APPLICATIONES MATHEMATICAE
23,3 (1995), pp. 305–318

Z. GRUDZIEŃ and D. SZYNAL (Lublin)

CHARACTERIZATIONS OF DISTRIBUTIONS BY MOMENTS OF ORDER STATISTICS WHEN THE SAMPLE SIZE IS RANDOM

Abstract. We give characterizations of the uniform distribution in terms of moments of order statistics when the sample size is random. Special cases of a random sample size (logarithmic series, geometrical, binomial, negative binomial, and Poisson distribution) are also considered.

1. Introduction and preliminaries. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with a common distribution function F. Denote by $X_{k:n}$ the kth order statistics of a sample $(X_1, \ldots, X_n)$. We write $X_{(n)} = \max(X_1, \ldots, X_n)$ and $X_{1:n} = \min(X_1, \ldots, X_n)$.

Characterizations of distributions via moments of order statistics when the sample size is fixed were treated in a great number of papers (cf. [1], [5], [6], [7] and references there). When the sample size is random, characterizations of that type have been considered in [7]. One of the results of [7] which we intend to generalize states that for characterizing the uniform distribution in terms of order statistics in the case when the sample size N has a logarithmic series distribution the finite set $\{EX, EX_{(N)}, EX^2_{(N)}\}$ is sufficient. Recall that in the case when the sample size is fixed the $U(0,1)$ distribution is characterized by the elements of the set $\{EX_{(2)}, EX^2\}$.

The aim of this paper is to generalize the above characterizations. Namely, we show that the uniform distribution on $(0,1)$ can be characterized by $EX^2_{(N)}$ and $E\frac{N}{N+1}X_{(N+1)}$, where N is a discrete random variable with $P[N = 1] > 0$. We note that in a special case when N has a logarithmic series distribution, $U(0,1)$ can be characterized by the set

1991 *Mathematics Subject Classification*: 62E10, 60E99.
Key words and phrases: order statistics, moments, uniform distribution, logarithmic series, geometrical, negative binomial, binomial, Poisson distribution, random sample size, recurrence formula.

$\{EX^2_{(N)}, E\frac{N}{N+1}X_{(N+1)}\}$ instead of $\{EX, EX_{(N)}, EX^2_{(N)}\}$. Moreover, we discuss characterization conditions for the uniform distribution when the probability function of the sample size N is given by the recurrence formula

$$P[N=n]=(a+b/n)P[N=n-1],\quad n\in\{2,3,\ldots\};\tag{1.1}$$

$a<1, a+b\ge 0$ (cf. [4], [8]).

2. Moments of order statistics with a random index. Let $\{X_n, n\ge 1\}$ be a sequence of i.i.d. random variables obeying a distribution F. Put

$$\psi_N(t)=Et^N,\quad |t|<1,\quad F^{-1}(t)=\inf\{x:F(x)\ge t\},\quad t\in(0,1).$$

LEMMA. *Let N be a positive integer-valued random variable independent of $\{X_n, n\ge 1\}$. If the probability function of N satisfies the recurrence relation* (1.1) *and the distribution function F has a finite mth moment for $m\ge 1$, then*

$$EX^m_{(N)}=aE\frac{N}{N+1}X^m_{(N+1)}+(a+b)E\frac{X^m_{(N+1)}}{N+1}+P[N=1]EX^m\tag{2.1}$$

and

$$EX^m_{1:N}=aE\frac{N}{N+1}X^m_{1:N+1}+(a+b)E\frac{X^m_{1:N+1}}{N+1}+P[N=1]EX^m.\tag{2.2}$$

Proof. Note that

$$EX^m_{(n)}=n\int_0^1(F^{-1}(t))^m t^{n-1}\,dt\qquad\text{(cf. [2]).}$$

Then

$$\begin{aligned}EX^m_{(N)}&=\sum_{n=1}^\infty E(\max(X_1,\ldots,X_n))^m P[N=n]\\&=\sum_{n=1}^\infty n\int_0^1(F^{-1}(t))^m t^{n-1}\,dt\,P[N=n]\\&=P[N=1]EX^m+a\int_0^1(F^{-1}(t))^m\sum_{n=2}^\infty nt^{n-1}P[N=n-1]\,dt\\&\quad+b\int_0^1(F^{-1}(t))^m\sum_{n=2}^\infty t^{n-1}P[N=n-1]\,dt\\&=P[N=1]EX^m+a\int_0^1(F^{-1}(t))^m t\psi'_N(t)\,dt\end{aligned}$$

$$+ (a+b) \int_0^1 (F^{-1}(t))^m \psi_N(t)\,dt$$

$$= aE\frac{N}{N+1}X^m_{(N+1)} + (a+b)E\frac{X^m_{(N+1)}}{N+1} + P[N=1]EX^m,$$

which gives (2.1). In a similar way we obtain (2.2).

