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A CLASS OF INTEGRABLE POLYNOMIAL VECTOR FIELDS

Abstract. We study the integrability of two-dimensional autonomous systems in the plane of the form $\dot{x} = -y + X_s(x, y)$, $\dot{y} = x + Y_s(x, y)$, where $X_s(x, y)$ and $Y_s(x, y)$ are homogeneous polynomials of degree s with $s \geq 2$. First, we give a method for finding polynomial particular solutions and next we characterize a class of integrable systems which have a null divergence factor given by a quadratic polynomial in the variable $(x^2 + y^2)^{s/2-1}$ with coefficients being functions of $\tan^{-1}(y/x)$.

1. Introduction. We consider two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = -y + X_s(x, y), \quad \dot{y} = x + Y_s(x, y), \tag{1.1}$$

where

$$X_s(x, y) = \sum_{k=0}^{s} a_k x^k y^{s-k}, \quad Y_s(x, y) = \sum_{k=0}^{s} b_k x^k y^{s-k}$$

are homogeneous polynomials of degree s, with $s \geq 2$, and with a_k and b_k, $k = 0, 1, \ldots, s$, being arbitrary real coefficients. Recently, these systems have been studied by several authors (see for instance [1], [3], [5], [6] and [8]), especially in order to obtain information about the number of small amplitude limit cycles and to determine the cyclicity of the origin (see for instance [4] and [7]). In this paper we study their integrability.

Our aim is to find solutions $W(x, y) = 0$ of system (1.1), where $W(x, y)$ is a null divergence factor (this notion will be defined below). In Theorem 1 we give an explicit method for obtaining such a factor, which is used in Theorem 2 to construct a particular class of integrable fields.

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We can write system (1.1) (see [2]) in polar coordinates $x = r\cos\varphi$ and $y = r\sin\varphi$ as

$$\dot r = P_s(\varphi) r^s, \qquad \dot\varphi = 1 + Q_s(\varphi) r^{s-1}, \tag{1.2}$$

where $P_s(\varphi)$ and $Q_s(\varphi)$ are trigonometric polynomials of the form

$$\begin{aligned} P_s(\varphi) &= R_{s+1}\cos((s+1)\varphi+\varphi_{s+1}) + R_{s-1}\cos((s-1)\varphi+\varphi_{s-1}) \\ &\quad + \ldots + \begin{cases} R_1\cos(\varphi+\varphi_1) & \text{if } s \text{ is even,} \\ R_0 & \text{if } s \text{ is odd,} \end{cases} \\ Q_s(\varphi) &= -R_{s+1}\sin((s+1)\varphi+\varphi_{s+1}) + \overline{R}_{s-1}\sin((s-1)\varphi+\overline{\varphi}_{s-1}) \\ &\quad + \ldots + \begin{cases} \overline{R}_1\sin(\varphi+\overline{\varphi}_1) & \text{if } s \text{ is even,} \\ \overline{R}_0 & \text{if } s \text{ odd,} \end{cases} \end{aligned}$$

where R_i, $\overline{R}_i$, φ_i and $\overline{\varphi}_i$ are real constants.

If we make the change of variable $R = r^{s-1}$, then system (1.2) becomes

$$\dot R = (s-1) P_s(\varphi) R^2, \qquad \dot\varphi = 1 + Q_s(\varphi) R. \tag{1.3}$$

In the study and determination of the first integrals for quadratic systems and homogeneous cubic systems (see [2]), we used a technique consisting in the research of polynomial particular solutions of system (1.3) of the form

$$V(R,\varphi) = 1 + V_1(\varphi)R + V_2(\varphi)R^2 + \ldots + V_p(\varphi)R^p = 0, \tag{1.4}$$

where $V_k(\varphi)$, $k = 1,\ldots,p$, are homogeneous trigonometric polynomials of degree $k(s-1)$ in the variables $\cos\varphi$ and $\sin\varphi$. The main results are the following.

A function $W(x,y)$ will be called a *null divergence factor* for system (1.1) if $W(x,y) = 0$ is a particular solution for this system and the divergence of the vector field

$$C = \left(\frac{-y + X_s(x,y)}{W(x,y)}, \frac{x + Y_s(x,y)}{W(x,y)} \right)$$

defined on $\mathbb{R}^2 \setminus \{(x,y) : W(x,y) = 0\}$ is zero.

