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A NOISY DUEL UNDER ARBITRARY MOTION. IX

1. Definitions and assumptions. In [13], [14] and in this paper an m versus n bullets noisy duel is considered in which duelists can move at will. It is assumed that Player I has greater maximal speed. The cases $m = 1, 2, 3$, $n = 1, 2, 3$ are solved. Let a be the point in which Player I is at the beginning of the duel, $0 \le a < 1$ (Player II is at 1). In contrast to [7]–[12] where the duels are solved for small a, now we solve the duels for any $0 \le a < 1$.

In this paper we consider the cases $m = 3$, $n = 1$; $m = 3$, $n = 2$; and $m = 3$, $n = 3$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys the opponent) if he fires when the distance between the players is $1 - s$. We assume that $P(s)$ is increasing and continuous in $[0, 1]$, has continuous second derivative in $(0, 1)$, and $P(s) = 0$ for $s \le 0$, $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The game is over if at least one player is destroyed or all bullets are shot. In the other case the duel lasts infinitely long and the payoff is zero.

The duel is noisy—each player hears every shot of his opponent.

As will be seen from the sequel, without loss of generality we can assume that Player II is motionless. It is also assumed that the maximal speed of Player I is 1 and that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

We suppose that between successive shots of the same player there has to pass a time $\widehat{\varepsilon} > 0$. We also assume that the reader knows the papers [7]–[14] and remembers the definitions, assumptions and results given there.

For other results in the theory of games of timing see [1]–[6], [15].

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2. The duel $(3,1)$**.** Consider the case where Player I has 3 bullets, Player II has one bullet and the duel begins when Player I is at the point a, $0 \le a < 1$. Let $Q(s) = 1 - P(s)$.

The duel $(3,1), \langle a \rangle$

Case 1: $Q(a) \ge Q(a_{31}) \cong 0.773459$. We define the following strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I. If Player II has not fired before, reach the point a_{31}, fire a shot at $\langle a_{31} \rangle$ and play optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I has reached the point a_{31}, fire a shot at a_{31}^{ε}. If he has not, do not fire.

We now have

$$v_{31}^{a} = \frac{1}{1 + 2P(a_{11})} = \frac{2\sqrt{2} + 1}{7} \cong 0.546918, \tag{1}$$
$$Q(a_{31}) = \frac{4 + \sqrt{2}}{2} \cong 0.773459.$$

The quantity a_{31}^{ε} is an absolutely continuous random variable taking values between $\langle a_{31} \rangle$ and $\langle a_{31} \rangle + \alpha(\varepsilon)$, where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The notation $\langle a_{31} \rangle$ refers to the first time when Player I reaches the point a_{31}.

v_{31}^{a} is the limit value of the duel $(3,1), \langle a \rangle$ (i.e. as $\widehat{\varepsilon} \to 0$).

In [8] it is proved that if $Q(a) \ge Q(a_{31})$ then the strategies ξ and η are optimal in limit for given a_{31}, and for this a_{31} the limit value of the game is given by (1).

Case 2: $0.749117 \cong Q(\widehat{a}_{31}) \le Q(a) \le Q(a_{31}) \cong 0.773459$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Fire a shot at $\langle a \rangle$ and play optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I has not fired before, fire a shot at a^{ε}.

We now have

$$v_{31}^{a} = P(a) + Q(a)\overset{1}{v}{}_{21}^{a} = 1 - (1 - v_{21})Q(a) \quad \text{ if } Q(\widehat{a}_{31}) \le Q(a) \le Q(a_{31});$$

$\overset{1}{v}{}_{mn}^{a}$ and $\overset{2}{v}{}_{mn}^{a}$ are the limit values of the duels $(m,n), \langle 1, a \wedge c, a \rangle$ and $(m,n), \langle 2, a, a \wedge c \rangle$ (see [8], Section 3).

Suppose that Player II also fires at $\langle a \rangle$. For such a strategy $\widehat{\eta}$, we obtain

$$K(\xi, \widehat{\eta}) \ge Q^2(a) - k(\widehat{\varepsilon}) \ge 1 - (1 - v_{21})Q(a) - k(\widehat{\varepsilon}).$$

This inequality holds if

$$Q^2(a) + (1 - v_{21})Q(a) - 1 \geq 0,$$

i.e. if $Q(a) \geq Q(\widehat{a}_{31}) \cong 0.749117$.

On the other hand, suppose that Player I has not fired a shot before $\langle a\rangle + \alpha(\varepsilon)$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi}, \eta) \leq -1 + 2Q(a) + k(\widehat{\varepsilon}) \leq 1 - (1 - v_{21})Q(a) + k(\widehat{\varepsilon})$$

if

$$Q(a) \leq \frac{2}{3 - v_{21}} = Q(a_{31}) \cong 0.773459.$$

The strategies ξ and η are optimal in limit if

$$0.749117 \cong Q(\widehat{a}_{31}) \leq Q(a) \leq Q(a_{31}) \cong 0.773459.$$

Case 3: $Q(a) \leq Q(\widehat{a}_{31})$. Define ξ and η as follows:

Strategy of Player I. Fire a shot at $\langle a\rangle$ and play optimally the resulting duel.

Strategy of Player II. Fire a shot at $\langle a\rangle$.

We now have

$$v_{31}^a = Q^2(a).$$

To prove that ξ and η are optimal in limit for $Q(a) \leq Q(\widehat{a}_{31})$, suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$,

$$\begin{aligned} K(\xi, \widehat{\eta}) &\geq P(a) + Q(a)\overset{1}{v}{}_{21}^{a} - k(\widehat{\varepsilon}) \\ &= \begin{cases} 1 - (1 - v_{21})Q(a) & \text{if } Q(a) \geq Q(a_{21}) \cong 0.707107, \\ 1 - 2Q(a) + 2Q^2(a) & \text{if } Q(a) \leq Q(a_{21}). \end{cases} \end{aligned}$$

Consider the following cases:

(a) $Q(a) \geq Q(a_{21}) \cong 0.707107$. In this case we should prove that

$$1 - (1 - v_{21})Q(a) \geq Q^2(a).$$

This inequality is satisfied if $Q(a) \leq Q(\widehat{a}_{31}) \cong 0.749117$.

(b) $Q(a) \leq Q(a_{21})$. In this case we have to prove that

$$1 - 2Q(a) + 2Q^2(a) \geq Q^2(a),$$

which always holds.

Now we define the duels $(m, n), \langle 1, a \wedge c, a\rangle$ and $(m, n), \langle 2, a, a \wedge c\rangle$. We have supposed that a time $\widehat{\varepsilon}$ has to elapse between successive shots of the same player. Let

$$(m, n), \langle 2, a, a \wedge c\rangle, \quad 0 < c \leq \widehat{\varepsilon},$$

be the duel in which Player I has m bullets, Player II has n bullets, but if $c < \widehat{\varepsilon}$, Player I can fire his bullets from time $\langle a\rangle$ on, and Player II from time $\langle a\rangle + c$ on. If $c = \widehat{\varepsilon}$ the rule is the same with the only exception that Player I is not allowed to fire at $\langle a\rangle$.

Similarly we define the duel $(m, n), \langle 1, a, a \wedge c\rangle$.

The duel $(3, 1), \langle 1, a \wedge c, a\rangle$

Case 1: $Q(a) \geq Q(a_{31}) \cong 0.773451$. In this case the strategies optimal in limit are the same as in the duel $(3, 1), \langle a\rangle$, Case 1.

Case 2: $Q(a) \leq Q(a_{31})$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel.

Strategy of Player II. Fire before $\langle a\rangle + c$.

We now have

$$\overset{1}{v}{}^{a}_{31} = -1 + 2Q(a).$$

The symbol $\rangle t\langle$ denotes the point at which Player I has been at time t. $\overset{1}{v}{}^{a}_{mn}$ and $\overset{2}{v}{}^{a}_{mn}$ are the limit values (i.e. as $\widehat{\varepsilon} \to 0$) of the games $(m, n), \langle 1, a \wedge c, a\rangle$ and $(m, n), \langle 2, a, a \wedge c\rangle$.

The proof that the strategies ξ and η are optimal in limit is omitted.

The duel $(3, 1), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \geq Q(a_{31})$. In this case the strategies optimal in limit are also the same as in the duel $(3, 1), \langle a\rangle$, Case 1.

Case 2: $Q(a) \leq Q(a_{31})$. Define ξ and η as follows:

Strategy of Player I. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel. If he has fired, play optimally the resulting duel $(2, 1), \langle 1, a \wedge c, a\rangle$.

We now have

$$\overset{2}{v}{}^{a}_{31} = P(a) + Q(a)\overset{1}{v}{}^{a}_{21}.$$

The proof that ξ and η are optimal in limit is also omitted.

3. Results for the duel $(3,1)$**.** We have

$$\overset{1}{v}{}^a_{31} = \begin{cases} v_{31} \cong 0.546918 & \text{if } Q(a) \ge Q(a_{31}) \cong 0.773459,\\ -1+2Q(a) & \text{if } Q(a) \le Q(a_{31}); \end{cases}$$

$$v^a_{31} = \begin{cases} v_{31} & \text{if } Q(a) \ge Q(a_{31}),\\ 1-(1-v_{21})Q(a) & \text{if } Q(a_{31}) \ge Q(a) \ge Q(\widehat{a}_{31}) \cong 0.749117,\\ Q^2(a) & \text{if } Q(a) \le Q(\widehat{a}_{31}); \end{cases}$$

$$\overset{2}{v}{}^a_{31} = \begin{cases} v_{31} & \text{if } Q(a) \ge Q(a_{31}),\\ 1-(1-v_{21})Q(a) & \text{if } Q(a_{31}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107,\\ 1-2Q(a)+2Q^2(a) & \text{if } Q(a) \le Q(a_{21}). \end{cases}$$

4. The duel $(3,2)$

The duel $(3,2), \langle a\rangle$

Case 1: $Q(a) \ge Q(a_{32}) \cong 0.833869$. To solve the duel $(3,2), \langle a\rangle$ for small a consider the duel $(m,n), \langle a\rangle$ for $m > n > 1$ and $a < a_{mn}$.

The duel $(m,n), \langle a\rangle$

Case 1: $Q(a) \ge Q(a_{mn})$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, reach the point a_{mn}, fire at a^{ε}_{mn} and play optimally the resulting duel. If he fired (say at a'), play optimally the resulting duel $(m,n-1), \langle 2, a', a' \wedge \widehat{\varepsilon}\rangle$.

Strategy of Player II. If Player I has not fired before, fire at $\langle a_{mn}\rangle$ and play optimally the resulting duel. If Player I has neither reached the point a_{mn} nor fired, do not fire either. If he has fired, play optimally the resulting duel.

The numbers a_{mn} and v^a_{mn} are obtained from the equations

$$Q(a_{mn}) = \frac{2}{2+v_{m,n-1}-v_{m-1,n}},$$
$$v^a_{mn} := v_{mn} = \frac{v_{m,n-1}-v_{m-1,n}}{2+v_{m,n-1}-v_{m-1,n}}$$

(see [12]).

For the duel $(3,2), \langle a\rangle$ we have

$$Q(a_{32}) \cong 0.833869, \qquad v_{32} \cong 0.289928.$$

For this a_{32} the strategies ξ and η are optimal in limit.

The duel $(3,2), \langle a\rangle$

Case 2: $0.831642 \cong Q(\widehat{a}_{32}) \le Q(a) \le Q(a_{32})$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at a^{ε} and play optimally afterwards. If he has fired, play optimally the resulting duel.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$v_{32}^{a} = -P(a) + Q(a)\overset{2}{v}{}_{31}^{a} = -1 + (1+v_{31})Q(a)$$

if

$$0.831642 \cong Q(\widehat{a}_{32}) \le Q(a) \le Q(a_{32}) \cong 0.833869.$$

Suppose that Player II has not fired before $\langle a\rangle + \alpha(\varepsilon)$. We obtain

$$K(\xi,\widehat{\eta}) \ge 1 - (1-v_{22})Q(a) - k(\widehat{\varepsilon}) \ge -1 + (1+v_{31})Q(a) - k(\widehat{\varepsilon}).$$

It follows that

$$Q(a) \le \frac{2}{2+v_{31}-v_{22}} = Q(a_{32}) \cong 0.833869. \tag{2}$$

On the other hand, suppose that Player I fires at $\langle a\rangle$. We obtain

$$\begin{aligned} K(\widehat{\xi},\eta) &\le Q^2(a)v_{21}^{a} + k(\widehat{\varepsilon}) = v_{21}Q^2(a) + k(\widehat{\varepsilon}) \\ &\le\ -1 + (1+v_{31})Q(a) + k(\widehat{\varepsilon}), \end{aligned}$$

which gives

$$v_{21}Q^2(a) - (1+v_{31})Q(a) + 1 \le 0.$$

Solving this inequality and taking into account the inequality (2) we deduce that the strategies ξ and η are optimal in limit if

$$0.831642 \cong Q(\widehat{a}_{32}) \le Q(a) \le Q(a_{32}) \cong 0.833869.$$

The duel $(3,2),\langle a\rangle$

Case 3: $Q(a) \le Q(\widehat{a}_{32}) \cong 0.831642$. Define ξ and η as follows:

Strategy of Player I. Fire at $\langle a\rangle$ and play optimally afterwards.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally afterwards.

We now have

$$\begin{aligned} v_{32}^{a} &= Q^2(a)v_{21}^{a} \\ &= \begin{cases} v_{21}Q^2(a) & \text{if } Q(\widehat{a}_{32}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107, \\ Q^2(a) - (1-v_{11})Q^3(a) & \\ & \text{if } Q(a_{21}) \ge Q(a) \ge Q(\widehat{a}_{21}) \cong 0.668179, \\ Q^4(a) & \text{if } Q(a) \le Q(\widehat{a}_{21}). \end{cases} \end{aligned} \tag{3}$$

Suppose that Player II does not fire at $\langle a\rangle$. We obtain

$$K(\xi,\widehat{\eta}) \ge P(a) + Q(a)\overset{1a}{v}_{22} - k(\widehat{\varepsilon})$$

$$= \begin{cases} 1-(1-v_{22})Q(a)-k(\widehat{\varepsilon}) & \text{if } 0.812085 \cong Q(a_{22}) \le Q(a), \\ 1-2Q(a)+(1+v_{21})Q^2(a)-k(\widehat{\varepsilon}) & \text{if } Q(a_{22}) \ge Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133, \\ 1-Q(a)+v_{11}Q^3(a)-k(\widehat{\varepsilon}) & \text{if } Q(\widehat{a}_{22}) \ge Q(a) \ge Q(a_{11}), \\ 1-Q(a)-Q^3(a)+2Q^4(a)-k(\widehat{\varepsilon}) & \text{if } Q(a_{11}) \ge Q(a) \ge 1/2, \\ 1-Q(a)-k(\widehat{\varepsilon}) & \text{if } Q(a) \le 1/2. \end{cases}$$

Consider the following cases:

(a) $Q(\widehat{a}_{32}) \ge Q(a) \ge Q(a_{22}) \cong 0.812085$. In this case we should prove that

$$1-(1-v_{22})Q(a) \ge v_{21}Q^2(a),$$

which holds if $Q(a) \le 0.835107$ and thus always in case (a).

(b) $0.781133 \cong Q(\widehat{a}_{22}) \le Q(a) \le Q(a_{22}) \cong 0.812085$. In this case we should prove that

$$1-2Q(a)+(1+v_{21})Q^2(a) \ge v_{21}Q^2(a),$$

which is always satisfied.

(c) $0.707107 \cong Q(a_{21}) \le Q(a) \le Q(\widehat{a}_{22})$. Now we should prove that

$$1-Q(a)+v_{11}Q^3(a) \ge v_{21}Q^2(a) \tag{4}$$

or

$$S(Q) = v_{11}Q^3(a) - v_{21}Q^2(a) - Q(a) + 1 \ge 0.$$

This function is decreasing in Q and $S(Q(\widehat{a}_{22})) \cong S(0.781133) > 0$. Thus the inequality is always satisfied in case (c).

(d) $0.668179 \cong Q(\widehat{a}_{21}) \le Q(a) \le Q(a_{21})$. In this case we should prove that

$$1-Q(a)+v_{11}Q^3(a) \ge Q^2(a)-(1-v_{11})Q^3(a).$$

This inequality can be rewritten in the form

$$(1-Q^2(a))(1-Q(a)) \ge 0$$

and is always satisfied.

(e) $Q(a_{11}) \le Q(a) \le Q(\widehat{a}_{21})$. In this case we should prove

$$1-Q(a)+Q^3(a)v_{11} \ge Q^4(a)$$

or

$$S(Q) = Q^4(a) - v_{11}Q^3(a) + Q(a) - 1 \le 0.$$

This function (of Q) is increasing and $S(Q(\widehat{a}_{21})) \cong S(0.668179) \ge 0$. Thus the inequality always holds in this interval.

(f) $1/2 \le Q(a) \le Q(a_{11})$. In this case we have

$$1 - Q(a) - Q^3(a) + 2Q^4(a) \ge Q^4(a)$$

or

$$(1 - Q(a))(1 - Q^3(a)) \ge 0,$$

which is always satisfied.

(g) $Q(a) \le 1/2$. In this case we have $1 - Q(a) \ge Q^4(a)$ or

$$S(Q) = Q^4(a) + Q(a) - 1 \le 0.$$

This function is increasing and $S(1/2) < 0$. Thus the inequality holds.

Now we suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi}, \eta) \le \; - P(a) + Q(a)\overset{2}{v}{}^{a}_{31} + k(\widehat{\varepsilon})$$
$$= \begin{cases} -1 + (1 + v_{31})Q(a) + k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a) \ge Q(a_{31}) \cong 0.773459, \\ -1 + 2Q(a) - (1 - v_{21})Q^2(a) + k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a_{31}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107, \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) + k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a) \le Q(a_{21}). \end{cases}$$

Consider the following cases:

(a) $Q(a) \ge Q(a_{31}) \cong 0.773459$. In this case we have

$$-1 + (1 + v_{31})Q(a) \le v_{21}Q^2(a),$$

which is satisfied if $Q(a) \le Q(\widehat{a}_{32}) \cong 0.831642$.

(b) $0.707107 \cong Q(a_{21}) \le Q(a) \le Q(a_{31}) \cong 0.773459$. Now we have

$$-1 + 2Q(a) - (1 - v_{21})Q^2(a) \le v_{21}Q^2(a).$$

This inequality is always satisfied.

(c) $0.668179 \cong Q(\widehat{a}_{21}) \le Q(a) \le Q(a_{21})$. Now we should prove that

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) \le Q^2(a) - (1 - v_{11})Q^3(a)$$

or that

$$S(Q) = (3 - v_{11})Q^3(a) - 3Q^2(a) + 2Q(a) - 1 \le 0.$$

This polynomial is increasing in Q in the considered case and $S(Q(\widehat{a}_{21})) \cong S(0.668179) > 0$. Thus the inequality holds.

(d) $Q(a) \le Q(\widehat{a}_{21}) \cong 0.668179$. In this case we obtain

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) \le Q^4(a).$$

This inequality can be rewritten in the form

$$(1 + Q^2(a))(1 - Q(a))^2 \ge 0$$

and is always satisfied.

Thus if $Q(a) \le Q(\widehat{a}_{32})$ then the strategies ξ and η are optimal in limit and the limit value of the game is given by (3).

The duel $(3,2), \langle 1, a \wedge c, a\rangle$

Case 1: $Q(a) \ge Q(a_{32}) \cong 0.833869$. The strategies ξ and η defined in the duel $(3,1), \langle a\rangle$, Case 1, are now also optimal in limit.

Case 2: $Q(a) \le Q(a_{32})$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel.

Strategy of Player II. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

We now have

$$\overset{1}{v}{}^{a}_{32} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{31}.$$

The proof that ξ and η are optimal in limit is omitted.

The duel $(3,2), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \ge Q(a_{32})$. The strategies ξ and η defined in the duel $(3,1), \langle a\rangle$, Case 1, are now also optimal in limit.

Case 2: $Q(a) \le Q(a_{32})$. Define ξ and η as follows:

Strategy of Player I. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel. If he has fired, also play optimally.

We now have

$$\overset{2}{v}{}^{a}_{32} = P(a) + Q(a)\overset{1}{v}{}^{a}_{22}$$

for $Q(a) \le Q(a_{32})$.

The proof that ξ and η are optimal in limit is omitted.

5. Results for the duel $(3,2)$**.** We have

$$\overset{1}{v}{}^{a}_{32} = \begin{cases} {}_{32} = 0.289928 & \text{if } Q(a) \ge Q(a_{32}) \cong 0.833869, \\ -1 + (1+v_{31})Q(a) & \text{if } Q(a_{32}) \ge Q(a) \ge Q(a_{31}) \cong 0.773459, \\ -1 + 2Q(a) - (1-v_{21})Q^2(a) & \\ & \text{if } Q(a_{31}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107, \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) & \\ & \text{if } Q(a) \le Q(a_{21}); \end{cases}$$

$$v^{a}_{32} = \begin{cases} v_{32} & \text{if } Q(a) \ge Q(a_{32}), \\ -1 + (1+v_{31})Q(a) & \text{if } Q(a_{32}) \ge Q(a) \ge Q(\widehat{a}_{32}) \cong 0.831642, \\ v_{21}Q^2(a) & \text{if } Q(\widehat{a}_{32}) \ge Q(a) \ge Q(a_{21}), \\ Q^2(a) - (1-v_{11})Q^3(a) & \\ & \text{if } Q(a_{21}) \ge Q(a) \ge Q(\widehat{a}_{21}) \cong 0.668179, \\ Q^4(a) & \text{if } Q(a) \le Q(\widehat{a}_{21}); \end{cases}$$

$$\overset{2}{v}{}^{a}_{32} = \begin{cases} v_{32} & \text{if } Q(a) \ge Q(a_{32}), \\ 1 - (1-v_{22})Q(a) & \text{if } Q(a_{32}) \ge Q(a) \ge Q(a_{22}) \cong 0.812085, \\ 1 - 2Q(a) + (1+v_{21})Q^2(a) & \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{21}), \\ 1 - 2Q(a) + 2Q^2(a) - (1-v_{11})Q^3(a) & \\ & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}), \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) & \\ & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

6. The duel $(3,3)$

The duel $(3,3), \langle a \rangle$

Case 1: $Q(a) \ge Q(a_{23}) \cong 0.882709$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, reach the point a_{23}, fire at a^{ε}_{23} and play optimally the resulting duel. If he fired (say at a'), play optimally the duel $(3,2), \langle 2, a', a' \wedge \widehat{\varepsilon} \rangle$.

Strategy of Player II. If Player I has not fired before, fire at $\langle a_{33} \rangle$ and play optimally the resulting duel. If he fired (say at a'), play optimally the resulting duel $(2,3), \langle 1, a' \wedge \widehat{\varepsilon}, a' \rangle$. If Player I has not reached the point a_{33}, do not fire.

In [8] it is proved that ξ and η are optimal in limit and

$$v^{a}_{33} = v_{33} \cong 0.129435$$

if $Q(a_{23}) \cong 0.882709,\ Q(a_{33}) \cong 0.875580$.

Case 2: $0.882709 \cong Q(a_{23}) \ge Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472$. Define ξ and η as follows:

STRATEGY OF PLAYER I. If Player II has not fired before, fire at a^{ε} and play optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I has not fired before, fire at $\langle\widehat{a}\rangle$ and play optimally the resulting duel.

We now have

$$\begin{aligned} v_{33}^{a} &= P(a) + Q(a)\overset{1}{v}{}_{23}^{a} = 1 - 2Q(a) + (1+v_{22})Q^2(a) \\ &= -P(\widehat{a}) + Q(\widehat{a})v_{32}. \end{aligned}$$

It follows that

$$Q(\widehat{a}) = \frac{1}{1+v_{32}}(2 - 2Q(a) + (1+v_{22})Q^2(a)) \le Q(a)$$

if $Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472$.

Suppose that Player II fires at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we obtain

$$\begin{aligned} K(\xi,\widehat{\eta}) &\ge -P(a) + Q(a)\overset{2}{v}{}_{32}^{a} - k(\widehat{\varepsilon}) = -1 + (1+v_{32})Q(a) - k(\widehat{\varepsilon}) \\ &\ge 1 - 2Q(a) + (1+v_{22})Q^2(a) - k(\widehat{\varepsilon}) \end{aligned}$$

provided that

$$Q(a) \ge Q(a_{32}) \cong 0.833869.$$

Solving the first inequality we obtain $Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472$.

Suppose that Player II fires after $\langle a\rangle + \alpha(\varepsilon)$ or does not fire at all. We then obtain

$$K(\xi,\widehat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}_{23}^{a} - k(\widehat{\varepsilon}) = v_{33}^{a} - k(\widehat{\varepsilon}).$$

On the other hand, suppose that Player I fires before $\langle\widehat{a}\rangle$, at a'. We then have

$$\begin{aligned} K(\widehat{\xi},\eta) &\le P(a') + Q(a')\overset{1}{v}{}_{23}^{a'} + k(\widehat{\varepsilon}) \\ &= 1 - 2Q(a') + (1+v_{22})Q^2(a') + k(\widehat{\varepsilon}) \\ &\le 1 - 2Q(a) + (1+v_{22})Q(a) + k(\widehat{\varepsilon}). \end{aligned}$$

This inequality is satisfied if $Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472$.

Finally, suppose that Player I does not fire before or at $\langle\widehat{a}\rangle$. We then obtain

$$K(\widehat{\xi},\eta) \le -P(\widehat{a}) + Q(\widehat{a})\overset{2}{v}{}_{32}^{a} + k(\widehat{\varepsilon}) = v_{33}^{a} + k(\widehat{\varepsilon}).$$

Thus if $Q(a_{23}) \ge Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472$ then the strategies ξ and η are optimal in limit and the limit value of the game is

$$v_{33}^{a} = 1 - 2Q(a) + (1+v_{22})Q^2(a).$$

Case 3: $0.860449 \cong Q(\check{a}_{33}) \le Q(a) \le Q(a_{33})$. Define ξ and η as follows:

STRATEGY OF PLAYER I. If Player II has not fired before, fire at a^{ε} and play optimally the resulting duel.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$v_{33}^a = -1 + (1+v_{32})Q(a)$$

for $v_{32} \cong 0.289928$.

Suppose that Player I fires at $\langle a\rangle$. We then obtain

$$K(\widehat{\xi},\eta) \le Q^2(a)v_{22}^a + k(\widehat{\varepsilon}) \le -1 + (1+v_{32})Q(a) + k(\widehat{\varepsilon})$$

if $v_{22}Q^2(a) - (1+v_{32})Q(a) + 1 \le 0$, i.e. if $Q(a) \ge Q(\breve{a}_{33}) \cong 0.860449$.

On the other hand, suppose that Player II does not fire before $\langle a\rangle + \alpha(\varepsilon)$. Then we obtain

$$\begin{aligned}K(\xi,\widehat{\eta}) &\ge P(a) + Q(a)\overset{1}{v}{}_{23}^{a} - k(\widehat{\varepsilon})\\ &= 1 - Q(a) + Q(a)(-1 + (1+v_{22})Q(a)) - k(\widehat{\varepsilon})\\ &= 1 - 2Q(a) + (1+v_{22})Q^2(a) - k(\widehat{\varepsilon}) \ge -1 + (1+v_{32})Q(a) - k(\widehat{\varepsilon})\end{aligned}$$

provided that $0.882709 \cong Q(a_{23}) \ge Q(a) \ge 0.812085$. This leads to the inequality

$$(1+v_{22})Q^2(a) - (3+v_{32})Q(a) + 2 \ge 0,$$

which is satisfied if $Q(a) \le Q(\widehat{a}_{33}) \cong 0.875472$.

Thus if $0.860449 \le Q(a) \le 0.875472$, then the strategies ξ and η are optimal in limit and the limit value of the game is

$$v_{33}^a = -1 + (1+v_{32})Q(a)$$

Case 4: $Q(a) \le Q(\breve{a}_{33}) \cong 0.860449$. Define ξ and η as follows:

Strategy of Player I. Fire at $\langle a\rangle$ and play optimally afterwards.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally afterwards.

We now have

$$\begin{aligned}v_{33}^a &= Q^2(a)v_{22}^a\\ &= \begin{cases} v_{22}Q^2(a) & \text{if } 0.860449 \cong Q(\breve{a}_{33}) \ge Q(a) \ge Q(a_{22}) = 0.812085,\\ -Q^2(a) + (1+v_{21})Q^3(a) & \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133,\\ v_{11}Q^4(a) & \text{if } Q(\widehat{a}_{22}) \ge Q(a) \ge Q(a_{11}),\\ 2Q^5(a) - Q^4(a) & \\ & \text{if } Q(a_{11}) \ge Q(a) \ge 1/2,\\ 0 & \text{if } Q(a) \le 1/2.\end{cases}\end{aligned}$$

Suppose that Player II did not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we obtain

$$K(\xi,\widehat{\eta}) \geq \begin{cases} 1-(1-v_{23})Q(a)-k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a)\geq Q(a_{23})\cong 0.882709, \\ 1-2Q(a)+(1+v_{22})Q^2(a)-k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a_{23})\geq Q(a)\geq Q(a_{22})\cong 0.812085, \\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+(1+v_{11})Q^4(a)-k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a_{22})\geq Q(a)\geq Q(a_{11}), \\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+2Q^4(a)-2Q^5(a) \\ \quad +Q^6(a)-k(\widehat{\varepsilon}) \quad \text{if } \ Q(a)\leq Q(a_{11}). \end{cases}$$

Consider the following cases:

(a) $Q(a)\geq Q(a_{23})\cong 0.882709$. In this case we should have

$$1-(1-v_{23})Q(a)\geq v_{22}Q^2(a),$$

which is satisfied if $Q(a)\leq 0.893715$.

(b) $0.812085\cong Q(a_{22})\leq Q(a)\leq Q(a_{23})$. In this case we need

$$1-2Q(a)+(1+v_{22})Q^2(a)\geq v_{22}Q^2(a)$$

which is always satisfied.

(c) $0.781133\cong Q(\widehat{a}_{22})\leq Q(a)\leq Q(a_{22})$. In this case we have

$$1-2Q(a)+2Q^2(a)-2Q^3(a)+(1+v_{11})Q^4(a)\geq -Q^2(a)+(1+v_{21})Q^3(a)$$

or

$$S(Q)=(1+v_{11})Q^4(a)-(3+v_{21})Q^3(a)+3Q^2(a)-2Q(a)+1\geq 0.$$

This function is decreasing in Q in the interval $[Q(\widehat{a}_{22}),Q(a_{22})]$ and $S(Q(a_{22}))\geq 0$. Thus the inequality holds.

(d) $Q(a_{11})\leq Q(a)\leq Q(\widehat{a}_{22})$. In this case we need

$$1-2Q(a)+2Q^2(a)-2Q^3(a)+(1+v_{11})Q^4(a)\geq v_{11}Q^4(a),$$

or

$$(1+Q^2(a))(1-Q(a))^2\geq 0,$$

which is always satisfied.

(e) $1/2\leq Q(a)\leq Q(a_{11})$. In this case we have

$$Q^6(a)-4Q^5(a)+3Q^4(a)-3Q^3(a)+2Q^2(a)-2Q(a)+1\geq 0,$$

which is always satisfied in this interval.

(f) $Q(a)\leq 1/2$. In this case we obtain

$$1-2Q(a)+2Q^2(a)-2Q^3(a)+2Q^4(a)-2Q^5(a)+Q^6(a)\geq 0,$$

which can be rewritten in the form

$$(1+Q^2(a)+Q^4(a))(1-Q(a))^2\geq 0.$$

Thus if $Q(a) \le 0.893715$ the inequality $K(\xi, \widehat{\eta}) \ge v_{33}^a - k(\widehat{\varepsilon})$ holds.

On the other hand, suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi}, \eta) \le -P(a) + Q(a)\overset{2}{v}{}_{32}^{a} + k(\widehat{\varepsilon})$$
$$= \begin{cases} -1 + & (1+v_{32})Q(a) + k(\widehat{\varepsilon}) \\ & \text{if } Q(a) \ge Q(a_{32}) \cong 0.833869, \\ -1 + & 2Q(a) - (1-v_{22})Q^2(a) + k(\widehat{\varepsilon}) \\ & \text{if } Q(a_{32}) \ge Q(a) \ge Q(a_{22}) \cong 0.812085, \\ -1 + & 2Q(a) - 2Q^2(a) + (1+v_{21})Q^3(a) + k(\widehat{\varepsilon}) \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107, \\ -1 + & 2Q(a) - 2Q^2(a) + 2Q^3(a) - (1-v_{11})Q^4(a) + k(\widehat{\varepsilon}) \\ & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}), \\ -1 + & 2Q(a) - 2Q^2(a) + 2Q^3(a) - 2Q^4(a) + 2Q^5(a) + k(\widehat{\varepsilon}) \\ & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

Consider the following cases:

(a) $Q(a) \ge Q(a_{32}) \cong 0.833869$. We should prove that

$$-1 + (1+v_{32})Q(a) \le v_{22}Q^2(a),$$

which is satisfied if $Q(a) \le Q(\check{a}_{33}) \cong 0.860449$.

(b) $0.812085 \cong Q(a_{22}) \le Q(a) \le Q(a_{32})$. We should prove that

$$-1 + 2Q(a) - (1-v_{22})Q^2(a) \le v_{22}Q^2(a),$$

which is always satisfied.

(c) $0.781133 \cong Q(\widehat{a}_{22}) \le Q(a) \le Q(a_{22})$. In this case we need

$$-1 + 2Q(a) - 2Q^2(a) + (1+v_{21})Q^3(a) \le -Q^2(a) + (1+v_{21})Q^3(a),$$

which is also always satisfied.

(d) $0.707107 \cong Q(a_{21}) \le Q(a) \le Q(\widehat{a}_{22})$. In this case we obtain the inequality

$$S(Q) = v_{11}Q^4(a) - (1+v_{21})Q^3(a) + 2Q^2(a) - 2Q(a) + 1 \ge 0.$$

This function is increasing in the considered interval and $S(Q(\widehat{a}_{22})) \cong S(0.781133) > 0$. Thus the inequality holds in this interval.

(e) $Q(a_{11}) \le Q(a) \le Q(a_{21})$. In this case we obtain

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - (1-v_{11})Q^4(a) \le v_{11}Q^4(a),$$

which is always satisfied.

(f) $1/2 \le Q(a) \le Q(a_{11})$. In this case we have

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - Q^4(a) = -(1+Q^2(a))(1-Q(a))^2 \le 0.$$

(g) Finally, let $Q(a) \leq 1/2$. We have

$$S(Q) = -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - 2Q^4(a) + 2Q^5(a) \leq 0.$$

The function $S(Q)$ is increasing in Q and $S(1/2) \leq 0$. Thus the inequality holds.

Therefore the inequality $K(\widehat{\xi}, \eta) \leq v^a_{33} + k(\widehat{\varepsilon})$ holds for $Q(a) \leq 0.860449$.
From the above it follows that if

$$Q(a) \leq Q(\check{a}_{33}) \cong 0.860449,$$

then the strategies ξ and η are optimal in limit.

The duel $(3,3), \langle 1, a \wedge c, a\rangle$. In Cases 1 and 2, i.e. if

$$Q(a) \geq Q(\widehat{a}_{33}) \cong 0.875472,$$

then the strategies optimal in limit are the same as in the duel $(3,3), \langle a\rangle$. The easy proof is omitted.

Case 3: $Q(a) \leq Q(\widehat{a}_{33}) \cong 0.875472$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel.

Strategy of Player II. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

We now have

$$\overset{1}{v}{}^a_{33} = -P(a) + Q(a)\overset{2}{v}{}^a_{32}$$

for $Q(a) \leq Q(\widehat{a}_{33}) \cong 0.875472$.
The proof of the limit optimality of ξ and η is omitted.

The duel $(3,3), \langle 2, a, a \wedge c\rangle$. Also here, in Cases 1 and 2 the strategies optimal in limit are the same as in the duel $(3,3), \langle a\rangle$. The proof is omitted.

Case 3: $Q(a) < Q(\widehat{a}_{33}) \cong 0.875472$. Define ξ and η as follows:

Strategy of Player I. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel.

We now have

$$\overset{2}{v}{}^a_{33} = P(a) + Q(a)\overset{1}{v}{}^a_{23}$$

for $Q(a) < Q(\widehat{a}_{33}) \cong 0.875472$. Also here, the proof of the limit optimality of ξ and η is omitted.

7. Results for the duel $(3,3)$**.** We have

$$\overset{1}{v}{}^{a}_{33} = \begin{cases} v_{33} \cong 0.129435 & \text{if } Q(a) \ge Q(a_{23}) \cong 0.882709, \\ 1-2Q(a)+(1+v_{22})Q^2(a) & \\ & \text{if } Q(a_{23}) \ge Q(a) \ge Q(\widehat{a}_{33}) \cong 0.875472, \\ -1+(1+v_{32})Q(a) & \text{if } Q(\widehat{a}_{33}) \ge Q(a) \ge Q(a_{32}) \cong 0.833869, \\ -1+2Q(a)-(1-v_{22})Q^2(a) & \\ & \text{if } Q(a_{32}) \ge Q(a) \ge Q(a_{22}) \cong 0.812085, \\ -1+2Q(a)-2Q^2(a)+(1+v_{21})Q^3(a) & \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{21}) \cong 0.707107, \\ -1+2Q(a)-2Q^2(a)+2Q^3(a)-(1-v_{11})Q^4(a) & \\ & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}), \\ -1+2Q(a)-2Q^2(a)+2Q^3(a)-2Q^4(a)+2Q^5(a) & \\ & \text{if } Q(a) \le Q(a_{11}); \end{cases}$$

$$v^{a}_{33} = \begin{cases} v_{33} & \text{if } Q(a) \ge Q(a_{23}), \\ 1-2Q(a)+(1+v_{22})Q^2(a) & \\ & \text{if } Q(a_{23}) \ge Q(a) \ge Q(\widehat{a}_{33}), \\ -1+(1+v_{32})Q(a) & \text{if } Q(\widehat{a}_{33}) \ge Q(a) \ge Q(\breve{a}_{33}) \cong 0.860449, \\ v_{22}Q^2(a) & \text{if } Q(\breve{a}_{33}) \ge Q(a) \ge Q(a_{22}), \\ -Q^2(a)+(1+v_{21})Q^3(a) & \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133, \\ v_{11}Q^4(a) & \text{if } Q(\widehat{a}_{22}) \ge Q(a) \ge Q(a_{11}), \\ 2Q^5(a)-Q^4(a) & \text{if } Q(a_{11}) \ge Q(a) \ge 1/2, \\ 0 & \text{if } Q(a) \le 1/2; \end{cases}$$

$$\overset{2}{v}{}^{a}_{33} = \begin{cases} v_{33} & \text{if } Q(a) \ge Q(a_{23}), \\ 1-2Q(a)+(1+v_{22})Q^2(a) & \\ & \text{if } Q(a_{23}) \ge Q(a) \ge Q(a_{22}), \\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+(1+v_{11})Q^4(a) & \\ & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{11}), \\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+2Q^4(a)-2Q^5(a)+Q^6(a) & \\ & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

For other noisy duels see [1], [3], [4], [7]–[14].

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