Remark 1. The relations (2.1) and (2.2) can be written in the form

$$EX^m_{(N)} = a\int_0^1 (F^{-1}(t))^m t\psi'_N(t)\,dt + (a+b)\int_0^1 (F^{-1}(t))^m \psi_N(t)\,dt + P[N=1]EX^m, \tag{2.3}$$

$$EX^m_{1:N} = a\int_0^1 (F^{-1}(t))^m (1-t)\psi'_N(1-t)\,dt + (a+b)\int_0^1 (F^{-1}(t))^m \psi_N(1-t)\,dt + EX^m P[N=1], \tag{2.4}$$

respectively.

Remark 2. In collective risk theory $X_{(N)}$ corresponds to the maximum amount of a claim among the random number of claims in a certain period (cf. [8]).

3. Characterizations of the uniform distribution

THEOREM 1. *Let $\{X_n, n\ge 1\}$ be a sequence of i.i.d. random variables with a common distribution function F such that $EX_1^2<\infty$. Suppose that N is a positive integer-valued random variable independent of $\{X_n, n\ge 1\}$ and has a probability function with $P[N=1]>0$. Then for given $\lambda>0$, $F(x)=x/\lambda, x\in(0,\lambda)$, iff*

$$EX^2_{(N)} - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2 E\frac{N}{N+2} = 0. \tag{3.1}$$

Proof. Let $F(x)=x/\lambda, x\in(0,\lambda)$. Then

$$EX^2_{(N)} = \int_0^1 (F^{-1}(t))^2\psi_N(t)\,dt = \int_0^1 t^2\psi_N(t)\,dt$$

and

$$E\frac{N}{N+1}X_{(N+1)} = \int_0^1 (F^{-1}(t))t\psi'_N(t)\,dt = \int_0^1 t^2\psi'_N(t)\,dt.$$

Taking into account that

$$E\frac{N}{N+2}=\int_0^1 t^2\psi'_N(t)\,dt$$

we obtain (3.1).

Now assume that (3.1) holds true, i.e.

$$\int_0^1 (F^{-1}(t))^2\psi'_N(t)\,dt-2\lambda\int_0^1 F^{-1}(t)t\psi'_N(t)\,dt+\lambda^2\int_0^1 t^2\psi'_N(t)\,dt=0.$$

Then $\int_0^1(F^{-1}(t)-\lambda t)^2\psi'_N(t)\,dt=0$. Therefore by the assumptions we conclude that $F^{-1}(t)=\lambda t$ a.e. on $(0,1)$.

COROLLARY 1 (cf. [6] for $\lambda=1$). *If* $P[N=1]=1$ *then* $F(x)=x$, $x\in(0,1)$, *iff*

$$EX_{(2)}-EX^2=1/3. \tag{3.2}$$

COROLLARY 2 (cf. [7]). *Let* $\{X_n,n\ge 1\}$ *be a sequence of i.i.d. random variables with a common distribution function* F *with finite second moment. Suppose that a random sample size* N *is independent of* $\{X_n,n\ge 1\}$ *and has the probability function*

$$P[N=n]=\frac{\alpha\theta^n}{n},\qquad n=1,2,\ldots;\ \theta\in(0,1),\ \alpha=-1/\ln(1-\theta). \tag{3.3}$$

Then $F(x)=x/\lambda$, $x\in(0,\lambda)$, *iff*

$$EX^2_{(N)}-2\lambda E\frac{N}{N+1}X_{(N+1)}-\lambda^2\alpha\left\{\frac{1}{2}+\frac{1}{\theta}+\frac{\ln(1-\theta)}{\theta^2}\right\}=0, \tag{3.4}$$

which is equivalent to

$$EX^2_{(N)}+2\lambda\left[\alpha EX-\frac{1}{\theta}EX_{(N)}\right]-\alpha\lambda^2\left\{\frac{1}{2}+\frac{1}{\theta}+\frac{\ln(1-\theta)}{\theta^2}\right\}=0. \tag{3.5}$$