We notice that if the divergence of a vector field is zero then system (1.1) defined for this vector field is integrable. In particular, if system (1.1) has a null divergence factor then this system is integrable and the origin is a center.

By using the functions x_i, $i = 1,\ldots,p$, defined implicitly by

$$V_1 = \sum_{j=1}^{p} x_j, \quad V_2 = \sum_{\substack{j,k=1 \\ j<k}}^{p} x_j x_k, \quad \ldots, \quad V_p = x_1 x_2 \ldots x_p, \tag{1.5}$$

the function (1.4) can be written as $V(R,\varphi) = \prod_{i=1}^{p}(1 + x_i(\varphi)R)$.

THEOREM 1. *If*

$$(V(R,\varphi))^\alpha = \Big(\prod_{i=1}^{p}(1+x_i(\varphi)R)\Big)^\alpha$$

is a null divergence factor for system (1.1) *with* α *a real number*, *then the functions* $x_i(\varphi)$, $i = 1,\dots,p$, *satisfy the following system of differential equations*:

$$\frac{dx_i}{dz} = \frac{\dfrac{x_i}{z-x_i}}{\dfrac{s+1}{s-1}+\alpha\displaystyle\sum_{j=1}^{p}\frac{x_j}{z-x_j}}, \qquad i=1,\dots,p, \tag{1.6}$$

where $z = Q_s(\varphi)$.

For $p = 2$ it is possible to find the general solution for system (1.6) and therefore to determine a null divergence factor for system (1.1). In this case, we have

THEOREM 2. *For* $s \in \mathbb{N}$ *with* $s \ge 2$, *and arbitrary* $k_1, k_2, \varphi_0 \in \mathbb{R}$, *the system of the form* (1.2) *with*

$$\begin{aligned} P_s(\varphi) &= 2(-k_1\cos^{s-2}(\varphi+\varphi_0)\sin^3(\varphi+\varphi_0)\\ &\quad + k_2\sin^{s-2}(\varphi+\varphi_0)\cos^3(\varphi+\varphi_0)),\\ Q_s(\varphi) &= (k_1\cos^{s-1}(\varphi+\varphi_0) - k_2\sin^{s-1}(\varphi+\varphi_0))\cos 2(\varphi+\varphi_0), \end{aligned} \tag{1.7}$$

is integrable.

In cartesian coordinates $x = r\cos(\varphi+\varphi_0)$ and $y = r\sin(\varphi+\varphi_0)$, we can write system (1.7) in the form

$$\begin{aligned} \dot{x} &= -y - k_1x^{s-1}y + k_2y^{s-2}(2x^2-y^2),\\ \dot{y} &= x + k_1x^{s-2}(x^2-2y^2) + k_2xy^{s-1}, \end{aligned} \tag{1.8}$$

with $s \in \mathbb{N}$ and $s \ge 2$. We notice that the origin is a center for system (1.8).

In Section 2 we give a method of obtaining particular solutions of system (1.1). Theorem 1 is proved in Section 3. Finally, Theorem 2 is proved in Section 4.

2. Particular solutions

PROPOSITION 1. *The function* (1.4) *is a particular solution of system* (1.3) *if the homogeneous trigonometric polynomials* $V_k(\varphi)$, $k = 1,\dots,p$, *satisfy the following differential system*:

$$\begin{aligned} V'_{k+1} + V'_kQ_s + k(s-1)V_kP_s &= V_kV'_1, \qquad k=1,\dots,p-1,\\ V'_pQ_s + p(s-1)V_pP_s &= V_pV'_1. \end{aligned} \tag{2.1}$$

where $' = d/d\varphi$.

