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AVERAGE COST MARKOV CONTROL PROCESSES WITH WEIGHTED NORMS: VALUE ITERATION

Abstract. This paper shows the convergence of the *value iteration* (or successive approximations) algorithm for average cost (AC) Markov control processes on Borel spaces, with possibly unbounded cost, under appropriate hypotheses on weighted norms for the cost function and the transition law. It is also shown that the aforementioned convergence implies strong forms of AC-optimality and the existence of forecast horizons.

1. Introduction. This paper deals with discrete-time Markov control processes (MCPs) on Borel spaces, with possibly unbounded costs, and the *average cost* (AC) criterion. Under suitable hypotheses on weighted norms for the one-stage cost function and the transition law, our main result (Theorem 2.6) shows the convergence of the *value iteration* (VI)—or successive approximations—algorithm. This result, which is very important in itself (see e.g. [1, 2, 5, 9, 11, 15, 16, 17] and their references for different types of applications of the VI algorithm), it is shown to have significant consequences, such as strong forms of AC-optimality (Corollaries 2.8, 2.9, 2.11) and the existence of forecast horizons (Corollary 2.12).

This paper is basically a sequel to [4], where, in particular, the existence of a solution to the *Average Cost Optimality Equation* (ACOE) is shown. This is in fact our point of departure: After introducing the necessary assumptions to obtain the ACOE (cf. Theorem 2.5), it is shown that an additional "topological recurrence" condition (cf. (2.19)) yields the convergence of the VI algorithm. The proofs of Theorem 2.6 and its corollaries are pre-

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sented in Sections 3 and 4 respectively. To conclude the paper, we present in Section 5 an example of a control system in which all the assumptions of Theorem 2.6 are satisfied.

2. Assumptions and main results. The discrete-time Markov control model (X, A, Q, c) we consider has been discussed by many authors, so our review can be brief. Our notation generally follows the companion paper [4], which also provides basic references on this topic.

The state space X and the action (or control) set A are both Borel spaces. For each $x \in X$, $A(x)$ denotes the set of feasible actions in x; $A(x)$ is a nonempty Borel subset of A. The set

$$\mathbb{K} := \{(x,a) \mid x \in X,\ a \in A(x)\} \tag{2.1}$$

is assumed to be a Borel subset of $X \times A$. The transition law Q is a stochastic kernel on X given $\mathbb{K}$, and the one-stage cost c is a real-valued measurable function on $\mathbb{K}$.

ASSUMPTION 2.1. (a) The cost c is nonnegative and $a \mapsto c(x,a)$ is l.s.c. (lower semicontinuous) on $A(x)$ for every $x \in X$; moreover, there exists a measurable function $v : X \to \mathbb{R}$ such that $\overline{v} := \inf_X v(x) > 0$,

$$\sup_{A(x)} c(x,a) \le v(x) \quad \forall x \in X, \tag{2.2a}$$

$$\int_X v(y)\, Q(dy \mid x,a) < \infty \quad \forall (x,a) \in \mathbb{K}, \tag{2.2b}$$

and the mapping

$$a \to \int_X v(y)\, Q(dy \mid x,a) \tag{2.2c}$$

is continuous on $A(x)$ for every $x \in X$;

(b) $A(x)$ is compact for every state x;

(c) $a \mapsto Q(B \mid x,a)$ is continuous on $A(x)$ for every $x \in X$ and $B \in \mathbb{B}_X$, where $\mathbb{B}_X$ denotes the Borel σ-algebra of X.

Let Δ be the class of all control policies and Δ_0 the subclass of *stationary* policies. We identify Δ_0 with the family of all measurable functions $f : X \to A$ such that $f(x) \in A(x)$ for $x \in X$.