Proof. Note that for the distribution (3.3) we get

$$E\frac{N}{N+2}=-\alpha\left\{\frac{1}{2}+\frac{1}{\theta}+\frac{\ln(1-\theta)}{\theta^2}\right\}. \tag{3.6}$$

Hence we have (3.4). Moreover, we see that the distribution (3.3) satisfies (1.1) with $a=\theta$, $b=-\theta$. Then by the Lemma we have

$$\theta E\frac{N}{N+1}X_{(N+1)}=EX_{(N)}-P[N=1]EX$$

or

$$E\frac{N}{N+1}X_{(N+1)}=EX_{(N)}/\theta-\alpha EX. \tag{3.7}$$

Putting (3.6) and (3.7) into (3.4) we get (3.5).

Other special cases characterizing the uniform distribution are given in the following corollaries.

COROLLARY 3. *Under the assumptions of Theorem* 1, $F(x) = x/\lambda$, $x \in (0, \lambda)$, *iff*

(i) $$EX_{(N)}^2 - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\left[1 + \frac{2p}{q^3}\left(\frac{p^2}{2} - 2p + \frac{3}{2} + \ln p\right)\right] = 0,$$

which is equivalent to

$$EX_{(N)}^2 - 2\lambda\left[\frac{EX_{(N)}}{q} - E\frac{X_{(N+1)}}{N+1} - \frac{pEX}{q}\right] + \lambda^2\left[1 + \frac{2p}{q^3}\left(\frac{p^2}{2} - 2p + \frac{3}{2} + \ln p\right)\right] = 0,$$

for N *having the probability function*

(3.8) $$P[N = n] = pq^{n-1}, \quad n = 1, 2, \ldots;\ 0 < p < 1,\ p + q = 1;$$

(ii) $$EX_{(N)}^2 - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\left\{\frac{1}{1-q^n} - 2\left[\frac{1-q^{n+2}}{n+2} - q\frac{1-q^{n+1}}{n+1}\right]/[p^2(1-q^n)]\right\} = 0,$$

which is equivalent to

$$EX_{(N)}^2 - 2\lambda\left[-\frac{qEX_{(N)}}{p} + nE\frac{X_{(N+1)}}{N+1} + \frac{nq^n}{1-q^n}EX\right] + \lambda^2\left\{\frac{1}{1-q^n} - 2\left[\frac{1-q^{n+2}}{n+2} - q\frac{1-q^{n+1}}{n+1}\right]/[p^2(1-q^n)]\right\} = 0,$$

for N *having the probability function*

(3.9) $$P[N = k] = \binom{n}{k}p^k q^{n-k}/(1-q^n), \quad k = 1, \ldots, n;\ p + q = 1,\ 0 < p < 1;$$

(iii) $$EX_{(N)}^2 - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\left\{\frac{\theta - 2}{\theta(1-e^{-\theta})} + \frac{2}{\theta^2}\right\} = 0,$$

which is equivalent to

$$EX_{(N)}^2 - 2\lambda\left\{EX_{(N+1)} + \left[\frac{e^{-\theta}EX}{1-e^{-\theta}} - EX_{(N)}\right]/\theta\right\} + \lambda^2\left\{\frac{\theta - 2}{\theta(1-e^{-\theta})} + \frac{2}{\theta^2}\right\} = 0,$$

for N *having the probability function*

(3.10) $$P[N = n] = e^{-\theta}\theta^n/[(1-e^{-\theta})n!], \quad n = 1, 2, \ldots;\ \theta > 0;$$

(iv) $$EX^2_{(N)} - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\Big\{\frac{1}{1-p^2} - 2p^2\Big[\frac{q}{p} + \ln p\Big]\Big/[q^2(1-p^2)]\Big\} = 0,$$

which is equivalent to

$$EX^2_{(N)} - 2\lambda\Big[\frac{EX_{(N)}}{q} - 2E\frac{X_{(N+1)}}{N+1} - \frac{2p^2EX}{1-p^2}\Big] + \lambda^2\Big\{\frac{1}{1-p^2} - 2p^2\Big[\frac{q}{p} + \ln p\Big]\Big/[q^2(1-p^2)]\Big\} = 0,$$