Proof. If we force $V(R,\varphi)=0$ to be a particular solution of system (1.3), then it must satisfy

$$\begin{aligned}(2.2)\qquad \dot V(R,\varphi)&=\lambda(R,\varphi)V(R,\varphi)\\ &=\lambda(R,\varphi)(1+V_1(\varphi)R+V_2(\varphi)R^2+\ldots+V_p(\varphi)R^p).\end{aligned}$$

Differentiating $V(R,\varphi)$ with respect to t we get

$$\begin{aligned}(2.3)\qquad \dot V(R,\varphi)&=\frac{\partial V}{\partial R}((s-1)P_s(\varphi)R)+\frac{\partial V}{\partial\varphi}(1+Q_s(\varphi)R^2)\\ &=(V_1(\varphi)+2V_2(\varphi)R+\ldots+pV_p(\varphi)R^{p-1})((s-1)P_s(\varphi)R^2)\\ &\quad+(V_1'(\varphi)R+V_2'(\varphi)R^2+\ldots+V_p'(\varphi)R^p)(1+Q_s(\varphi)R)\\ &=V_1'(\varphi)R+\sum_{k=1}^{p-1}(V_{k+1}'(\varphi)\\ &\quad+V_k'(\varphi)Q_s(\varphi)+k(s-1)V_k(\varphi)P_s(\varphi))R^{k+1}\\ &\quad+(V_p'(\varphi)Q_s(\varphi)+p(s-1)V_p(\varphi)P_s(\varphi))R^{p+1},\end{aligned}$$

and if we equate the terms on the right-hand side of (2.2) and (2.3) it results first in $\lambda(R,\varphi)=V_1'(\varphi)R$, and considering this relationship we obtain (2.1). We notice that $V(R,\varphi)$ satisfies $\dot V=(V_1'(\varphi)R)V$. ∎

Proposition 2. *In order to find a particular solution of system* (1.1) *of the form* (1.4) *it is sufficient to find p different solutions of the differential equation*

$$(x-Q_s(\varphi))\frac{dx}{d\varphi}=(s-1)P_s(\varphi)x$$

so that the functions $V_k(\varphi)$, $k=1,\ldots,p$, obtained from relations (1.6) *be homogeneous trigonometric polynomials of degree $k(s-1)$.*

Proof. By using the functions x_i introduced in (1.5) we define

$$(2.4)\qquad V_0^i=1,\quad V_1^i=\sum_{\substack{j=1\\ j\ne i}}^{p}x_j,\quad V_2^i=\sum_{\substack{j,k=1\\ j<k\\ j,k\ne i}}^{p}x_jx_k,\quad\ldots$$
$$\ldots,\quad V_{p-1}^i=x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_p,$$

for $i=1,\ldots,p$. From (1.5) and (2.4) we get

$$V_k=\frac{1}{k}\sum_{i=1}^{p}V_{k-1}^ix_i,\qquad V_k'=\sum_{i=1}^{p}V_{k-1}^ix_i',$$

where $k = 1, \dots, p$. Then we can write system (1.5) as

$$\sum_{i=1}^{p} V_k^i x_i' + \Big(\sum_{i=1}^{p} V_{k-1}^i x_i'\Big) Q_s + (s-1)\Big(\sum_{i=1}^{p} V_{k-1}^i x_i\Big) P_s = V_k\Big(\sum_{i=1}^{p} x_i'\Big),$$
$$\Big(\sum_{i=1}^{p} V_{p-1}^i x_i'\Big) Q_s + (s-1)\Big(\sum_{i=1}^{p} V_{p-1}^i x_i\Big) P_s = V_p\Big(\sum_{i=1}^{p} x_i'\Big),$$

where $k = 1, \dots, p-1$. On the other hand, from the equalities

$$V_k \sum_{i=1}^{p} x_i' - \sum_{i=1}^{p} V_k^i x_i' = \sum_{i=1}^{p} (V_k - V_k^i) x_i' = \sum_{i=1}^{p} V_{k-1}^i x_i x_i', \qquad k = 1, \dots, p-1,$$
$$V_p \sum_{i=1}^{p} x_i' = \sum_{i=1}^{p} V_p x_i' = \sum_{i=1}^{p} V_{p-1}^i x_i x_i',$$

system (2.4) can be written as

$$\begin{aligned} &\sum_{i=1}^{p} V_{k-1}^i (x_i' Q_s + (s-1) x_i P_s - x_i x_i') = 0, \qquad k = 1, \dots, p-1, \\ &\sum_{i=1}^{p} V_{p-1}^i (x_i' Q_s + (s-1) x_i P_s - x_i x_i') = 0. \end{aligned} \tag{2.5}$$