As in [4], if $f \in \Delta_0$, we write

$$c(x, f(x)) =: c(x,f) \quad \text{and} \quad Q(\cdot \mid x, f(x)) = Q(\cdot \mid x, f) = Q_f(\cdot \mid x). \tag{2.3}$$

Let $h : X \to \mathbb{R}$ be a given measurable function. Then for each $\delta \in \Delta$ and $x \in X$, (2.4) $$J_n(\delta, x, h) := E_x^\delta\Big[\sum_{t=0}^{n-1} c(x_t, a_t) + h(x_n)\Big]$$

is the expected *n-stage cost* when using the policy δ, given the initial state $x_0 = x$ and the *terminal cost* function h. The *optimal* n-stage cost is

(2.5) $$J_n^*(x, h) := \inf_{\Delta} J_n(\delta, x, h).$$

If $h(\cdot) \equiv 0$ we write

(2.6) $$J_n(\delta, x, 0) := J_n(\delta, x) \quad \text{and} \quad J_n^*(x, 0) := v_n(x).$$

The long-run expected *average cost* (AC) when using a policy δ, given the initial state $x_0 = x$, is

(2.7) $$J(\delta, x) := \limsup_{n\to\infty} \frac{1}{n} J_n(\delta, x).$$

A policy δ^* is said to be *AC-optimal* if

(2.8) $$J(\delta^*, x) = \inf_{\Delta} J(\delta, x) =: J^*(x) \quad \forall x \in X,$$

and J^* thus defined is called the *optimal AC-function*.

To obtain AC-optimal policies we impose the following two assumptions.

ASSUMPTION 2.2. For every stationary policy $f \in \Delta_0$ the (state) Markov process defined by the stochastic kernel Q_f in (2.3) is positive Harris-recurrent [12], i.e., it is Harris-recurrent and has an invariant probability measure q_f:

(2.9) $$q_f(B) = \int_X Q_f(B \mid x)\, q_f(dx) \quad \forall B \in \mathbb{B}_X.$$

ASSUMPTION 2.3. There exists a probability measure ν on X and a nonnegative number $\alpha < 1$ for which the following holds: For every $f \in \Delta_0$ there exists a nonnegative function $h_f \le 1$ on X such that for all $x \in X$ and $B \in \mathbb{B}_X$:

(a) $Q_f(B \mid x) \ge h_f(x)\nu(B)$;

(b) $\int_X v(y)\, Q_f(dy \mid x) \le h_f(x)\|\nu\|_v + \alpha v(x)$, where v is the function in (2.2) and

$$\|\nu\|_v := \int_X v(x)\, \nu(dx) < \infty;$$

(c) $\inf_{\Delta_0} \int_X h_f(x)\, \nu(dx) =: \gamma > 0$.

Let Φ_v be the normed linear space of all measurable functions ϕ on X with

(2.10) $$\|\phi\|_v := \sup_X \frac{|\,\phi(x)\,|}{v(x)} < \infty.$$

Under Assumptions 2.1–2.3 it is shown in [4, Theorem 2.6] that there exist a constant $\varrho^* \ge 0$, a function $\phi^* \in \Phi_v$ and a stationary policy f^* such that

$$\varrho^* + \phi^*(x) \ge \min_{A(x)} \Big[c(x,a) + \int_X \phi^*(y)\, Q(dy \mid x,a)\Big] \tag{2.11}$$

$$= c(x,f^*) + \int_X \phi^*(y)\, Q(dy \mid x, f^*) \quad \forall x \in X,$$

and

$$J(f^*,x) = J^*(x) = \varrho^* \quad \forall x \in X. \tag{2.12}$$

In other words, the pair (ϱ^*,ϕ^*) is a *solution* of the so-called *average cost optimality inequality* (ACOI) (2.11), while (2.12) states that f^* *is AC-optimal and that the optimal AC-function is the constant* ϱ^*.

To get equality in (2.11) we need an additional assumption, where we use the following notation: If d_1 and d_2 denote the metrics on X and A respectively, we define a metric d on $\mathbb{K}$ as

$$d((x,a),(x',a')) := \max\{d_1(x,x'), d_2(a,a')\}$$

for all (x,a) and (x',a') in $\mathbb{K}$. Furthermore, Ψ denotes the class of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(s) \to 0$ as $s \downarrow 0$.

ASSUMPTION 2.4. (a) The compact-valued multifunction $x \mapsto A(x)$ is continuous with respect to the Hausdorff metric;

(b) for each $x \in X$, there exist functions ψ_x^c and ψ_x^Q in Ψ such that for all $a \in A(x)$ and $(x',a') \in \mathbb{K}$:

(i) $|c(k) - c(k')| \le \psi_x^c[d(k,k')]$, and
(ii) $\|Q(\cdot \mid k) - Q(\cdot \mid k')\|_v \le \psi_x^Q[d(k,k')]$,

where $k := (x,a)$, $k' := (x',a')$, and, for any finite signed measure μ on X,

$$\|\mu\|_v := \int_X v(x)|\mu|(dx), \tag{2.13}$$

with $|\mu|$:= total variation of μ.