or

$$EX^2_{(N)} - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\Big\{\frac{1}{1-p^3} + 2p^3\Big[-\frac{1}{2} + \frac{1}{p} - \frac{1}{2p^2}\Big]\Big/[q^2(1-p^3)]\Big\} = 0,$$

which is equivalent to

$$EX^2_{(N)} - 2\lambda\Big[\frac{EX_{(N)}}{q} - 3E\frac{X_{(N+1)}}{N+1} - \frac{3p^3EX}{1-p^3}\Big] + \lambda^2\Big\{\frac{1}{1-p^3} + 2p^3\Big[-\frac{1}{2} + \frac{1}{p} - \frac{1}{2p^2}\Big]\Big/[q^2(1-p^3)]\Big\} = 0,$$

or

$$EX^2_{(N)} - 2\lambda E\frac{N}{N+1}X_{(N+1)} + \lambda^2\Big\{\frac{1}{1-p^r} + 2\Big(\frac{p-p^r}{1-r} - \frac{p^2-p^r}{2-r}\Big)\Big/[q^2(1-p^r)]\Big\} = 0,$$

which is equivalent to

$$EX^2_{(N)} - 2\lambda\Big[\frac{EX_{(N)}}{q} - rE\frac{X_{(N+1)}}{N+1} - \frac{rp^rEX}{1-p^r}\Big] + \lambda^2\Big\{\frac{1}{1-p^r} + 2\Big(\frac{p-p^r}{1-r} - \frac{p^2-p^r}{2-r}\Big)\Big/[q^2(1-p^r)]\Big\} = 0,$$

for N having the probability function

$$P[N=n] = \binom{n+r-1}{n}p^rq^n/(1-p^r), \qquad n = 1,2,\ldots, \tag{3.11}$$

with $r = 2, 3$ and an integer $r \ge 4$, respectively.

Characterization conditions in the spirit of (3.5) (cf. [7]) with the probability function (1.1) are as follows.

Theorem 2. *Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables obeying a distribution F with finite second moment. Further, let N be de-*

fined by (1.1) *and be independent of* $\{X_n, n \geq 1\}$. *Then*, *for given* $\lambda > 0$, $F(x) = x/\lambda$ *on* $(0, \lambda)$ *if and only if*

$$EX^2_{(N)} - 2\lambda\left[EX_{(N)} - (a+b)E\frac{X_{(N+1)}}{N+1} - P[N=1]EX\right]/a + \lambda^2 E\frac{N}{N+2} = 0 \tag{3.12}$$

whenever $a \neq 0$ *and* $a \neq -b$,

$$EX^2_{(N)} - 2\lambda[EX_{(N)} - P[N=1]EX]/a + \lambda^2 E\frac{N}{N+2} = 0 \tag{3.13}$$

whenever $a \neq 0$ *and* $a = -b$, *and*

$$EX^2_{(N)} - 2\lambda[EX_{(N+1)} + (P[N=1]EX - EX_{(N)})/b] + \lambda^2 E\frac{N}{N+2} = 0 \tag{3.14}$$

whenever $a = 0$.

P r o o f. The Lemma of Section 2 and Theorem 1 lead us after simple evaluations to (3.12)–(3.14).

The conditions for $U(0, \lambda)$ in terms of first order statistics are as follows.

THEOREM 1′. *Let a sequence* $\{X_n, n \geq 1\}$ *and a random variable* N *be as in Theorem* 1. *Then for* $\lambda > 0$, $F(x) = x/\lambda$, $x \in (0, \lambda)$, *iff*

$$EX^2_{1:N} + 2\lambda\left[E\frac{N}{N+1}X_{1:N+1} - EX_{1:N}\right] + \lambda^2\left[1 - 2E\frac{N}{N+1} + E\frac{N}{N+2}\right] = 0.$$

COROLLARY 1′ (cf. [6]). *If* $P[N=1] = 1$ *then* $F(x) = x$, $x \in (0,1)$, *iff* $EX^2 + EX_{1:2} - 2EX + 1/3 = 0$, *which is equivalent to* $EX_{2:2} - EX^2 = 1/3$.

