty$, *then there exists a global solution* Q, *classical in* $(0, 1) \times (0, \infty)$.

(ii) *If* $M + 2M^* > 8\pi$, *then solutions cannot be defined globally in time.*

Proof. (i) Since the idea of the proof resembles that of Theorem 1(i) we only mention the main technical tools. Of course, we do not expect any result similar to that in Theorem 1(ii), because even the stationary solutions have unbounded derivatives.

A counterpart of the function (13),

$$q(y) = \frac{Ay^\gamma}{y^\gamma + B}, \tag{21}$$

with $\gamma = 1 - M^*/(4\pi)$ and $B > 0$, is a supersolution of (19) provided that $A \le 8\pi - 2M^*$. The condition $\frac{d}{dy}Q_0(0) < \infty$ guarantees that a suitably small $B > 0$ can be chosen so that $q(y) \ge Q_0(y)$. This means that q is a supersolution of the problem (19), (11).

(ii) We apply the idea of the proof of Theorem 3 in [4]. Integrating (19) on $[0, 1]$ we obtain

$$\frac{d}{dt}\Big(\int_0^1 Q\,dy\Big) = 4yQ_y|_0^1 - 4\int_0^1 Q_y\,dy + \frac{1}{2\pi}\int_0^1 (Q^2)_y\,dy + \frac{1}{\pi}\int_0^1 M^*Q_y\,dy,$$

hence

$$\frac{d}{dt}\Big(\int_0^1 Q\,dy\Big) \ge -4M + \frac{M^2}{2\pi} + \frac{MM^*}{\pi} = \frac{M}{2\pi}(M + 2M^* - 8\pi). \tag{22}$$

It is clear that under the condition $M + 2M^* > 8\pi$ positive nondecreasing solutions of (19), (11) cannot be global in time because (22) implies an unbounded growth of the integral $\int_0^1 Q$, while $\int_0^1 Q \le M$ must hold.

Note that multiplying (18) by r and integrating on $[0, R]$, we obtain the condition $M + 2M^* > 2n\sigma_n R^{n-2}$ as a generalization of the condition in (ii) for the finite time blow up in the n-dimensional problem (cf. [4, Th. 3] for $M^* = 0$).

Another proof of (ii) (in the spirit of that of Theorem 2(i)) can be given under a mild supplementary assumption on the behavior of Q_0 near $y = 0$. Namely, if $M + 2M^* > 8\pi$, then $\underline{Q}(y, t) = Ay^\gamma(y^\gamma + (b - ct)^2)^{-1} + \varepsilon y$, with an $A > M$ and suitable $b, c, \varepsilon > 0$, is a subsolution of the problem (19), (11).

Remark. It is of interest to note that for $n = 2$ local-in-time solutions to (19), (11) can be defined only if $M^* < 4\pi$. Indeed, positive nondecreasing solutions to (19), (11) are solutions of the linear parabolic equation $q_t = 4yq_{yy} + \pi^{-1}(M^* + Q)q_y$. The standard theory of diffusion processes shows that this equation can have a solution satisfying the boundary conditions $q(0,t) = 0$, $q(1,t) = M$ only if $M^* < 4\pi$ (see e.g. [8, Ch. 15, Sec. 6, Example 6, p. 238, or Elementary Problem 28, p. 381]).

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