Remark. If the compact sets $A(x)$ do not depend on x, i.e., $A(x) \equiv A$ for all x in X, then Assumption 2.4 may be replaced by the following:

ASSUMPTION 2.4′. For each x in X there exist functions ψ_x^c and ψ_x^Q such that for all $x' \in X$:

(i) $\sup_A |c(x,a) - c(x',a)| \le \psi_x^c[d_1(x,x')]$, and
(ii) $\sup_A \|Q(\cdot \mid x,a) - Q(\cdot \mid x',a)\|_v \le \psi_x^Q[d_1(x,x')]$.

Then Theorem 2.8 of [4] states the following:

THEOREM 2.5. *If Assumptions* 2.1–2.4 *hold*, *then*:

(a) *There exists a canonical triplet* (ϱ^*, ϕ^*, f^*), *where* $\varrho^* \geq 0$ *is a constant*, ϕ^* *is a continuous function in* Φ_v, *and* f^* *is a stationary policy*; *that is*, (ϱ^*, ϕ^*, f^*) *satisfies the average cost optimality equation* (*ACOE*):

$$\varrho^* + \phi^*(x) = \min_{A(x)} \Big[c(x,a) + \int_X \phi^*(y)\, Q(dy \mid x, a)\Big] \tag{2.14}$$
$$= c(x, f^*) + \int_X Q^*(y)\, Q(dy \mid x, f^*) \quad \forall x \in X;$$

(b) (2.12) *holds true*, *i.e.*, f^* *is AC-optimal and the constant* ϱ^* *is the optimal AC-function.*

Equivalently (see e.g. [4, 8, 9, 11]), Theorem 2.5(a) says that for every $x \in X$ and $n = 1, 2, \ldots,$

$$J_n(f^*, x, \phi^*) = J_n^*(x, \phi^*) = n\varrho^* + \phi^*(x), \tag{2.15}$$

where J_n and J_n^* are the functions defined in (2.4)–(2.6).

In [4] we obtained Theorem 2.5 by the so-called "vanishing discount" approach in which the AC problem is studied via β-discounted cost problems in the limit as $\beta \uparrow 1$. In contrast, basically the *main problem* we are concerned with in this paper is to obtain $(\varrho^*, \phi^*(\cdot))$ in (2.14) by the *value iteration* (VI) algorithm, which is the following. Let v_n be the optimal n-stage cost in (2.6), i.e.,

$$v_n(x) := \inf_{\Delta} J_n(\delta, x), \quad x \in X,\ n \geq 1; \quad v_0(\cdot) := 0, \tag{2.16}$$

and let $z \in X$ be an arbitrary (but *fixed*) state. Define a sequence of constants j_n and a sequence of functions ϕ_n as

$$j_n := v_n(z) - v_{n-1}(z) \quad \text{and} \quad \phi_n(x) := v_n(x) - v_n(z), \quad x \in X. \tag{2.17}$$

Then the VI algorithm is said to converge if, as $n \to \infty$,

$$j_n \to \varrho^* \quad \text{and} \quad \phi_n(x) \to \phi^*(x) \quad \forall x \in X. \tag{2.18}$$

The following result states that (2.18) holds under the hypotheses of Theorem 2.5 and the additional condition (2.19) in which $q_f, f \in \Delta_0$, is the probability measure in Assumption 2.2.

THEOREM 2.6. *If Assumptions* 2.1–2.4 *hold and*, *in addition*, f^* *is such that*

$$q_{f^*}(U) > 0 \quad \textit{for each nonempty open set } U \subset X, \tag{2.19}$$

then the VI algorithm converges and, *moreover*, *the convergence* $\phi_n \to \phi^*$ *in* (2.18) *is uniform on compact sets.*

Remark. An obvious sufficient condition for (2.19) is that $Q(U \mid x, a) > 0$ for every open set $U \subset X$, $x \in X$ and $a \in A(x)$. Other sufficient conditions may be found, e.g., in [10].

Theorem 2.6 has important consequences. To state them, let us first recall the following definitions (cf. [8, 9, 11]).