COROLLARY 2′ (cf. [7]). *Under the assumptions of Theorem* 1′, $F(x) = x/\lambda$, $x \in (0, \lambda)$, *iff*

$$EX^2_{1:N} + 2\lambda\left[E\frac{N}{N+1}X_{1:N+1} - EX_{1:N}\right] + \lambda^2\alpha\left[\frac{3}{2} - \frac{1}{\theta} - \left(1 - \frac{1}{\theta}\right)^2 \ln(1-\theta)\right] = 0, \tag{3.15}$$

which is equivalent to

$$EX^2_{1:N} - 2\lambda\left[\alpha EX + \left(1 - \frac{1}{\theta}\right)EX_{1:N}\right] = -\alpha\lambda^2\left[\frac{3}{2} - \frac{1}{\theta} - \left(1 - \frac{1}{\theta}\right)^2 \ln(1-\theta)\right], \tag{3.16}$$

for N *having the probability function* (3.3).

Proof. Note that for the distribution (3.3) we get

$$1-2E\frac{N}{N+1}+E\frac{N}{N+2}=\alpha\left[\frac{3}{2}-\frac{1}{\theta}-\left(1-\frac{1}{\theta}\right)^2\ln(1-\theta)\right]. \tag{3.17}$$

Hence we have (3.15). Moreover, for the distribution (3.3) by the Lemma we get

$$E\frac{N}{N+1}X_{1:N+1}-EX_{1:N}=-\left(1-\frac{1}{\theta}\right)EX_{1:N}-\alpha EX. \tag{3.18}$$

Putting (3.17) and (3.18) into (3.15) we obtain (3.16).

Corollary 3′. *Let $\{X_n, n\ge 1\}$ be a sequence of i.i.d. random variables obeying a distribution function F with finite second moment. Suppose that a random sample size N is independent of $\{X_n, n\ge 1\}$. Then for given $\lambda>0$, $F(x)=x/\lambda$, $x\in(0,\lambda)$, iff*

(i) $$EX_{1:N}^2+2\lambda\left[E\frac{N}{N+1}X_{1:N+1}-EX_{1:N}\right] +\lambda^2\left[2p\ln p\left(\frac{1}{q^3}-\frac{1}{q^2}\right)+\frac{p^3-4p^2+3p}{q^3}-\frac{2p}{q}\right]=0,$$

which is equivalent to

$$EX_{1:N}^2-2\lambda\left[\left(1-\frac{1}{q}\right)EX_{1:N}+E\frac{X_{1:N+1}}{N+1}+\frac{p}{q}EX\right] +\lambda^2\left[2p\ln p\left(\frac{1}{q^3}-\frac{1}{q^2}\right)+\frac{p^3-4p^2+3p}{q^3}-\frac{2p}{q}\right]=0,$$

for N having the probability function (3.8);

(ii) $$EX_{1:N}^2+2\lambda\left[E\frac{N}{N+1}X_{1:N+1}-EX_{1:N}\right] +\lambda^2\left\{\frac{q^n}{q^n-1}+2\left[\frac{1-q^{n+1}}{n+1}\left(\frac{1}{p}+\frac{q}{p^2}\right)-\frac{1-q^{n+2}}{p^2(n+2)}\right]/(1-q^n)\right\}=0,$$

which is equivalent to

$$EX_{1:N}^2-2\lambda\left[\frac{EX_{1:N}}{p}-nE\frac{X_{1:N+1}}{N+1}-\frac{q^n}{1-q^n}EX\right] +\lambda^2\left\{\frac{q^n}{q^n-1}+2\left[\frac{1-q^{n+1}}{n+1}\left(\frac{1}{p}+\frac{q}{p^2}\right)-\frac{1-q^{n+2}}{p^2(n+2)}\right]/(1-q^n)\right\}=0,$$

for N having the probability function (3.9);

(iii) $$EX_{1:N}^2 + 2\lambda\Big[E\frac{N}{N+1}X_{(N+1)} - EX_{1:N}\Big] + \Big(\frac{\lambda}{\theta}\Big)^2\Big(2 - \frac{\theta(\theta+2)e^{-\theta}}{1-e^{-\theta}}\Big) = 0,$$

which is equivalent to

$$EX_{1:N}^2 - 2\lambda\Big[\Big(1+\frac{1}{\theta}\Big)EX_{1:N} + \frac{e^{-\theta}EX}{1-e^{-\theta}}\Big] + \Big(\frac{\lambda}{\theta}\Big)^2\Big(2 - \frac{\theta(\theta+2)e^{-\theta}}{1-e^{-\theta}}\Big) = 0,$$

for N *having the probability function* (3.10);

