Abstract. We study the integrability of two-dimensional autonomous systems in the plane of the form \( \dot{x} = -y + X_s(x, y) \), \( \dot{y} = x + Y_s(x, y) \), where \( X_s(x, y) \) and \( Y_s(x, y) \) are homogeneous polynomials of degree \( s \) with \( s \geq 2 \). First, we give a method for finding polynomial particular solutions and next we characterize a class of integrable systems which have a null divergence factor given by a quadratic polynomial in the variable \( (x^2 + y^2)^{s/2-1} \) with coefficients being functions of \( \tan^{-1}(y/x) \).

1. Introduction. We consider two-dimensional autonomous systems of differential equations of the form
\[
\dot{x} = -y + X_s(x, y), \quad \dot{y} = x + Y_s(x, y),
\]
where
\[
X_s(x, y) = \sum_{k=0}^{s} a_k x^k y^{s-k}, \quad Y_s(x, y) = \sum_{k=0}^{s} b_k x^k y^{s-k}
\]
are homogeneous polynomials of degree \( s \), with \( s \geq 2 \), and with \( a_k \) and \( b_k \), \( k = 0, 1, \ldots, s \), being arbitrary real coefficients. Recently, these systems have been studied by several authors (see for instance [1], [3], [5], [6] and [8]), especially in order to obtain information about the number of small amplitude limit cycles and to determine the cyclicity of the origin (see for instance [4] and [7]). In this paper we study their integrability.

Our aim is to find solutions \( W(x, y) = 0 \) of system (1.1), where \( W(x, y) \) is a null divergence factor (this notion will be defined below). In Theorem 1 we give an explicit method for obtaining such a factor, which is used in Theorem 2 to construct a particular class of integrable fields.

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We can write system (1.1) (see [2]) in polar coordinates \( x = r \cos \varphi \) and \( y = r \sin \varphi \) as
\[
(1.2) \quad \dot{r} = P_s(\varphi) r^s, \quad \dot{\varphi} = 1 + Q_s(\varphi) r^{s-1},
\]
where \( P_s(\varphi) \) and \( Q_s(\varphi) \) are trigonometric polynomials of the form
\[
P_s(\varphi) = R_{s+1} \cos((s+1)\varphi + \varphi_{s+1}) + R_{s-1} \cos((s-1)\varphi + \varphi_{s-1})
+ \ldots + \begin{cases} R_1 \cos(\varphi + \varphi_1) & \text{if } s \text{ is even}, \\ R_0 & \text{if } s \text{ is odd}, \end{cases}
\]
\[
Q_s(\varphi) = -R_{s+1} \sin((s+1)\varphi + \varphi_{s+1}) + R_{s-1} \sin((s-1)\varphi + \varphi_{s-1})
+ \ldots + \begin{cases} R_1 \sin(\varphi + \varphi_1) & \text{if } s \text{ is even}, \\ R_0 & \text{if } s \text{ odd}, \end{cases}
\]
where \( R_i, R_i, \varphi_i \) and \( \varphi_i \) are real constants.

If we make the change of variable \( R = r^{s-1} \), then system (1.2) becomes
\[
(1.3) \quad \dot{R} = (s-1) P_s(\varphi) R^2, \quad \dot{\varphi} = 1 + Q_s(\varphi) R.
\]

In the study and determination of the first integrals for quadratic systems and homogeneous cubic systems (see [2]), we used a technique consisting in the research of polynomial particular solutions of system (1.3) of the form
\[
(1.4) \quad V(R, \varphi) = 1 + V_1(\varphi) R + V_2(\varphi) R^2 + \ldots + V_p(\varphi) R^p = 0,
\]
where \( V_k(\varphi), k = 1, \ldots, p, \) are homogeneous trigonometric polynomials of degree \( k(s-1) \) in the variables \( \cos \varphi \) and \( \sin \varphi \). The main results are the following.

A function \( W(x, y) \) will be called a null divergence factor for system (1.1) if \( W(x, y) = 0 \) is a particular solution for this system and the divergence of the vector field
\[
C = \left( \frac{-y + X_s(x, y)}{W(x, y)}, \frac{x + Y_s(x, y)}{W(x, y)} \right)
\]
defined on \( \mathbb{R}^2 \setminus \{(x, y) : W(x, y) = 0\} \) is zero.

