APPLICATIONES MATHEMATICAE
23,3 (1995), pp. 279–284

R. RÓŻAŃSKI (Wrocław)

ON A STRONGLY CONSISTENT ESTIMATOR OF THE SQUARED L_2-NORM OF A FUNCTION

Abstract. A kernel estimator of the squared L_2-norm of the intensity function of a Poisson random field is defined. It is proved that the estimator is asymptotically unbiased and strongly consistent. The problem of estimating the squared L_2-norm of a function disturbed by a Wiener random field is also considered.

Introduction. The theory of kernel estimation of functions has been rapidly and successfully developed for the last three decades. Many authors encouraged with these results tried to extend the theory to other areas of statistics, for example, the statistics of stochastic processes. In particular, Ramlau-Hansen (1983) investigated properties of kernel estimators of the intensity function of a point process in the so-called multiplicative intensity model. Following this paper and the results of Koronacki and Wertz (1987) Różański (1992) obtained some results concerning recursive kernel estimation of the intensity function of a Poisson random field and asymptotic sequential confidence bounds for the intensity function. The construction of the sequential bounds is based on the assumption that there exists a strongly consistent estimator of the squared L_2-norm of the intensity function. In the present paper we construct such an estimator using Schuster's (1974) idea.

1. Estimation of the squared L_2-norm of the intensity function of a Poisson random field

DEFINITION 1.1. Let (Ω, F, P) be a probability space. A mapping N from Ω into the space of all non-negative, integer-valued measures defined

1991 *Mathematics Subject Classification*: 62M09, 62G05.
Key words and phrases: Poisson random field, Wiener random field, stochastic integral with respect to a p-parameter martingale, kernel estimator, asymptotic unbiasedness, strong consistency.

on $B_{\mathbb{R}^p}$, the σ-algebra of Borel subsets of $\mathbb{R}^p$, is called a *Poisson random measure* if the following conditions are satisfied:

(i) for each bounded Borel set $B \in B_{\mathbb{R}^p}$,

$$P(\{\omega : N(B,\omega) = k\}) = \frac{\left(\int_B \lambda(u)\,du\right)^k}{k!} \exp\Big(-\int\limits_B \lambda(u)\,du\Big),$$

(ii) for every finite collection of disjoint bounded Borel sets $B_1, \ldots, B_n$ the random variables $N(B_1), \ldots, N(B_n)$ are independent.

Let $z = (x_1, \ldots, x_p) \in \mathbb{R}_+^p = [0,\infty) \times \ldots \times [0,\infty)$ and $[0,z] = [0,x_1] \times \ldots \times [0,x_p]$. Then $N_z = N([0,z])$ is called a *Poisson random field*.

If $F_z, z \in \mathbb{R}_+^p$, is the σ-algebra generated by the random variables N_u, $u \le z$, then the random field N_z can be written in the form

$$N_z = \int\limits_{[0,z]} \lambda(u)\,du + M_z,$$

where (M_z, F_z) is a martingale (see Cairoli and Walsh (1975) for the definition). The function λ is called the intensity function.

In Różański (1992) a recursive modification of the Ramlau-Hansen (1983) kernel estimator of the intensity function was defined. The estimator has the form

$$\widehat{\lambda}_n(N_{1,z}, \ldots, N_{n,z}, z) = \widehat{\lambda}(z) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{b_k^p} \int\limits_I K\left(\frac{z-u}{b_k}\right) N_k(du)$$

where $N_{1,z}, \ldots, N_{n,z}$ are independent copies of the Poisson field N_z observed on the unit cube $I = [0,1] \times \ldots \times [0,1]$ and K is a kernel function, that is, a probability density function. We also assume that K has support contained in $Q = [-1,1] \times \ldots \times [-1,1]$ and that $b_n = n^{-\beta}$, where $\beta p < 1$.

Strong consistency, asymptotic unbiasedness and asymptotic normality of this estimator have been proved. Moreover, asymptotic sequential confidence bounds for the intensity function based on the integrated squared error have also been studied. The appropriate stopping times have been constructed with a strong consistent estimator δ_n of $\int_I \lambda^2(u)\,du$.

We will define such an estimator following Schuster's (1974) ideas and we will prove that it is asymptotically unbiased and strongly consistent.