If we set $X_i = x_i' Q_s + (s-1) x_i P_s - x_i x_i'$, $i = 1, \dots, p$, system (2.5) becomes

$$\sum_{i=1}^{p} V_{k-1}^i X_i = 0, \qquad k = 1, \dots, p. \tag{2.6}$$

This is a linear system of p equations in the variables $X_1, \dots, X_p$. The matrix A of system (2.6) is given by

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ V_1^1 & V_1^2 & \dots & V_1^p \\ \dots & \dots & \dots & \dots \\ V_{p-1}^1 & V_{p-1}^2 & \dots & V_{p-1}^p \end{pmatrix},$$

and a straightforward computation shows that

$$\det A = \prod_{\substack{i,j=1 \\ i<j}}^{p} (x_i - x_j).$$

If we assume that the variables $x_1, x_2, \dots, x_p$ are pairwise different, then $\det A \neq 0$. Therefore, the only possible solution of the linear system (2.6) is $X_i = 0$ for $i = 1, \dots, p$, that is,

$$x_i' Q_s + (s-1) x_i P_s - x_i x_i' = 0, \qquad i = 1, \dots, p.$$

This completes the proof. ■

We note that system (1.3) and the system of Proposition 2 are equivalent if we make the change of variable $R=-1/x$.

3. Null divergence factors. If system (1.1) is written in polar coordinates (see (1.2)), then the function $(V(R,\varphi))^\alpha$ is a null divergence factor for system (1.1) if

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{P_s(\varphi)r^{s+1}}{(V(R,\varphi))^\alpha}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1+Q_s(\varphi)r^{s-1}}{(V(R,\varphi))^\alpha}\right)=0. \tag{3.1}$$

Now assume that the function $V(R,\varphi)$ is of the form given in (1.4) with $R=r^{s-1}$. Then if we develop the expression (3.1) with respect to the powers of R, we have

$$\begin{gathered}(s+1)P_s+Q_s'-\alpha V_1'=0,\\ V_{k+1}'+V_k'Q_s+k(s-1)V_kP_s=V_kV_1',\qquad k=1,\dots,p-1,\\ V_p'Q_s+p(s-1)V_pP_s=V_pV_1'.\end{gathered} \tag{3.2}$$

System (3.2) coincides with system (1.5) except that the value of V_1 in system (3.2) is determined as a function of $P_s(\varphi),Q_s'(\varphi)$ and α.

P r o o f o f T h e o r e m 1. From (1.6), Proposition 2 and (2.5), system (3.2) takes the form

$$x_i'=\frac{(s-1)P_sx_i}{x_i-Q_s},\qquad i=1,\dots,p, \tag{3.3}$$

with the condition

$$(s+1)P_s+Q_s'-\alpha\sum_{i=1}^{p}x_i'=0. \tag{3.4}$$

If we take $z=Q_s$ as independent variable instead of φ, then we have

$$\frac{dx_i}{d\varphi}=\frac{dx_i}{dQ_s}\frac{dQ_s}{d\varphi},\qquad i=1,\dots,p,$$

and (3.4) goes over to

$$(s+1)P_s+Q_s'-\alpha Q_s'\sum_{i=1}^{p}\frac{dx_i}{dz}=0,$$

which gives

$$P_s=\frac{1}{s+1}\left(\alpha\sum_{i=1}^{p}\frac{dx_i}{dz}-1\right)Q_s'. \tag{3.5}$$

By inserting the expression (3.5) in system (3.3), and considering the change of variable $z = Q_s$ we can write

$$\frac{dx_i}{dz} = \frac{s-1}{s+1} \cdot \frac{\alpha \sum_{j=1}^{p} \frac{dx_j}{dz} - 1}{x_i - z} x_i, \qquad i = 1, \ldots, p.$$

Then, isolating dx_i/dz, $i = 1, \ldots, p$, in the above system we get

$$\frac{dx_i}{dz} = \frac{\prod_{\substack{j=1 \\ j \neq i}}^{p} (z - x_j)}{\frac{s+1}{s-1} \prod_{j=1}^{p} (z - x_j) + \alpha \sum_{j=1}^{p} \Big(\prod_{\substack{k=1 \\ k \neq j}}^{p} (z - x_k) \Big) x_j} x_i, \qquad i = 1, \ldots, p.$$

If we divide the numerator and denominator of this fraction by the product $\prod_{j=1}^{p} (z - x_j)$, we obtain system (1.6). ■

Note that system (1.6) is symmetric with respect to the variables x_i, $i = 1, \ldots, p$.

We want to find functions of the form

$$U(x_1, \ldots, x_p, z) \equiv H(x_1, \ldots, x_p) + zG(x_1, \ldots, x_p) \tag{3.6}$$

so that, for system (1.6), $dU/dz = 0$.

Proposition 3. *In order to find functions of the form* (3.6) *for system* (1.6) *it is sufficient to find solutions of the partial differential system*

$$\begin{gathered} \frac{\partial H}{\partial x_i} + x_i \frac{\partial G}{\partial x_i} + \alpha G = 0, \qquad i = 1, \ldots, p, \\ \sum_{i=1}^{p} x_i \frac{\partial G}{\partial x_i} + \frac{s+1}{s-1} G = 0. \end{gathered} \tag{3.7}$$

Proof. If we differentiate (3.6) with respect to z, we have

$$\frac{dU}{dz} = \sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{dx_i}{dz} + z \sum_{i=1}^{p} \frac{\partial G}{dx_i} \frac{dx_i}{dz} + G = 0.$$

By replacing the value of dx_i/dz, $i = 1, \ldots, p$, given in (1.6) in the previous expression it becomes

$$\sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{x_i}{z - x_i} + z \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} \frac{x_i}{z - x_i} + \left(\frac{s+1}{s-1} + \alpha \sum_{i=1}^{p} \frac{x_i}{z - x_i} \right) G = 0,$$

or

$$\sum_{i=1}^{p}\frac{\partial H}{\partial x_i}\frac{x_i}{z-x_i}+\sum_{i=1}^{p}\frac{\partial G}{\partial x_i}\left(x_i+\frac{x_i^2}{z-x_i}\right)+\left(\frac{s+1}{s-1}+\alpha\sum_{i=1}^{p}\frac{x_i}{z-x_i}\right)G$$
$$=\sum_{i=1}^{p}\frac{x_i\dfrac{\partial H}{\partial x_i}+x_i^2\dfrac{\partial G}{\partial x_i}+\alpha x_i G}{z-x_i}+\sum_{i=1}^{p}x_i\frac{\partial G}{\partial x_i}+\frac{s+1}{s-1}G=0.$$

In order that this last expression be null it is sufficient that conditions (3.7) hold. Notice that these conditions are not necessary in order that the previous expression be null. ∎

4. Quadratic null divergence factors. We now consider the case $p=2$, that is to say,

$$V(R,\varphi)=1+V_1(\varphi)R+V_2(\varphi)R^2.$$

In this case, system (3.2) takes the form

$$\begin{aligned}(s+1)P_s+Q_s'-\alpha V_1'&=0,\\ V_2'+V_1'Q_s+(s-1)V_1P_s&=V_1V_1',\\ V_2'Q_s+2(s-1)V_2P_s&=V_2V_1',\end{aligned}$$

where $V_1=x_1+x_2$, $V_2=x_1x_2$, and system (1.7) goes over to

$$\frac{dx_1}{dz}=\frac{\dfrac{x_1}{z-x_1}}{a+\alpha\left(\dfrac{x_1}{z-x_1}+\dfrac{x_2}{z-x_2}\right)},\qquad \frac{dx_2}{dz}=\frac{\dfrac{x_2}{z-x_2}}{a+\alpha\left(\dfrac{x_1}{z-x_1}+\dfrac{x_2}{z-x_2}\right)}, \tag{4.1}$$

with $a=(s+1)/(s-1)$.

In this case, we want to obtain functions of the form

$$U(x_1,x_2,z)=H(x_1,x_2)+zG(x_1,x_2) \tag{4.2}$$

so that, for system (4.1), $dU/dz=0$.

By applying Proposition 3, the functions $H(x_1,x_2)$ and $G(x_1,x_2)$ have to satisfy the system

$$\begin{aligned}\frac{\partial H}{\partial x_1}+x_1\frac{\partial G}{\partial x_1}+\alpha G&=0,\\ \frac{\partial H}{\partial x_2}+x_2\frac{\partial G}{\partial x_2}+\alpha G&=0,\\ x_1\frac{\partial G}{\partial x_1}+x_2\frac{\partial G}{\partial x_2}+aG&=0.\end{aligned} \tag{4.3}$$

If we make the change of variable $u = x_2/x_1$ and we take the functions G and H as follows:

$$G(x_1, x_2) = x_1^{-a} g(u), \qquad H(x_1, x_2) = x_1^{1-a} h(u),$$

then the third equation of system (4.3) is satisfied identically, and the system takes the form

$$\begin{aligned} &\alpha g + u\frac{dg}{du} + \frac{dh}{du} = 0, \\ (4.4) \qquad &(1-a)h = [(a-\alpha) - \alpha u]g + u(1-u)\frac{dg}{du}. \end{aligned}$$

If we differentiate the second equation of (4.4) with respect to u, we have

$$(1-a)\frac{dh}{du} = -\alpha g + [(a-\alpha+1) - (\alpha+2)u]\frac{dg}{du} + u(1-u)\frac{d^2g}{du^2}.$$

By replacing the value of dh/du obtained from the first equation of system (4.4) in the previous expression, we find

$$(4.5) \qquad u(1-u)\frac{d^2g}{du^2} + [(1+a-\alpha) - (1+\alpha+a)u]\frac{dg}{du} - a\alpha g = 0.$$

The relation (4.5) is a hypergeometric second order linear differential equation. We will study it for the particular case $a - \alpha = 1/2$. This relation is satisfied by certain integrable systems (1.1) in the quadratic case $s = 2$.

Since $a - \alpha = 1/2$, the equation (4.5) can be written as

$$u(1-u)\frac{d^2g}{du^2} + \left(\frac{3}{2} - \left(2a + \frac{1}{2}\right)u\right)\frac{dg}{du} - a\left(a - \frac{1}{2}\right)g = 0.$$

The general solution of this equation is given by

$$g(u) = u^{-1/2}[C_1(1+\sqrt{u})^{2(1-a)} + C_2(1-\sqrt{u})^{2(1-a)}],$$

where C_1 and C_2 are arbitrary constants. For this $g(u)$ we have

$$h(u) = C_1(1+\sqrt{u})^{2(1-a)} - C_2(1-\sqrt{u})^{2(1-a)}.$$

By going back through the change of variables it is easy to see that

$$\begin{aligned} G(x_1, x_2) &= (x_1x_2)^{-1/2}(C_1(\sqrt{x_1} + \sqrt{x_2})^{2(1-a)} + C_2(\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}), \\ H(x_1, x_2) &= C_1(\sqrt{x_1} + \sqrt{x_2})^{2(1-a)} - C_2(\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}. \end{aligned}$$

Therefore

$$\begin{aligned} U_1(x_1, x_2, z) &= (\sqrt{x_1} + \sqrt{x_2})^{2(1-a)}(1 + z/\sqrt{x_1x_2}), \\ U_2(x_1, x_2, z) &= (\sqrt{x_1} - \sqrt{x_2})^{2(1-a)}(1 - z/\sqrt{x_1x_2}) \end{aligned}$$

are two independent functions of the form (4.2) for system (4.1), which we can write in the form

$$U_1(x_1,x_2,z)=(x_1+x_2+2\sqrt{x_1x_2})^{1-a}(1+z/\sqrt{x_1x_2}),$$
$$U_2(x_1,x_2,z)=(x_1+x_2-2\sqrt{x_1x_2})^{1-a}(1-z/\sqrt{x_1x_2}).$$

As $V_1=x_1+x_2$, $V_2=x_1x_2$, $z=Q_s$ and $a=(s+1)/(s-1)$ we can write

$$U_1(V_1,V_2,z)=(V_1+2\sqrt{V_2})^{-2/(s-1)}(1+Q_s/\sqrt{V_2}),$$
$$U_2(V_1,V_2,z)=(V_1-2\sqrt{V_2})^{-2/(s-1)}(1-Q_s/\sqrt{V_2}),$$

that is,

$$\begin{aligned}(V_1+2\sqrt{V_2})^{-2(s-1)}(1+Q_s/\sqrt{V_2})&=K_1,\\(V_1-2\sqrt{V_2})^{-2/(s-1)}(1-Q_s/\sqrt{V_2})&=K_2,\end{aligned}\tag{4.5}$$

where K_1 and K_2 are arbitrary constants.

Proof of Theorem 2. We can write system (4.5) as

$$\begin{aligned}V_1+2\sqrt{V_2}&=\overline{K}_1(1+Q_s/\sqrt{V_2})^{(s-1)/2},\\V_1-2\sqrt{V_2}&=\overline{K}_2(1-Q_s/\sqrt{V_2})^{(s-1)/2}.\end{aligned}\tag{4.6}$$

By multiplying the two equations, we have

$$V_1^2-4V_2=\overline{K}_1\overline{K}_2(1-Q_s^2/V_2)^{(s-1)/2}.\tag{4.7}$$

As V_1 and V_2 are homogeneous trigonometric polynomials of degrees $s-1$ and $2(s-1)$ respectively, the left-hand side of (4.7) is a trigonometric polynomial of degree $2(s-1)$. So the right-hand side of (4.7) must have the same degree. In particular, V_2 is a divisor of Q_s^2. On the other hand, if we square the first equation of (4.6), and we develop the right-hand side of that equation according to the Newton binomial, and group the terms with or without the factors $\sqrt{V_2}$, we see that $\sqrt{V_2}$ is a homogeneous trigonometric polynomial of degree $s-1$, and a divisor of Q_s. Hence $X_2=Q_s/\sqrt{V_2}$ is a homogeneous trigonometric polynomial of degree $(s+1)-(s-1)=2$, and we can write system (4.6) as

$$\begin{aligned}V_1+2\sqrt{V_2}&=\overline{K}_1(1+X_2)^{(s-1)/2},\\V_1-2\sqrt{V_2}&=\overline{K}_2(1-X_2)^{(s-1)/2}.\end{aligned}\tag{4.8}$$

By subtracting both equations of system (4.8), we have

$$4\sqrt{V_2}=\overline{K}_1(1+X_2)^{(s-1)/2}-\overline{K}_2(1-X_2)^{(s-1)/2},$$

and then

$$Q_s=X_2\sqrt{V_2}=\frac{1}{4}X_2(\overline{K}_1(1+X_2)^{(s-1)/2}-\overline{K}_2(1-X_2)^{(s-1)/2}).$$

If s is even, the trigonometric polynomials $1+X_2$ and $1-X_2$ must be the squares of first degree homogeneous trigonometric polynomials in order to satisfy system (4.8). In the case where s is odd, this condition is not necessary, but we can also impose it. We can easily prove that

$$X_2(\varphi)=\cos 2(\varphi+\varphi_0)=\cos 2\omega,$$

where φ_0 is arbitrary; it follows that

$$1+X_2=2\cos^2\omega,\qquad 1-X_2=2\sin^2\omega, \tag{4.9}$$

and

$$Q_s(\varphi)=(k_1\cos^{s-1}\omega-k_2\sin^{s-1}\omega)\cos 2\omega,$$

where $k_1=\frac{1}{4}\overline{K}_1 2^{(s-1)/2}$, $k_2=\frac{1}{4}\overline{K}_2 2^{(s-1)/2}$. By inserting the values obtained in (4.9) into system (4.8) we have

$$V_1+2\sqrt{V_2}=4k_1\cos^{s-1}\omega,\qquad V_1-2\sqrt{V_2}=4k_2\sin^{s-1}\omega.$$

Therefore we obtain

$$V_1=2(k_1\cos^{s-1}\omega+k_2\sin^{s-1}\omega),\qquad V_2=(k_1\cos^{s-1}\omega-k_2\sin^{s-1}\omega)^2.$$

Finally, P_s is obtained from the first equation of (3.2):

$$\begin{aligned}P_s&=\frac{1}{s+1}(-Q_s'+\alpha V_1')=\frac{1}{s+1}\bigg(-Q_s'+\frac{s+3}{2(s-1)}V_1'\bigg)\\&=2(-k_1\cos^{s-2}\omega\sin^3\omega+k_2\sin^{s-2}\omega\cos^3\omega).\end{aligned}$$

This completes the proof of the theorem. ∎

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