We notice that if the divergence of a vector field is zero then system (1.1) defined for this vector field is integrable. In particular, if system (1.1) has a null divergence factor then this system is integrable and the origin is a center.

By using the functions \( x_i, i = 1, \ldots, p, \) defined implicitly by
\[
(1.5) \quad V_1 = \sum_{j=1}^p x_j, \quad V_2 = \sum_{j,k=1 \atop j < k}^p x_j x_k, \quad \ldots, \quad V_p = x_1 x_2 \ldots x_p,
\]
the function (1.4) can be written as \( V(R, \varphi) = \prod_{i=1}^p (1 + x_i(\varphi) R) \).
Theorem 1. If
\[
(V(R, \varphi))^\alpha = \left( \prod_{i=1}^{p} (1 + x_i(\varphi)R) \right)^\alpha
\]
is a null divergence factor for system (1.1) with \( \alpha \) a real number, then the functions \( x_i(\varphi), i = 1, \ldots, p \), satisfy the following system of differential equations:
\[
dx_i \over dz = \frac{x_i}{s + 1 + \alpha \sum_{j=1}^{p} \frac{x_j}{z - x_j}}, \quad i = 1, \ldots, p,
\]
where \( z = Q_s(\varphi) \).

For \( p = 2 \) it is possible to find the general solution for system (1.6) and therefore to determine a null divergence factor for system (1.1). In this case, we have

Theorem 2. For \( s \in \mathbb{N} \) with \( s \geq 2 \), and arbitrary \( k_1, k_2, \varphi_0 \in \mathbb{R} \), the system of the form (1.2) with
\[
P_s(\varphi) = 2(-k_1 \cos^{s-2}(\varphi + \varphi_0) \sin^3(\varphi + \varphi_0) \\
+ k_2 \sin^s(\varphi + \varphi_0) \cos^3(\varphi + \varphi_0)),
\]
\[
Q_s(\varphi) = (k_1 \cos^{s-1}(\varphi + \varphi_0) - k_2 \sin^{s-1}(\varphi + \varphi_0)) \cos(\varphi + \varphi_0),
\]
is integrable.

In cartesian coordinates \( x = r \cos(\varphi + \varphi_0) \) and \( y = r \sin(\varphi + \varphi_0) \), we can write system (1.7) in the form
\[
\dot{x} = -y - k_1 x^{s-1} y + k_2 y^{s-2}(2x^2 - y^2),
\]
\[
\dot{y} = x + k_1 x^{s-2}(x^2 - 2y^2) + k_2 xy^{s-1},
\]
with \( s \in \mathbb{N} \) and \( s \geq 2 \). We notice that the origin is a center for system (1.8).

In Section 2 we give a method of obtaining particular solutions of system (1.1). Theorem 1 is proved in Section 3. Finally, Theorem 2 is proved in Section 4.

2. Particular solutions

Proposition 1. The function (1.4) is a particular solution of system (1.3) if the homogeneous trigonometric polynomials \( V_k(\varphi), k = 1, \ldots, p \), satisfy the following differential system:
\[
V_k' + V_k'Q_s + k(s - 1)V_kP_s = V_kV_1', \quad k = 1, \ldots, p - 1,
\]
\[
V_p'Q_s + p(s - 1)V_pP_s = V_pV_1'.
\]
where \( ' = d/d\varphi \).
Proof. If we force $V(R, \varphi) = 0$ to be a particular solution of system (1.3), then it must satisfy

$$
V(R, \varphi) = \lambda(R, \varphi)(1 + V_1(\varphi)R + V_2(\varphi)R^2 + \ldots + V_p(\varphi)R^p).
$$