THEOREM 1.1. *If λ is a continuous function on the unit cube I and $N_{1,z}, \ldots, N_{n,z}$ are independent copies of the Poisson random field N_z observed on I then the estimator*

$$\delta_n = \frac{1}{n-1} \sum_{j=2}^{n} \frac{1}{j-1} \sum_{k=1}^{j-1} \frac{1}{b_k^p} \int\limits_I \int\limits_I K\left(\frac{z-u}{b_k}\right) N_k(du)\, N_j(dz)$$

is an asymptotically unbiased, strongly consistent estimator for $\int_I \lambda^2(u)\,du$.

Proof. First we show the asymptotic unbiasedness of the estimator δ_n. We have

$$E\delta_n = \frac{1}{n-1}\sum_{j=2}^{n}\frac{1}{j-1}\sum_{k=1}^{j-1}\frac{1}{b_k^p}E\int_I\int_I K\Big(\frac{z-u}{b_k}\Big)\,N_k(du)\,N_j(dz)$$
$$= \frac{1}{n-1}\sum_{j=2}^{n}\int_I \lambda(z)\Big[\frac{1}{j-1}\sum_{k=1}^{j-1}\frac{1}{b_k^p}\int_I K\Big(\frac{z-u}{b_k}\Big)\lambda(u)\,du\Big]\,dz.$$

Since the function λ is continuous, we have

$$A_j(z) = \frac{1}{j-1}\sum_{k=1}^{j-1}\frac{1}{b_k^p}\int_I K\Big(\frac{z-u}{b_k}\Big)\lambda(u)\,du \underset{j\to\infty}{\longrightarrow} \lambda(z).$$

Let $B_j(z)=\lambda(z)A_j(z)$. By the Lebesgue theorem we conclude that

$$\int_I B_j(z)\,dz \underset{j\to\infty}{\longrightarrow} \int_I \lambda^2(z)\,dz.$$

Thus the same is true for the sequence of mean averages and

$$\frac{1}{n-1}\sum_{j=2}^{n}\int_I B_j(z)\,dz \underset{n\to\infty}{\longrightarrow} \int_I \lambda^2(z)\,dz,$$

which ends the proof of asymptotic unbiasedness.

To prove the strong consistency of δ_n, write

$$\delta_n - E\delta_n = \frac{1}{n-1}\sum_{j=2}^{n}\frac{1}{j-1}\sum_{k=1}^{j-1}\frac{1}{b_k^p}E\int_I\int_I K\Big(\frac{z-u}{b_k}\Big)\,M_k(du)\,M_j(dz),$$

where (M_z, F_z) is a martingale. Let

$$U_j = \frac{1}{j-1}\sum_{k=1}^{j-1}\frac{1}{b_k^p}\int_I\int_I K\Big(\frac{z-u}{b_k}\Big)\,M_k(du)\,M_j(dz).$$

For each j, the mean value of U_j is equal to zero. Thus

$$\operatorname{Var} U_j = EU_j^2 = \frac{1}{(j-1)^2}\sum_{k=1}^{j-1}\sum_{l=1}^{j-1}\frac{1}{b_k^p b_l^p}E\int_I\int_I K\Big(\frac{z-u}{b_k}\Big)\,M_k\,(du)M_j(dz)$$
$$\times\int_I\int_I K\Big(\frac{z'-u'}{b_l}\Big)\,M_l(du')\,M_j(dz')$$
$$\le \frac{C}{(j-1)^2}\sum_{k=1}^{j-1}\frac{1}{b_k^p} \le C_1,$$

where C_1 is a constant. We have used formula 2.5 of Cairoli and Walsh (1975) to obtain the above inequality. In the same way we can conclude that the random variables U_j are orthogonal. Thus applying the Rademacher–Men'shov strong law of large numbers (see Reveš (1968)) we obtain the assertion.

Remark 1.1. In Różański (1992) the estimator of the squared L_2-norm of the intensity λ is defined in a little bit different but asymptotically equivalent form. Namely, introducing $N_{0,z} = 0$ almost surely and $b_0 = 1$, we can define

$$\delta_n' = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{(j-1)\vee 1}\sum_{k\le j-1}\frac{1}{b_k^p}\int_I\int_I K\left(\frac{z-u}{b_k}\right)N_k(du)\,N_j(dz).$$

Remark 1.2. The estimator δ_n can be applied to estimation of the mean value of the energy of electrons reaching an anode with intensity $\lambda(t)$ (Gardiner (1984)).

2. Estimation of the L_2-norm of a function disturbed by a space-time white noise

Definition 2.1. A family $\{W(B)\}_{B\in B_{\mathbb{R}^p}}$ of random variables is called a *p-parameter Wiener measure* if the following conditions are satisfied:

(i) for every finite collection of disjoint bounded Borel sets $B_1,\ldots,B_n$ the random variables $W(B_1),\ldots,W(B_n)$ are independent,

(ii) $EW(B) = 0$ and $EW(A\cap B) = \mathrm{vol}(A\cap B)$ for all bounded Borel sets A, B.

The family $\{W_z\}_{z\in\mathbb{R}_+^p} = \{W([0,z])\}_{z\in\mathbb{R}_+^p}$ is then called a *Wiener random field*.

Let us assume that we observe n independent copies $X_{1,z},\ldots,X_{n,z}$, $z\in I$, of the random field

$$X_z = \int_{[0,z]} f(u)\,du + W_z, \qquad z\in I,$$

where $f(z)$ is an unobservable deterministic function. We will estimate the squared L_2-norm of the function f using an appropriate version of the estimator δ_n defined in the previous section. The estimator is proved to be asymptotically unbiased and strongly consistent.

Theorem 2.1. *If f is a continuous function on the unit cube I and $X_{1,z},\ldots,X_{n,z}$ are independent copies of the random field X_z observed on I then the estimator*

$$\delta_n = \frac{1}{n-1}\sum_{j=2}^{n}\frac{1}{j-1}\sum_{k=1}^{n}\frac{1}{b_k^p}\int_I\int_I K\left(\frac{z-u}{b_k}\right)X_k(du)\,X_j(dz)$$

is an asymptotically unbiased, strongly consistent estimator for $\int_I f^2(u)\,du$.

P r o o f. Using the same methods as in the proof of Theorem 1.1 we can conclude that

$$E\delta_n = \frac{1}{n-1}\sum_{j=2}^{n}\frac{1}{j-1}\sum_{k=1}^{n}\frac{1}{b_k^p}\int_I\int_I K\left(\frac{z-u}{b_k}\right)f(u)\,du\,f(z)\,dz$$

tends to $\int_I f^2(u)\,du$ and

$$\delta_n - E\delta_n = \frac{1}{n-1}\sum_{j=2}^{n}\frac{1}{j-1}\sum_{k=1}^{n}\frac{1}{b_k^p}\int_I\int_I K\left(\frac{z-u}{b_k}\right)W_k(du)\,W_j(dz)$$

tends to zero almost surely as $n\to\infty$.

It is worth noting that if F_{z_1,z_2} denotes the σ-algebra generated by $V_{u,v} = W_k(u)W_j(v)$ for $(u,v)\le(z_1,z_2)$ then $(V_{z_1,z_2},F_{z_1,z_2})$ is a martingale. Thus the stochastic integral

$$\int_I\int_I K\left(\frac{z-u}{b_k}\right)W_k(du)\,W_j(dz)$$

can be understood as a stochastic integral with respect to the martingale V_{z_1,z_2} (Cairoli and Walsh (1975)).

R e m a r k 2.1. Let us consider a signal $f(t)$ that is disturbed by a white noise and can be described by the equation

$$dX(t) = f(t)\,dt + dW(t).$$

The signal X may be observed on an oscilloscope screen. Assume that we can make n repetitions of the experiment, independent of one another, under the same physical conditions. Then the estimator δ_n can be applied to estimate the energy of the signal $f(t)$.

References

R. Cairoli and J. B. Walsh (1975), *Stochastic integrals in the plane*, Acta Math. 134, 111–183.

C. W. Gardiner (1984), *Handbook for Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer Series in Synergetics, Berlin.

J. Koronacki and W. Wertz (1987), *A global stopping rule for recursive density estimators*, Statist. Planning Inference 20, 23–39.

H. Ramlau-Hansen (1983), *Smoothing counting process intensities by means of kernel functions*, Ann. Statist. 12, 453–466.

P. Reveš (1968), *Laws of Large Numbers*, Academic Press, New York.

R. Różański (1992), *Recursive estimation of intensity function of a Poisson random field*, J. Statist. Planning Inference 33, 165–174.

E. F. Schuster (1974), *On the rate of convergence of an estimate of a probability density*, Scand. Actuar. J., 103–107.

ROMAN RÓŻAŃSKI
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

Received on 27.7.1994