DEFINITION 2.7. A policy δ^* is said to be:

(a) *strong AC-optimal* if

$$J(\delta^*, x) \le \liminf_{n\to\infty} \frac{J_n(\delta, x)}{n} \quad \forall \delta \in \Delta,\ x \in X;$$

(b) *F-strong AC-optimal* ("F" for Flynn—see [3]) if

$$\lim_{n\to\infty} \frac{J_n(\delta^*, x) - v_n(x)}{n} = 0 \quad \forall x \in X.$$

COROLLARY 2.8. *Under the hypotheses of Theorem* 2.6:

(a) $\lim_{n\to\infty} v_n(x)/n = \varrho^*$ *for all* $x \in X$; *in fact*,

$$\left| \frac{v_n(x)}{n} - \varrho^* \right| \le c_1 \frac{v(x)}{n} \tag{2.20}$$

for all $n \ge 1$, $x \in X$, *and some constant* c_1;

(b) *the canonical policy* f^* *in* (2.14)–(2.15) *is F-strong AC-optimal*;

(c) $J(f^*, x) := \limsup_{n\to\infty} J_n(f^*, x)/n = \lim_{n\to\infty} J_n(f^*, x)/n = \varrho^*$ *for all* $x \in X$;

(d) f^* *is strong AC-optimal.*

On the other hand, from elementary Dynamic Programming (see e.g. [1, 2, 9]) it is well known that, under Assumption 2.1, the functions v_n in (2.16) can be iteratively obtained as

$$v_n(x) = \min_{A(x)} \Big[c(x,a) + \int_X v_{n-1}(y)\, Q(dy \mid x, a) \Big] \tag{2.21}$$

for all $x \in X$ and $n = 1, 2, \ldots$, with $v_0(\cdot) := 0$, which, incidentally, motivates the name of *value iteration* (VI) *functions* for the v_n. Moreover—again under Assumption 2.1 (cf. e.g. Lemma 4.2 in [4])—for every $n \ge 1$ there exists a stationary policy $f_n \in \Delta_0$ such that $f_n(x) \in A(x)$ realizes the minimum in (2.21) for all x in X, i.e.,

$$v_n(x) = c(x, f_n) + \int_X v_{n-1}(y)\, Q(dy \mid x, f_n) \quad \forall x \in X. \tag{2.22}$$

The f_n form a sequence that "converges" to a canonical policy $\widehat{f} \in \Delta_0$ in the following sense.

COROLLARY 2.9. *Under the hypotheses of Theorem* 2.6 *there exists a stationary policy* $\widehat{f}$ *such that*:

(a) *for every* $x \in X$, $\widehat{f}(x)$ *is an accumulation point of* $\{f_n(x)\}$;

(b) $\widehat{f}$ *is AC-optimal and* $(\varrho^*, \phi^*, \widehat{f})$ *is a canonical triplet*, *i.e.*, (2.14)–(2.15) *hold when* f^* *is replaced by* $\widehat{f}$.

It also turns out that the "VI policies" f_n are *asymptotically optimal* in the sense of (2.25) below. To state this in precise terms, let us first note the following result (proved in Section 3), which in fact is also used to prove Theorem 2.6.

LEMMA 2.10. *Under Assumptions* 2.1–2.3, *for every stationary policy* $f \in \Delta_0$ *the average cost* $J(f, \cdot)$ *is a constant* $J(f)$ *given by*

$$J(f, x) = J(f) := \int_X c(y, f)\, q_f(dy) \quad \forall x \in X. \tag{2.23}$$

Then we have (with $\|\cdot\|_v$ being the norm in (2.10)):

COROLLARY 2.11. *Suppose that the hypotheses of Theorem* 2.6 *hold and*, *moreover*, *the convergence* $\phi_n \to \phi^*$ *in* (2.18) *is such that*, *as* $n \to \infty$,

$$\|\phi_n - \phi^*\|_v \to 0. \tag{2.24}$$

Then

$$J(f_n) \to \varrho^*. \tag{2.25}$$

Finally, we give conditions for the existence of *forecast horizons* N, which is an important issue in some applications (see e.g. [6, 13]).