Differentiating $V(R, \varphi)$ with respect to $t$ we get

$$
\dot{V}(R, \varphi) = \frac{\partial V}{\partial R}((s - 1)P_s(\varphi)R) + \frac{\partial V}{\partial \varphi}(1 + Q_s(\varphi)R^2)
$$

$$
= (V_1(\varphi) + 2V_2(\varphi)R + \ldots + pV_p(\varphi)R^{p-1})((s - 1)P_s(\varphi)R^2)
$$

$$
+ (V'_1(\varphi)R + V'_2(\varphi)R^2 + \ldots + V'_p(\varphi)R^p)(1 + Q_s(\varphi)R)
$$

$$
= V'_1(\varphi)R + \sum_{k=1}^{p-1} (V'_{k+1}(\varphi)
$$

$$
+ V'_k(\varphi)Q_s(\varphi) + k(s - 1)V_k(\varphi)P_s(\varphi))R^{k+1}
$$

$$
+ (V''_p(\varphi)Q_s(\varphi) + p(s - 1)V'_p(\varphi)P_s(\varphi))R^{p+1},
$$

and if we equate the terms on the right-hand side of (2.2) and (2.3) it results first in $\lambda(R, \varphi) = V'_1(\varphi)R$, and considering this relationship we obtain (2.1). We notice that $V(R, \varphi)$ satisfies $\dot{V} = (V'_1(\varphi)R)\dot{V}$. □

Proposition 2. In order to find a particular solution of system (1.1) of the form (1.4) it is sufficient to find $p$ different solutions of the differential equation

$$
(x - Q_s(\varphi))\frac{dx}{d\varphi} = (s - 1)P_s(\varphi)x
$$

so that the functions $V_k(\varphi), k = 1, \ldots, p$, obtained from relations (1.6) be homogeneous trigonometric polynomials of degree $k(s - 1)$.

Proof. By using the functions $x_i$ introduced in (1.5) we define

$$
\begin{align*}
V_0^i &= 1, & V_1^i &= \sum_{j=1, j \neq i}^{p} x_j, & V_2^i &= \sum_{j,k=1, j < k}^{p} x_j x_k, & \ldots , \\
& & \ldots , & V_{p-1}^i &= x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_p,
\end{align*}
$$

for $i = 1, \ldots, p$. From (1.5) and (2.4) we get

$$
V_k = \frac{1}{k} \sum_{i=1}^{p} V_{k-1}^i x_i, \quad V'_k = \sum_{i=1}^{p} V_{k-1}^i x'_i,
$$
where \( k = 1, \ldots, p \). Then we can write system (1.5) as
\[
\sum_{i=1}^{p} V_{i}^{k} x_{i}' + \left( \sum_{i=1}^{p} V_{k-1}^{i} x_{i}' \right) Q_{s} + (s-1) \left( \sum_{i=1}^{p} V_{k-1}^{i} x_{i}' \right) P_{s} = V_{k} \left( \sum_{i=1}^{p} x_{i}' \right),
\]
where \( k = 1, \ldots, p \). On the other hand, from the equalities
\[
V_{k} \left( \sum_{i=1}^{p} x_{i}' \right) - \left( \sum_{i=1}^{p} V_{k}^{i} x_{i}' \right) = 0, \quad k = 1, \ldots, p-1,
\]
system (2.4) can be written as
\[
\sum_{i=1}^{p} V_{k-1}^{i} (x_{i}' Q_{s} + (s-1)x_{i}' P_{s} - x_{i}' x_{i}') = 0, \quad k = 1, \ldots, p-1,
\]
(2.5)
\[
\sum_{i=1}^{p} V_{p-1}^{i} (x_{i}' Q_{s} + (s-1)x_{i}' P_{s} - x_{i}' x_{i}') = 0.
\]
If we set \( X_{i} = x_{i}' Q_{s} + (s-1)x_{i}' P_{s} - x_{i}' x_{i}' \), \( i = 1, \ldots, p \), system (2.5) becomes
\[
\sum_{i=1}^{p} V_{k-1}^{i} X_{i} = 0, \quad k = 1, \ldots, p.
\]
(2.6)
This is a linear system of \( p \) equations in the variables \( X_{1}, \ldots, X_{p} \). The matrix \( A \) of system (2.6) is given by
\[
A = \left( \begin{array}{cccc}
V_{1}^{1} & 1 & \cdots & 1 \\
V_{1}^{2} & V_{2}^{2} & \cdots & V_{2}^{p} \\
\vdots & \ldots & \ldots & \vdots \\
V_{p-1}^{1} & V_{p-1}^{2} & \cdots & V_{p-1}^{p}
\end{array} \right),
\]
and a straightforward computation shows that
\[
\det A = \prod_{i,j=1}^{p} (x_{i} - x_{j}).
\]
If we assume that the variables \( x_{1}, x_{2}, \ldots, x_{p} \) are pairwise different, then \( \det A \neq 0 \). Therefore, the only possible solution of the linear system (2.6) is
\[
X_{i} = 0 \quad \text{for} \quad i = 1, \ldots, p,
\]
that is,
\[
x_{i}' Q_{s} + (s-1)x_{i}' P_{s} - x_{i}' x_{i}' = 0, \quad i = 1, \ldots, p.
\]
This completes the proof.
We note that system (1.3) and the system of Proposition 2 are equivalent if we make the change of variable $R = -1/x$.

3. Null divergence factors. If system (1.1) is written in polar coordinates (see (1.2)), then the function $(V(R, \varphi))^\alpha$ is a null divergence factor for system (1.1) if

$$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{P_s(\varphi) r^{s+1}}{(V(R, \varphi))^\alpha} \right) + \frac{\partial}{\partial \varphi} \left( \frac{1 + Q_s(\varphi) r^{s-1}}{(V(R, \varphi))^\alpha} \right) = 0. \tag{3.1}$$

Now assume that the function $V(R, \varphi)$ is of the form given in (1.4) with $R = r^{s-1}$. Then if we develop the expression (3.1) with respect to the powers of $R$, we have

$$\begin{align*}
(s + 1)P_s + Q'_s - \alpha V'_1 &= 0, \\
V'_{k+1} + V'_k Q_s + k(s-1)V_k P_s &= V_k V'_1, \quad k = 1, \ldots, p - 1, \\
V'_p Q_s + p(s-1)V_p P_s &= V_p V'_1. \tag{3.2}
\end{align*}$$

System (3.2) coincides with system (1.5) except that the value of $V_1$ in system (3.2) is determined as a function of $P_s(\varphi), Q'_s(\varphi)$ and $\alpha$.

Proof of Theorem 1. From (1.6), Proposition 2 and (2.5), system (3.2) takes the form

$$x'_i = \frac{(s-1)P_s x_i}{x_i - Q'_s}, \quad i = 1, \ldots, p, \tag{3.3}$$

with the condition

$$\begin{align*}
(s + 1)P_s + Q'_s - \alpha \sum_{i=1}^{p} x'_i &= 0. \tag{3.4}
\end{align*}$$

If we take $z = Q'_s$ as independent variable instead of $\varphi$, then we have

$$\frac{dx_i}{d\varphi} = \frac{dx_i}{dQ'_s} \frac{dQ'_s}{d\varphi}, \quad i = 1, \ldots, p,$$

and (3.4) goes over to

$$\begin{align*}
(s + 1)P_s + Q'_s - \alpha Q'_s \sum_{i=1}^{p} \frac{dx_i}{dz} &= 0, \\
\frac{dz}{d\varphi} &= \left( \alpha \sum_{i=1}^{p} \frac{dx_i}{dz} - 1 \right) Q'_s. \tag{3.5}
\end{align*}$$

By inserting the expression (3.5) in system (3.3), and considering the change
of variable \( z = Q_x \) we can write
\[
\frac{dx_i}{dz} = \frac{s - 1}{s + 1} \frac{\alpha \sum_{j=1}^{p} \frac{dx_j}{dz} - 1}{x_i - z} x_i, \quad i = 1, \ldots, p.
\]
Then, isolating \( \frac{dx_i}{dz}, i = 1, \ldots, p \), in the above system we get
\[
\frac{dx_i}{dz} = \prod_{j=1}^{p} (z - x_j) \frac{s + 1}{s - 1} \sum_{j=1}^{p} \prod_{k=1}^{p} (z - x_k) x_j, \quad i = 1, \ldots, p.
\]

If we divide the numerator and denominator of this fraction by the product \( \prod_{j=1}^{p} (z - x_j) \), we obtain system (1.6).

Note that system (1.6) is symmetric with respect to the variables \( x_i, i = 1, \ldots, p \).

We want to find functions of the form
\[
U(x_1, \ldots, x_p, z) \equiv H(x_1, \ldots, x_p) + zG(x_1, \ldots, x_p)
\]
so that, for system (1.6), \( dU/dz = 0 \).

**Proposition 3.** In order to find functions of the form (3.6) for system (1.6) it is sufficient to find solutions of the partial differential system
\[
\frac{\partial H}{\partial x_i} + x_i \frac{\partial G}{\partial x_i} + \alpha G = 0, \quad i = 1, \ldots, p,
\]
(3.7)
\[
\sum_{i=1}^{p} x_i \frac{\partial G}{\partial x_i} + \frac{s + 1}{s - 1} G = 0.
\]

**Proof.** If we differentiate (3.6) with respect to \( z \), we have
\[
\frac{dU}{dz} = \sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{dx_i}{dz} + \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} \frac{dx_i}{dz} + G = 0.
\]
By replacing the value of \( \frac{dx_i}{dz}, i = 1, \ldots, p \), given in (1.6) in the previous expression it becomes
\[
\sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{x_i}{z - x_i} + \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} \frac{x_i}{z - x_i} + \left( \frac{s + 1}{s - 1} + \alpha \sum_{i=1}^{p} \frac{x_i}{z - x_i} \right) G = 0,
\]
or
\[
\sum_{i=1}^{p} \frac{\partial H}{\partial x_i} x_i \frac{x_i}{z - x_i} + \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} \left( x_i + \frac{x_i^2}{z - x_i} \right) + \frac{s + 1}{s - 1} + \alpha \sum_{i=1}^{p} \frac{x_i}{z - x_i} G
\]

\[
= \sum_{i=1}^{p} x_i \frac{\partial H}{\partial x_i} + x_i \frac{\partial G}{\partial x_i} + \alpha x_i G + \sum_{i=1}^{p} x_i \frac{\partial G}{\partial x_i} + \frac{s + 1}{s - 1} G = 0.
\]

In order that this last expression be null it is sufficient that conditions (3.7) hold. Notice that these conditions are not necessary in order that the previous expression be null.

### 4. Quadratic null divergence factors.

We now consider the case \( p = 2 \), that is to say,

\[ V(R, \varphi) = 1 + V_1(\varphi)R + V_2(\varphi)R^2. \]

In this case, system (3.2) takes the form

\[
(s + 1)P_s + Q'_s - \alpha V'_1 = 0, \quad V'_2 + V'_1 Q'_s + (s - 1)V_1 P_s = V_1 V'_1, \quad V'_2 Q'_s + 2(s - 1)V_2 P_s = V_2 V'_1,
\]

where \( V_1 = x_1 + x_2, \ V_2 = x_1 x_2 \), and system (1.7) goes over to

\[
\begin{align*}
\frac{dx_1}{dz} &= \frac{x_1}{a + \alpha \left( \frac{x_1}{z - x_1} + \frac{x_2}{z - x_2} \right)}, \\
\frac{dx_2}{dz} &= \frac{x_2}{a + \alpha \left( \frac{x_1}{z - x_1} + \frac{x_2}{z - x_2} \right)},
\end{align*}
\]

with \( a = (s + 1)/(s - 1) \).

In this case, we want to obtain functions of the form

\[ U(x_1, x_2, z) = H(x_1, x_2) + zG(x_1, x_2) \]

so that, for system (4.1), \( dU/dz = 0 \).

By applying Proposition 3, the functions \( H(x_1, x_2) \) and \( G(x_1, x_2) \) have to satisfy the system

\[
\begin{align*}
\frac{\partial H}{\partial x_1} + x_1 \frac{\partial G}{\partial x_1} + \alpha G &= 0, \\
\frac{\partial H}{\partial x_2} + x_2 \frac{\partial G}{\partial x_2} + \alpha G &= 0, \\
x_1 \frac{\partial G}{\partial x_1} + x_2 \frac{\partial G}{\partial x_2} + aG &= 0.
\end{align*}
\]
If we make the change of variable $u = x_2/x_1$ and we take the functions $G$ and $H$ as follows:

$$G(x_1, x_2) = x_1^{-a}g(u), \quad H(x_1, x_2) = x_1^{1-a}h(u),$$

then the third equation of system (4.3) is satisfied identically, and the system takes the form

$$\alpha g + u \frac{dg}{du} + \frac{dh}{du} = 0,$$

(4.4)

$$(1 - a)h = [(a - \alpha) - \alpha u]g + u(1 - u) \frac{dg}{du}.$$

If we differentiate the second equation of (4.4) with respect to $u$, we have

$$(1 - a) \frac{dh}{du} = -\alpha g + [(a - \alpha + 1) - (\alpha + 2)u] \frac{dg}{du} + u(1 - u) \frac{d^2g}{du^2}.$$

By replacing the value of $dh/du$ obtained from the first equation of system (4.4) in the previous expression, we find

$$(4.5) \quad u(1 - u) \frac{d^2g}{du^2} + [(1 + a - \alpha) - (1 + \alpha + a)u] \frac{dg}{du} - a^2g = 0.$$

The relation (4.5) is a hypergeometric second order linear differential equation. We will study it for the particular case $a - \alpha = 1/2$. This relation is satisfied by certain integrable systems (1.1) in the quadratic case $s = 2$.

Since $a - \alpha = 1/2$, the equation (4.5) can be written as

$$u(1 - u) \frac{d^2g}{du^2} + \left(\frac{3}{2} - \left(2a + \frac{1}{2}\right)u\right) \frac{dg}{du} - a\left(a - \frac{1}{2}\right)g = 0.$$

The general solution of this equation is given by

$$g(u) = u^{-1/2}[C_1(1 + \sqrt{u})^{2(1-a)} + C_2(1 - \sqrt{u})^{2(1-a)}],$$

where $C_1$ and $C_2$ are arbitrary constants. For this $g(u)$ we have

$$h(u) = C_1(1 + \sqrt{u})^{2(1-a)} - C_2(1 - \sqrt{u})^{2(1-a)}.$$

By going back through the change of variables it is easy to see that

$$G(x_1, x_2) = (x_1 x_2)^{-1/2}(C_1(\sqrt{x_1} + \sqrt{x_2})^{2(1-a)} + C_2(\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}),$$

$$H(x_1, x_2) = C_1(\sqrt{x_1} + \sqrt{x_2})^{2(1-a)} - C_2(\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}.$$

Therefore

$$U_1(x_1, x_2, z) = (\sqrt{x_1} + \sqrt{x_2})^{2(1-a)}(1 + z/\sqrt{x_1 x_2}),$$

$$U_2(x_1, x_2, z) = (\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}(1 - z/\sqrt{x_1 x_2}).$$
are two independent functions of the form (4.2) for system (4.1), which we can write in the form
\[ U_1(x_1, x_2, z) = (x_1 + x_2 + 2\sqrt{x_1 x_2})^{1-a}(1 + z/\sqrt{x_1 x_2}), \]
\[ U_2(x_1, x_2, z) = (x_1 + x_2 - 2\sqrt{x_1 x_2})^{1-a}(1 - z/\sqrt{x_1 x_2}). \]
As \( V_1 = x_1 + x_2, \ V_2 = x_1 x_2, \ z = Q_s \) and \( a = (s + 1)/(s - 1) \) we can write
\[ U_1(V_1, V_2, z) = (V_1 + 2\sqrt{V_2})^{-2/(s-1)}(1 + Q_s/\sqrt{V_2}), \]
\[ U_2(V_1, V_2, z) = (V_1 - 2\sqrt{V_2})^{-2/(s-1)}(1 - Q_s/\sqrt{V_2}), \]
that is,
\begin{align*}
(4.5) \\
(V_1 + 2\sqrt{V_2})^{-2/(s-1)}(1 + Q_s/\sqrt{V_2}) &= K_1, \\
(V_1 - 2\sqrt{V_2})^{-2/(s-1)}(1 - Q_s/\sqrt{V_2}) &= K_2,
\end{align*}
where \( K_1 \) and \( K_2 \) are arbitrary constants.

**Proof of Theorem 2.** We can write system (4.5) as
\begin{align*}
(4.6) \\
V_1 + 2\sqrt{V_2} &= K_1(1 + Q_s/\sqrt{V_2})^{(s-1)/2}, \\
V_1 - 2\sqrt{V_2} &= K_2(1 - Q_s/\sqrt{V_2})^{(s-1)/2}.
\end{align*}
By multiplying the two equations, we have
\begin{equation}
(4.7) \quad V_1^2 - 4V_2 = K_1 K_2 (1 - Q_s^2/V_2)^{(s-1)/2}.
\end{equation}
As \( V_1 \) and \( V_2 \) are homogeneous trigonometric polynomials of degrees \( s - 1 \) and \( 2(s - 1) \) respectively, the left-hand side of (4.7) is a trigonometric polynomial of degree \( 2(s - 1) \). So the right-hand side of (4.7) must have the same degree. In particular, \( V_2 \) is a divisor of \( Q_s^2 \). On the other hand, if we square the first equation of (4.6), and we develop the right-hand side of that equation according to the Newton binomial, and group the terms with or without the factors \( \sqrt{V_2} \), we see that \( \sqrt{V_2} \) is a homogeneous trigonometric polynomial of degree \( s - 1 \), and a divisor of \( Q_s \). Hence \( X_2 = Q_s/\sqrt{V_2} \) is a homogeneous trigonometric polynomial of degree \( (s + 1) - (s - 1) = 2 \), and we can write system (4.6) as
\begin{align*}
(4.8) \\
V_1 + 2\sqrt{V_2} &= K_1(1 + X_2)^{(s-1)/2}, \\
V_1 - 2\sqrt{V_2} &= K_2(1 - X_2)^{(s-1)/2}.
\end{align*}
By subtracting both equations of system (4.8), we have
\[ 4\sqrt{V_2} = K_1(1 + X_2)^{(s-1)/2} - K_2(1 - X_2)^{(s-1)/2}, \]
and then
\[ Q_s = X_2\sqrt{V_2} = \frac{1}{4}X_2(K_1(1 + X_2)^{(s-1)/2} - K_2(1 - X_2)^{(s-1)/2}). \]
If $s$ is even, the trigonometric polynomials $1 + X_2$ and $1 - X_2$ must be the squares of first degree homogeneous trigonometric polynomials in order to satisfy system (4.8). In the case where $s$ is odd, this condition is not necessary, but we can also impose it. We can easily prove that

$$X_2(\varphi) = \cos 2(\varphi + \varphi_0) = \cos 2\omega,$$

where $\varphi_0$ is arbitrary; it follows that

(4.9) $$1 + X_2 = 2 \cos^2 \omega, \quad 1 - X_2 = 2 \sin^2 \omega,$$

and

$$Q_s(\varphi) = (k_1 \cos^{s-1} \omega - k_2 \sin^{s-1} \omega) \cos 2\omega,$$

where $k_1 = \frac{1}{4} K_1 2^{(s-1)/2}$, $k_2 = \frac{1}{4} K_2 2^{(s-1)/2}$. By inserting the values obtained in (4.9) into system (4.8) we have

$$V_1 + 2\sqrt{V_2} = 4k_1 \cos^{s-1} \omega, \quad V_1 - 2\sqrt{V_2} = 4k_2 \sin^{s-1} \omega.$$

Therefore we obtain

$$V_1 = 2(k_1 \cos^{s-1} \omega + k_2 \sin^{s-1} \omega), \quad V_2 = (k_1 \cos^{s-1} \omega - k_2 \sin^{s-1} \omega)^2.$$

Finally, $P_s$ is obtained from the first equation of (3.2):

$$P_s = \frac{1}{s+1}(-Q_s' + \alpha V_1') = \frac{1}{s+1} \left(-Q_s' + \frac{s+3}{2(s-1)} V_1'\right)$$

$$= 2(-k_1 \cos^{s-2} \omega \sin^{s} \omega + k_2 \sin^{s-2} \omega \cos^{s} \omega).$$

This completes the proof of the theorem.

References

[1] N. N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center of type $(R)$, Mat. Sb. 30 (72) (1952), 181–196 (in Russian); English transl.: Amer. Math. Soc. Transl. 100 (1954), 397–413.