(iv) $$EX_{1:N}^2 + 2\lambda\Big[E\frac{N}{N+1}X_{1:N+1} - EX_{1:N}\Big] - \lambda^2p^2\Big[1 + \frac{2\ln p}{q^2} + \frac{2}{q}\Big]/(1-p^2) = 0,$$

which is equivalent to

$$EX_{1:N}^2 - 2\lambda\Big[\Big(1-\frac{1}{q}\Big)EX_{1:N} + 2E\frac{X_{1:N+1}}{N+1} + \frac{2p^2EX}{1-p^2}\Big] + \lambda^2\Big\{p^2\Big[1 - \frac{2\ln p}{q^2}\Big] + \frac{2p(q^2-1)}{q}\Big\}/(1-p^2) = 0,$$

or

$$EX_{1:N}^2 + 2\lambda\Big[E\frac{N}{N+1}X_{1:N+1} - EX_{1:N}\Big] + \lambda^2\Big\{p^3\Big[1 - 2\Big(\frac{1}{2} + q - \frac{1}{2p^2}\Big)/q + 2\Big(-\frac{1}{2} + \frac{1}{p} - \frac{1}{2p^2}\Big)/q^2\Big]/(1-p^3)\Big\} = 0,$$

which is equivalent to

$$EX_{1:N}^2 - 2\lambda\Big[\Big(1-\frac{1}{q}\Big)EX_{1:N} + 3E\frac{X_{1:N+1}}{N+1} + \frac{3p^3EX}{1-p^3}\Big] + \lambda^2\Big\{p^3\Big[1 - 2\Big(\frac{1}{2} + q - \frac{1}{2p^2}\Big)/q + 2\Big(-\frac{1}{2} + \frac{1}{p} - \frac{1}{2p^2}\Big)/q^2\Big]/(1-p^3)\Big\} = 0,$$

or

$$EX_{1:N}^2 + 2\lambda\Big[E\frac{N}{N+1}X_{1:N+1} - EX_{1:N}\Big] + \lambda^2\Big\{\frac{2(p^2-p^r)}{(r-2)q^2} - \frac{2(p-p^r)p}{(r-1)q^2} - p^r\Big\}/(1-p^r) = 0,$$

which is equivalent to

$$EX_{1:N}^2 - 2\lambda\bigg[\Big(1-\frac{1}{q}\Big)EX_{1:N} + rE\frac{X_{1:N+1}}{N+1} + \frac{rp^rEX}{1-p^r}\bigg] + \lambda^2\bigg\{\frac{2(p^2-p^r)}{(r-2)q^2} - \frac{2(p-p^r)p}{(r-1)q^2} - p^r\bigg\}/(1-p^r) = 0,$$

for N having the probability function (3.11), *with $r = 2, 3$ and an integer $r \geq 4$, respectively.*

THEOREM 2′. *Let a sequence $\{X_n, n \geq 1\}$ and a random variable N be as in Theorem 2. Then for given $\lambda > 0$, $F(x) = x/\lambda$, $x \in (0,\lambda)$, iff*

$$EX_{1:N}^2 - 2\lambda\bigg[\Big(1-\frac{1}{a}\Big)EX_{1:N} + \Big(1+\frac{b}{a}\Big)E\frac{X_{1:N+1}}{N+1} + \frac{P[N=1]EX}{a}\bigg] + \lambda^2\bigg[1 - 2E\frac{N}{N+1} + E\frac{N}{N+2}\bigg] = 0$$

whenever $a \neq 0$, $a \neq -b$,

$$EX_{1:N}^2 - 2\lambda\bigg[\frac{P[N=1]EX}{a} + \Big(1-\frac{1}{a}\Big)EX_{1:N}\bigg] + \lambda^2\bigg[1 - 2E\frac{N}{N+1} + E\frac{N}{N+2}\bigg] = 0$$

whenever $a \neq 0$, $a = -b$, and

$$EX_{1:N}^2 - 2\lambda\bigg[\Big(1+\frac{1}{b}\Big)EX_{1:N} + \frac{P[N=1]EX}{b}\bigg] + \lambda^2\bigg[1 - 2E\frac{N}{N+1} + E\frac{N}{N+2}\bigg] = 0$$

whenever $a = 0$.

Now we see that $E\frac{N}{N+1}X_{(N+1)}^2$ and $E\frac{N}{N+2}X_{(N+2)}$ characterize $U(0,\lambda)$, $\lambda > 0$.

THEOREM 3. *Under the assumptions of Theorem* 1, *for given $\lambda > 0$, $F(x) = x/\lambda$, $x \in (0,\lambda)$, iff*

$$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2 E\frac{N}{N+3} = 0. \tag{3.19}$$

Proof. Let $F(x) = x/\lambda$, $x \in (0,\lambda)$. Then $F^{-1}(t) = \lambda t$, $t \in (0,1)$ and

$$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2 E\frac{N}{N+3} = \lambda^2\int_0^1 t^3\psi_N'(t)\,dt - 2\lambda^2\int_0^1 t^3\psi_N'(t)\,dt + \lambda^2 E\frac{N}{N+3}.$$

But

$$E\frac{N}{N+3} = \int_0^1 t^3 \psi'_N(t)\,dt.$$

Hence we have (3.19).

Assume now that (3.19) is satisfied. Then

$$\int_0^1 (F^{-1}(t) - \lambda t)^2 t \psi'_N(t)\,dt = 0$$

and by the assumptions of Theorem 3 we get $F^{-1}(t) = \lambda t$ a.e. on $(0, \lambda)$.

COROLLARY 4 (cf. [6]). *If* $P[N=1] = 1$, *then* $F(x) = x$, $x \in (0,1)$, *iff*

$$EX_{(2)}^2 - \tfrac{4}{3}EX_{(3)} + \tfrac{1}{2} = 0.$$

COROLLARY 5. *If the random sample size* N *has the distribution* (3.3) *then* $F(x) = x/\lambda$, $x \in (0, \lambda)$, *iff*

$$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} - \lambda^2\alpha\left[\frac{\ln(1-\theta)}{\theta^3} + \frac{1}{\theta^2} + \frac{1}{2\theta} - \frac{1}{3}\right] = 0.$$

COROLLARY 6. *Under the assumptions of Theorem* 3, $F(x) = x/\lambda$, $x \in (0, \lambda)$, *iff*

(i) $$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left[1 + \frac{3p}{q^4}\left(\ln p - 3p + \frac{3}{2}p^2 - \frac{p^3}{3} + \frac{11}{6}\right)\right] = 0,$$

for N *having the probability function* (3.8);

(ii) $$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left\{\frac{1}{1-q^n} - \frac{3}{p^3}\left[\frac{1-q^{n+3}}{n+3} - \frac{2q}{n+2}(1-q^{n+2}) + \frac{q^2}{n+1}(1-q^{n+1})\right]\Big/(1-q^n)\right\} = 0$$

for N *having the probability function* (3.9);

(iii) $$E\frac{N}{N+1}X_{(N+1)}^2 - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left[\left(1 - \frac{3}{\theta} + \frac{6}{\theta^2}\right)\Big/(1-e^{-\theta}) - \frac{6}{\theta^3}\right] = 0$$

for N having the probability function (3.10);

(iv) $$E\frac{N}{N+1}X^2_{(N+1)} - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left[1 + \frac{3p^2}{q^3(1-p^2)}\left(p^2 - \frac{p^3}{3} - \frac{1}{p} - 2\ln p + \frac{1}{3}\right)\right] = 0$$

or

$$E\frac{N}{N+1}X^2_{(N+1)} - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left\{1 + \frac{3p^3}{q^3(1-p^3)}\left[\frac{2}{p} - \frac{1}{2p^2} + \ln p - p + p^2 - \frac{p^3}{3} - \frac{7}{6}\right]\right\} = 0$$

or

$$E\frac{N}{N+1}X^2_{(N+1)} - 2\lambda E\frac{N}{N+2}X_{(N+2)} + \lambda^2\left\{\frac{1}{1-p^r} + \frac{3}{q^3(1-p^r)}\left[\frac{p-p^r}{1-r} - 2\left(\frac{p^2-p^r}{2-r}\right) + \frac{p^3-p^r}{3-r}\right]\right\} = 0$$

for N having the probability function (3.10), *with $r = 2, 3$ and an integer $r \geq 4$, respectively.*

THEOREM 3′. *Under the assumptions of Theorem 2′, for given $\lambda > 0$, $F(x) = x/\lambda$, $x \in (0, \lambda)$, iff*

$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\left(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\right) + \lambda^2\left(E\frac{N}{N+3} - 2E\frac{N}{N+2} + E\frac{N}{N+1}\right) = 0.$$