COROLLARY 2.12. *Suppose that Assumptions* 2.1–2.4 *hold and let* (ϱ^*, ϕ^*, f^*) *be as in Theorem* 2.5. *Also suppose that*, *for every* x *in* X, *the control constraint set* $A(x)$ *is finite and*, *moreover*, $f^*(x)$ *is the unique minimizer of* (2.14). *Then for any initial state* $x \in X$ *there exists an integer* N *such that* $f_n(x) = f^*(x)$ *for* $n \ge N$; *that is*, *in* (2.22) *we have*

$$v_n(x) = c(x, f^*) + \int_X v_{n-1}(y)\, Q(dy \mid x, f^*) \quad \forall n \ge N.$$

In Section 5 we show an example in which the hypotheses of Theorem 2.6 are all true. First, the theorem itself and its corollaries are proved in Sections 3 and 4 respectively.

3. Proof of Theorem 2.6.

The main idea behind the proof of Theorem 2.6 is basically the same originally used by White [17] (cf. [5, 9, 11, 16]). Namely, one considers the "error" functions

$$e_n(x) := n\varrho^* + \phi^*(x) - v_n(x), \quad x \in X,\ n = 0, 1, \ldots, \tag{3.1}$$

with $(\varrho^*, \phi^*(\cdot))$ and v_n as in (2.14)–(2.15) and (2.6) respectively. Then the idea is to show that e_n converges uniformly on compact sets to a constant, say c_2, i.e.,

$$\lim_{n\to\infty} e_n(x) = c_2 \quad \forall x \in X. \tag{3.2}$$

Finally, with z as in (2.17) and observing that the function ϕ^* can be chosen so that $\phi^*(z) = 0$ (see [4, Remark 5.3]) we may rewrite ϕ_n and j_n as

$$\phi_n(x) = \phi^*(x) - (e_n(x) - e_n(z)) \tag{3.3}$$

and

$$j_n = \varrho^* - (e_n(z) - e_{n-1}(z)); \tag{3.4}$$

thus (3.2) implies (2.18).

The remainder of this section is dedicated to proving (3.2), but, first, as a further motivation for the proof, note that (2.21) can be equivalently written as

$$j_n + \phi_n(x) = \min_{A(x)} \Big[c(x,a) + \int_X \phi_{n-1}(y)\, Q(dy \mid x,a)\Big], \tag{3.5}$$

which is of the same form as the ACOE (2.14). This clearly suggests that (2.18) should yield the ACOE in the limit as $n \to \infty$.

Lemma 3.1. *Suppose that Assumptions* 2.1–2.3 *hold. Then there are constants* c_3, c_4 *and* c_5 *such that for all* $x \in X$ *and* $n = 0, 1, \ldots$:

(a) $\sup_{\Delta_0} E_x^f v(x_n) \le c_3 v(x)$;
(b) $\sup_{\Delta_0} E_x^f |\phi^*(x_n)| \le c_4 v(x)$;
(c) $\sup_n \|e_n\|_v \le c_4$;
(d) $\sup_n \|\phi_n\|_v \le c_5$.

Proof. Let $f \in \Delta_0$ be an arbitrary policy and $x \in X$ an arbitrary initial state.

(a) By the Markov property and Assumption 2.3(b),

$$\begin{aligned} E^f[v(x_n)|x_0, \ldots, x_{n-1}] &= \int_X v(y)\, Q_f(dy \mid x_{n-1}) \\ &\le h_f(x_{n-1})\|\nu\|_v + \alpha v(x_{n-1}). \end{aligned}$$

Hence

$$E_x^f v(x_n) \le \|\nu\|_v + \alpha E_x^f v(x_{n-1}).$$

Iteration of this inequality yields

$$\begin{aligned} E_x^f v(x_n) &\le \|\nu\|_v(1 + \ldots + \alpha^{n-1}) + \alpha^n v(x) \\ &\le \frac{\|\nu\|_v}{1-\alpha} + v(x) \\ &\le c_3 v(x), \quad \text{with } c_3 := \frac{\|\nu\|_v}{(1-\alpha)\bar{v}} + 1. \end{aligned}$$

Since c_3 is independent of f and x, we obtain (a).

(b) $E_x^f|\phi^*(x_n)| \le \|\phi^*\|_v E_x^f v(x_n) \le c_3\|\phi^*\|_v v(x)$, which yields (b) with $c_4 := c_3\|\phi^*\|_v$.

(c) Note that, from (2.15) and (2.4)–(2.5), we may rewrite e_n in (3.1) as

$$\begin{aligned} e_n(x) &= J_n(f^*, x, \phi^*) - v_n(x) \\ &= J_n(f^*, x) + E_x^{f^*}\phi^*(x_n) - v_n(x) \qquad (3.6) \\ &= \inf_{\Delta} J_n(\delta, x, \phi^*) - v_n(x) \\ &= \inf_{\Delta}[J_n(\delta, x) + E_x^{\delta}\phi^*(x_n)] - v_n(x). \qquad (3.7) \end{aligned}$$

Since $v_n(x) \le J_n(f^*, x)$ (see (2.6)), (3.6) and part (b) yield

$$e_n(x) \ge E_x^{f^*}\phi^*(x_n) \ge -c_4 v(x) \quad \forall x \in X,\ n \ge 0.$$

Similarly, from (3.7),

$$\begin{aligned} e_n(x) &\le \inf_{\Delta} J_n(\delta, x) + \sup_{\Delta_0} E_x^f \phi^*(x_n) - v_n(x) \\ &= \sup_{\Delta_0} E_x^f \phi^*(x_n) \le c_4 v(x) \quad \forall x \in X,\ n \ge 0. \end{aligned}$$

Hence, $|e_n(x)| \le c_4 v(x)$ for $x \in X$ and $n \ge 0$, which proves (c).

(d) From (c), $|e_n(z)| \le c_4 v(z) \le v(z)v(x)/\overline{v}$ for $n \ge 0$, with $\overline{v} := \inf_x v(x) > 0$ (see Assumption 2.1(a)). Therefore, part (d) follows from (c) and (3.3), with $c_5 := \|\phi^*\|_v + c_4(1 + v(z)/\overline{v})$. ■

LEMMA 3.2. *Under Assumptions* 2.1–2.4, *the family of functions* $\{\phi_n : n = 0, 1 \ldots\}$ (*hence* $\{e_n, n = 0, 1, \ldots\}$) *is pointwise bounded and equicontinuous on X.*

Proof. Pointwise boundedness of $\{\phi_n\}$ follows from Lemma 3.1(d) since $|\phi_n(x)| \le c_5 v(x)$ for every $x \in X$ and all n. On the other hand, from (3.5),

$$\phi_n(x) = -j_n + \min_{A(x)} \Big[c(x, a) + \int_X \phi_{n-1}(y)\, Q(dy \mid x, a)\Big].$$

Therefore, using Lemma 3.1(d) again, the equicontinuity of $\{\phi_n\}$ follows from Lemma 6.1 of [4].

Finally, the pointwise boundedness of $\{e_n\}$ follows from Lemma 3.1(c), whereas from (3.3),

$$|e_n(x) - e_n(y)| \le |\phi^*(x) - \phi^*(y)| + |\phi_n(x) - \phi_n(y)| \quad \forall x, y \in X,$$

so that the equicontinuity of $\{e_n\}$ follows from that of $\{\phi_n\}$ and the continuity of ϕ^* (see Theorem 2.5). ■

We will next prove a result that implies Lemma 2.10.

LEMMA 3.3. *Under Assumptions* 2.1–2.3, *there exist positive constants* $\overline{c}$ *and* η, *with* $\eta < 1$, *such that*

$$\sup_{\Delta_0} \left| \frac{J_n(f,x)}{n} - J(f) \right| \le \frac{\overline{c}v(x)}{n(1-\eta)} \tag{3.8}$$

for all $x \in X$ *and* $n = 1, 2, \ldots,$ *with* $J(f)$ *as in* (2.23).

Remark 3.4. (3.8) implies that in (2.7) we may replace "lim sup" by "lim" if δ is a stationary policy, i.e., for every $f \in \Delta_0$ and $x \in X$:

$$J(f,x) = \lim_{n\to\infty} \frac{J_n(f,x)}{n} = J(f). \tag{3.9}$$

Proof of Lemma 3.3. Under Assumptions 2.2 and 2.3, [4, Lemma 3.4] shows the existence of constants $\overline{c} > 0$ and $0 < \eta < 1$ satisfying

$$\sup_{\Delta_0} \|Q^t(\cdot \mid x, f) - q_f\|_v \le \overline{c}v(x)\eta^t \tag{3.10}$$

for all $x \in X$ and $t = 0, 1, \ldots,$ where we have used the notation (2.13), and $\|q_f\|_v < \infty$. Hence,

$$\left| \frac{J_n(f,x)}{n} - J(f) \right| = \left| \frac{1}{n} \sum_{t=0}^{n-1} E_x^f c(x_t, f) - \int_X c(y,f)\, q_f(dy) \right|$$
$$\le \frac{1}{n} \sum_{t=0}^{n-1} \left| \int_X c(y,f)\, [Q^t(dy \mid x, f) - q_f(dy)] \right|.$$

Thus, since $\sup_X c(x,f)/v(x) \le 1$ for all $f \in \Delta_0$ (see (2.2a)),

$$\left| \frac{J_n(f,x)}{n} - J(f) \right| \le \frac{1}{n} \sum_{t=0}^{n-1} \|Q^t(\cdot \mid x, f) - q_f\|_v.$$

This inequality and (3.10) yield (3.8). ■

Finally, to complete the proof of Theorem 2.6 we have:

LEMMA 3.5. *Under the hypotheses of Theorem* 2.6, *there exists a constant* c_2 *for which* (3.2) *holds and the convergence is uniform on compact sets.*

Proof. By Lemma 3.2 (on $\{e_n\}$) and the Ascoli Theorem (see e.g. [14], p. 179) there is a subsequence $\{e_{n(i)}\}$ of $\{e_n\}$ and a continuous function u such that

$$\lim_{i\to\infty} e_{n(i)}(x) = u(x) \quad \forall x \in X, \tag{3.11}$$

and the convergence is uniform on compact sets. Moreover, by Lemma 3.1(c), u is in Φ_v.

On the other hand, a straightforward induction argument (cf. [11, Lemma 5.4] or [9, Lemma 5.6.5]) yields

$$\int_X e_n(y)\, Q^m(dy \mid x, f^*) \le e_{n+m}(x) \tag{3.12}$$

for every $x \in X$ and $n, m \ge 0$. Now in (3.12) fix n and let $m \to \infty$ through values of m for which (3.11) holds. This, together with (3.10), implies

$$\int_X e_n(y)\, q_{f^*}(dy) \le u(x) \quad \forall x \in X.$$

Now replace n by $n(i)$ and let $i \to \infty$ to obtain, by the Dominated Convergence Theorem (recall Lemma 3.1(c)),

$$\int_X u(y)\, q_{f^*}(dy) \le u(x) \quad \forall x \in X.$$

Therefore, $\int_X u(y)\, q_{f^*}(dy) = c_2$, where $c_2 := \inf_X u(x)$; or, equivalently,

$$\int_X [u(y) - c_2]\, q_{f^*}\,(dy) = 0.$$

As $u(\cdot) - c_2 \ge 0$, we see that $u(x) = c_2$ for q_{f^*}-almost all $x \in X$, i.e., $q_{f^*}(U) = 0$, where $U := \{x : u(x) > c_2\}$. Observe that U is an open set, since $u(\cdot)$ is continuous; hence, by (2.19), U is empty. In other words,

$$u(x) = c_2 \quad \text{for all } x \in X. \tag{3.13}$$

Summarizing, (3.11) and (3.13) show that the subsequence $\{e_{n(i)}\}$ satisfies the conclusion of the lemma. Furthermore, a completely similar argument shows that any subsequence of $\{e_n\}$ has in turn a subsequence converging uniformly on compact sets to a constant c_2', which necessarily—using (3.12) again—equals c_2. Hence $\{e_n\}$ itself converges to c_2 uniformly on compact sets. ■

Lemma 3.5 completes the proof of Theorem 2.6.

4. Proofs of the corollaries

Proof of Corollary 2.8. (a) From (3.1) and Lemma 3.1(c), for all $x \in X$ and $n \ge 1$,

$$\left|\frac{v_n(x)}{n} - \varrho^*\right| \le \frac{|\phi^*(x)| + |e_n(x)|}{n} \le \frac{c_1 v(x)}{n}$$

for some constant c_1.

(b) From (3.6) and Lemma 3.1(b), (c),

$$0 \le J_n(f^*, x) - v_n(x) \le |e_n(x)| + E_x^{f^*}|\phi^*(x_n)| \le 2c_4 v(x);$$

hence

$$0 \le \frac{J_n(f^*,x) - v_n(x)}{n} \le \frac{2c_4 v(x)}{n}. \tag{4.1}$$

(c) From (2.15), $J_n(f^*,x) - n\varrho^* = \phi^*(x) - E_x^{f^*}\phi^*(x_n)$. Thus Lemma 3.1(b) yields

$$\left|\frac{J_n(f^*,x)}{n} - \varrho^*\right| \le \frac{c_6 v(x)}{n} \tag{4.2}$$

for some constant c_6, which proves (c).

Moreover, from Lemma 3.3 (see also (3.9)),

$$J(f^*,x) = \varrho^* = \int_X c(y,f^*)\, q_{f^*}(dy) \quad \forall x \in X. \tag{4.3}$$

(d) From parts (c) and (b) and the definition of v_n in (2.6),

$$\begin{aligned} J(f^*,x) &= \liminf \frac{J_n(f^*,x)}{n} = \liminf \frac{v_n(x)}{n} \\ &\le \liminf \frac{J_n(\delta,x)}{n} \quad \forall \delta \in \Delta \text{ and } x \in X. \ \blacksquare \end{aligned}$$

Proof of Corollary 2.9. The existence of $\widehat{f} \in \Delta_0$ satisfying part (a) is ensured by Schäl's [15, Proposition 12.2], and the proof that $\widehat{f}$ satisfies (b) can be done as in the proofs of [7, Theorem 4.2] or [11, Theorem 6.1]. ■

Proof of Corollary 2.11. Let D be the *AC-discrepancy function* defined as

$$D(x,a) := c(x,a) + \int_X \phi^*(y)\, Q(dy \mid x,a) - \phi^*(x) - \varrho^* \tag{4.4}$$

for all $x \in X$ and $a \in A(x)$. Observe that we can write the ACOE (2.14) as

$$\min_{A(x)} D(x,a) = 0 \quad \forall x \in X,$$

so that, in particular, D is a nonnegative function.

If f is a stationary policy we write $D(x,f) := D(x,f(x))$, $x \in X$.

For any stationary policy $f \in \Delta_0$, (2.9) and (2.23) yield

$$\int_X D(y,f)\, q_f(dy) = J(f) - \varrho^*. \tag{4.5}$$

On the other hand, integration with respect to q_f in Assumption 2.3(b) shows that

$$\int_X v(y)\, q_f(dy) \le b_0 \quad \text{with} \quad b_0 := \frac{\|\nu\|_v}{1-\alpha}. \tag{4.6}$$

Thus, combining (4.5)–(4.6),

$$0 \le J(f) - \varrho^* \le \|D(\cdot, f)\|_v \int_X v(y)\, q_f(dy) \le b_0 \|D(\cdot, f)\|_v. \tag{4.7}$$

Now, let $f_n \in \Delta_0$ be as (2.22) or, equivalently, from (3.5), for all $x \in X$,

$$j_n + \phi_n(x) = c(x, f_n) + \int_X \phi_{n-1}(y)\, Q(dy \mid x, f_n). \tag{4.8}$$

Then, from (4.4) and (4.8),

$$\begin{aligned} (4.9)\quad D(x, f_n) &= c(x, f_n) + \int_X \phi^*(y)\, Q(dy \mid x, f_n) - \phi^*(x) - \varrho^* \\ &= (j_n - \varrho^*) + (\phi_n(x) - \phi^*(x)) - \int_X (\phi_{n-1}(y) - \phi^*(y))\, Q(dy \mid x, f_n) \\ &\le |j_n - \varrho^*| + \|\phi_n - \phi^*\|_v v(x) + \|\phi_{n-1} - \phi^*\|_v \int_X v(y)\, Q(dy \mid x, f_n). \end{aligned}$$

Note also that Assumption 2.3(b) yields that $\int_X v(y)\, Q(dy \mid x, f_n)$ is bounded above by $v(x)$ times a constant independent of n. Hence (4.9), (2.24) and (2.18) imply $\|D(\cdot, f_n)\|_v \to 0$ as $n \to \infty$, which combined with (4.7) yields (2.25), i.e.,

$$0 \le J(f_n) - \varrho^* \le b_0 \|D(\cdot, f_n)\|_v \to 0 \quad \text{as } n \to \infty. \ \blacksquare$$

Finally, to conclude this section we observe that the proof of Corollary 2.12 is—except for minor, obvious changes—the same as the proof of Theorem 4.4 in [6].

5. