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RECURRENCE RELATIONS  
WITH PERIODIC COEFFICIENTS  
AND CHEBYSHEV POLYNOMIALS

*Abstract.* We show that polynomials defined by recurrence relations with periodic coefficients may be represented with the help of Chebyshev polynomials of the second kind.

**Introduction.** The class of orthogonal polynomials studied in this paper served as a starting point for several authors [1–4] when studying more general classes of orthogonal polynomials or continued fractions. The aim of this note is to show that, on the other hand, this class can be described with the help of the classical Chebyshev polynomials.

Let  $\{P_n\}_n$ ,  $\deg P_n = n$ , be a sequence of polynomials defined by a three-term recurrence

$$(1) \quad \begin{aligned} \forall n > 0: \quad P_n(z) &= (z + b_{n-1})P_{n-1}(z) - a_{n-1}P_{n-2}(z), \\ P_{-1} &\equiv 0, \quad P_0 \equiv 1, \end{aligned}$$

where the coefficients are periodic with period  $p$ :

$$(2) \quad \forall n \geq 0: \quad a_{n+p} = a_n, \quad b_{n+p} = b_n.$$

Note that for convenience we define  $a_0 := a_p$ . The case  $p = 1$  immediately leads to the classical Chebyshev polynomials. We will show that, for arbitrary  $p$ , the polynomials  $P_n$  may also be expressed with the help of the Chebyshev polynomials of the second kind. For the special case  $p = 2$  and  $b_n = 0$ , such a representation has been given in [1, p. 91]. The case of asymptotically periodic recurrence coefficients was studied in [2].

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The results of the first section do not require any additional assumptions, hence they remain valid also in the case of complex coefficients. In contrast, for our main result of Section II we will restrict ourselves to the case of real coefficients.

**I. Some properties induced by the periodicity.** In the considerations to follow we will often omit the argument and write simply  $P_n$  instead of  $P_n(z)$ .

PROPERTY 1. *The polynomial  $P_{p-1}$  divides the polynomial  $P_{2p-1}$ :*

$$(3) \quad \forall p > 0: \quad P_{2p-1}(z) = Q_p(z)P_{p-1}(z),$$

where

$$(4) \quad \begin{aligned} Q_1(z) &= P_1(z), \\ Q_2(z) &= P_2(z) - a_0, \\ Q_p(z) &= P_p(z) - a_0 \left[ \prod_{i=1}^{p-2} (z + b_i) + R_{p-4}(z) \right], \quad p \geq 3, \end{aligned}$$

and  $R_k$  is a polynomial of degree  $k$  ( $R_{-1} \equiv 0$ ).

Proof. The case  $p = 1$  is trivial. For  $p \geq 2$ , using (1) and (2) we obtain

$$\begin{aligned} \begin{vmatrix} P_{2p-1} & P_p \\ P_{p-1} & P_0 \end{vmatrix} &= \begin{vmatrix} (z + b_{2p-2})P_{2p-2} - a_{2p-2}P_{2p-3} & P_p \\ (z + b_{p-2})P_{p-2} - a_{p-2}P_{p-3} & P_0 \end{vmatrix} \\ &= (z + b_{p-2}) \begin{vmatrix} P_{2p-2} & P_p \\ P_{p-2} & P_0 \end{vmatrix} - a_{p-2} \begin{vmatrix} P_{2p-3} & P_p \\ P_{p-3} & P_0 \end{vmatrix}. \end{aligned}$$

In the case  $p = 2$ , the first term vanishes, and the assertion follows immediately:

$$\begin{vmatrix} P_3 & P_2 \\ P_1 & 1 \end{vmatrix} = -a_0 \begin{vmatrix} P_1 & P_2 \\ 0 & 1 \end{vmatrix}, \quad \text{that is, } P_3 = (P_2 - a_0)P_1.$$

If  $p \geq 3$ , we apply the recurrence relation (1) for  $P_{2p-2}$  and  $P_{p-2}$ , so that together with the periodicity (2) the above expression can be rewritten as

$$[(z + b_{p-2})(z + b_{p-3}) - a_{p-2}] \begin{vmatrix} P_{2p-3} & P_p \\ P_{p-3} & P_0 \end{vmatrix} - a_{p-3}(z + b_{p-2}) \begin{vmatrix} P_{2p-4} & P_p \\ P_{p-4} & P_0 \end{vmatrix}.$$

As before, for  $p = 3$  the left hand term vanishes yielding the assertion, whereas for  $p > 3$  we continue to transform the left hand term by applying the recurrence relation (1) for  $P_{2p-3}$  and  $P_{p-3}$ . The assertion (3) for arbitrary  $p$  follows after  $p - 1$  such transformations. ■

Later we shall obtain a more general result than (3) (formula (14)):

PROPERTY 1 (generalized). *The polynomial  $P_{p-1}$  divides the polynomial  $P_{kp-1}$  for any positive integer  $k$ .*

PROPERTY 2.

$$(5) \quad \forall n \geq -1 : \quad \begin{vmatrix} P_{2p+n}(z) & P_{2p-1}(z) \\ P_{p+n}(z) & P_{p-1}(z) \end{vmatrix} = -aP_{p-1}(z)P_n(z),$$

where  $a = \prod_{i=0}^{p-1} a_i$ .

Proof. Property 2 will be shown by induction on  $n \geq -1$ , the case  $n = -1$  is trivial. For  $n = 0$ , we get with the help of (1) and (2),

$$\begin{aligned} \begin{vmatrix} P_{2p} & P_{2p-1} \\ P_p & P_{p-1} \end{vmatrix} &= \begin{vmatrix} (z + b_{p-1})P_{2p-1} & P_{2p-1} \\ (z + b_{p-1})P_{p-1} & P_{p-1} \end{vmatrix} + a_{p-1} \begin{vmatrix} P_{2p-1} & P_{2p-2} \\ P_{p-1} & P_{p-2} \end{vmatrix} \\ &= \dots = \prod_{i=0}^{p-1} a_i \begin{vmatrix} P_p & P_{p-1} \\ P_0 & P_{-1} \end{vmatrix} \equiv -aP_{p-1}. \end{aligned}$$

Supposing now that assertion (5) holds for  $n \leq k$ , one obtains for  $n = k + 1$ :

$$\begin{aligned} \begin{vmatrix} P_{2p+k+1} & P_{2p-1} \\ P_{p+k+1} & P_{p-1} \end{vmatrix} &= (z + b_k) \begin{vmatrix} P_{2p+k} & P_{2p-1} \\ P_{p+k} & P_{p-1} \end{vmatrix} - a_k \begin{vmatrix} P_{2p+k-1} & P_{2p-1} \\ P_{p+k} & P_{p-1} \end{vmatrix} \\ &= -a(z + b_k)P_{p-1}P_k + aa_kP_{p-1}P_{k-1} = -aP_{p-1}P_{k+1}. \quad \blacksquare \end{aligned}$$

PROPERTY 3.

$$(6) \quad \forall n \geq -1 : \quad P_{2p+n}(z) = Q_p(z)P_{p+n}(z) - aP_n(z).$$

This is clearly formula (5) modified by using (4). The similar result was already derived in [2] and [3].

PROPERTY 4. *Let the sequence of polynomials  $\{V_n\}_n$  be defined by*

$$(7) \quad \forall n \geq 0 : \quad V_{n+1}(z) = zV_n(z) - aV_{n-1}(z), \quad V_{-1} \equiv 0, \quad V_0 \equiv 1.$$

Then

$$(8) \quad \forall n \geq -1, \forall k \geq 1 : \quad P_{kp+n}(z) = P_{p+n}(z)V_{k-1}[Q_p(z)] - aP_n(z)V_{k-2}[Q_p(z)].$$

Proof. The assertion will be shown by induction on  $k \geq 1$ . The case  $k = 1$  is trivial, and for  $k = 2$  equation (8) coincides with (6). Suppose the result is true for  $k$  and let us verify it for  $k + 1$ . Replacing  $n$  by  $n + p$  in (8) yields

$$\begin{aligned} P_{(k+1)p+n}(z) &= P_{kp+(n+p)}(z) \\ &= P_{2p+n}(z)V_{k-1}[Q_p(z)] - aP_{n+p}(z)V_{k-2}[Q_p(z)]. \end{aligned}$$

By (6) this equals

$$\begin{aligned}
& [Q_p(z)P_{p+n}(z) - aP_n(z)]V_{k-1}[Q_p(z)] - aP_{p+n}(z)V_{k-2}[Q_p(z)] \\
&= P_{p+n}(z)\{Q_p(z)V_{k-1}[Q_p(z)] - aV_{k-2}[Q_p(z)]\} - aP_n(z)V_{k-1}[Q_p(z)] \\
&= P_{p+n}(z)V_k[Q_p(z)] - aP_n(z)V_{k-1}[Q_p(z)],
\end{aligned}$$

leading to the result for  $k + 1$ . ■

**II. A connection between the polynomials  $P_n$  and the Chebyshev polynomials of the second kind  $U_n$ .** In the sequel, the periodic coefficients  $a_n$  are supposed to be positive, and  $b_n$  real:

$$(9) \quad \exists p \geq 1, \forall n \geq 0 : \quad a_{n+p} = a_n > 0, \quad b_{n+p} = b_n \in \mathbb{R}.$$

The Chebyshev polynomials of the second kind  $U_n$  are defined by the recurrence relation

$$(10) \quad \forall n \geq 0 : \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad U_{-1} \equiv 0, \quad U_0 \equiv 1,$$

or, explicitly, by

$$U_n(x) = \frac{\sin[(n+1)\arccos(x)]}{\sin[\arccos(x)]}, \quad x \in [-1, +1].$$

In fact, the polynomials  $V_n$  introduced in (7) coincide, up to a simple transformation, with the Chebyshev polynomials of the second kind:

$$(11) \quad V_n(x) = (\sqrt{a})^n U_n\left(\frac{x}{2\sqrt{a}}\right).$$

Therefore, equation (8) leads immediately to our main result:

**THEOREM.** *The polynomials  $P_j$  defined by the recurrence relation (1) with coefficients satisfying the conditions (9) can be expressed as follows:*

$$(12) \quad \forall k \geq 1, \forall n \geq -1 :$$

$$P_{kp+n}(x) = (\sqrt{a})^{k-1} \left\{ P_{p+n}(x) U_{k-1} \left[ \frac{Q_p(x)}{2\sqrt{a}} \right] - \sqrt{a} P_n(x) U_{k-2} \left[ \frac{Q_p(x)}{2\sqrt{a}} \right] \right\},$$

where  $Q_p(x) = P_{2p-1}(x)/P_{p-1}(x)$  is defined by (4) and  $a = \prod_{i=0}^{p-1} a_i$ .

**Remark 1.** As we have already mentioned in the introduction, for the particular case  $p = 1$ , the polynomials  $P_k$  coincide, up to a simple transformation, with the Chebyshev polynomials. More precisely, from (12) with  $n = 0$  we conclude that

$$\begin{aligned}
P_k(x) &= (\sqrt{a})^{k-1} \left\{ P_1(x) U_{k-1} \left[ \frac{P_1(x)}{2\sqrt{a}} \right] - \sqrt{a} U_{k-2} \left[ \frac{P_1(x)}{2\sqrt{a}} \right] \right\} \\
&= (\sqrt{a})^k \left\{ 2 \frac{P_1(x)}{2\sqrt{a}} U_{k-1} \left[ \frac{P_1(x)}{2\sqrt{a}} \right] - U_{k-2} \left[ \frac{P_1(x)}{2\sqrt{a}} \right] \right\},
\end{aligned}$$

that is,

$$(13) \quad P_k(x) = (\sqrt{a})^k U_k \left[ \frac{P_1(x)}{2\sqrt{a}} \right].$$

Remark 2. Taking  $n = -1$  in (12) leads to

$$(14) \quad P_{kp-1}(x) = (\sqrt{a})^{k-1} P_{p-1}(x) U_{k-1} \left[ \frac{Q_p(x)}{2\sqrt{a}} \right],$$

generalizing Property 1.

**Conclusion.** In the proof given above we have not required the part of the condition (9) that the coefficients  $b_j$  are assumed to be real. However, this assumption completes the conditions on the coefficients of a three-term recurrence (1) to generate a sequence of orthogonal polynomials. Finally, we notice that the result of our Theorem may simplify the study of zeros of the polynomials  $P_n$  since the zeros of the Chebyshev polynomials are explicitly known.

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