COROLLARY 4′ (cf. [6]). *If $P[N = 1] = 1$, then $F(x) = x$, $x \in (0, 1)$, iff*

$$EX^2_{1:2} - 2EX_{1:2} + \tfrac{4}{3}EX_{1:3} + \tfrac{1}{6} = 0,$$

which is equivalent to

$$EX^2_{1:2} - \tfrac{2}{3}EX_{2:3} + \tfrac{1}{6} = 0.$$

COROLLARY 5′. *If the random sample size N has the distribution* (3.3), *then $F(x) = x/\lambda$, $x \in (0, \lambda)$, iff*

$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\left(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\right) + \lambda^2\alpha\left[\frac{3}{2} - \frac{1}{\theta} - \left(1 - \frac{1}{\theta}\right)^2\ln(1-\theta) - \frac{\theta}{3}\right]/\theta = 0.$$

Corollary 6′. *Under the assumptions of Theorem* 3′, $F(x) = x/\lambda$, $x \in (0,\lambda)$, *iff*

(i)
$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\Big(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\Big[\ln p\Big(\frac{3p}{q^4} - \frac{4p}{q^3} + \frac{p}{q^2}\Big) + \frac{3p}{q^4}\Big(\frac{3}{2}p^2 - 3p - \frac{p^3}{3} + \frac{11}{6}\Big) - \frac{2p}{q^3}(p^2 - 4p + 3) + \frac{p}{q}\Big] = 0$$

for N having the probability function (3.8);

(ii)
$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\Big(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\Big[-\frac{3}{p^3}\frac{1-q^{n+3}}{n+3} + \frac{2}{n+2}(1-q^{n+2})\Big(\frac{3q}{p^3} + \frac{2}{p^2}\Big) + \frac{1-q^{n+1}}{n+1}\Big(\frac{3}{p^3} - \frac{4q}{p^2} - \frac{1}{p}\Big)\Big]\Big/(1-q^n) = 0$$

for N having the probability function (3.9);

(iii)
$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\Big(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\Big[\Big(\frac{6}{\theta} + 1\Big)\Big/[\theta(1-e^{-\theta})] - \Big(\frac{6}{\theta^3} + \frac{4}{\theta^2} + \frac{1}{\theta}\Big)\Big] = 0$$

for N having the probability function (3.10);

(iv)
$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\Big(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\Big[\frac{3p^2}{q^3}\Big(-q - 2\ln p - \frac{q}{p}\Big) + 4p^2\Big(\frac{q}{p} + \ln p\Big)\Big/q^2 - p\Big]\Big/(1-p^2) = 0$$

or

$$E\frac{N}{N+1}X^2_{1:N+1} + 2\lambda\Big(E\frac{N}{N+2}X_{1:N+2} - E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\Big[\frac{3p^3}{q^3}\Big(\frac{2}{p} + \ln p - \frac{1}{2p^2} - \frac{3}{2}\Big) - \frac{4p^3}{q^2}\Big(-\frac{1}{2} + \frac{1}{p} - \frac{1}{2p^2}\Big) + \frac{p^3}{q}\Big(\frac{1}{2} - \frac{1}{2p^2}\Big)\Big]\Big/(1-p^3) = 0$$

or

$$E\frac{N}{N+1}X^2_{1:N+1}+2\lambda\Big(E\frac{N}{N+2}X_{1:N+2}-E\frac{N}{N+1}X_{1:N+1}\Big) + \lambda^2\bigg[\frac{p-p^r}{1-r}\Big(\frac{3}{q^3}-\frac{4}{q^2}+\frac{1}{q}\Big)+\frac{p^2-p^r}{2-r}\Big(\frac{4}{q^2}-\frac{6}{q^3}\Big) - \frac{3(p^3-p^r)}{q^3(3-r)}\bigg]\Big/(1-p^r)=0$$

for N having the probability function (3.11) *with* $r = 2, 3$ *and an integer* $r \geq 4$, *respectively.*

Acknowledgments. The authors are very grateful to the referee for valuable suggestions leading to new characterizing conditions and improvement of the previous presentation.

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ZOFIA GRUDZIEŃ AND DOMINIK SZYNAL
INSTITUTE OF MATHEMATICS
UNIVERSITY OF M. CURIE-SKŁODOWSKA
PL. M. CURIE-SKŁODOWSKIEJ 1
20-031 LUBLIN, POLAND

Received on 1.9.1994;
revised version on 22.12.1994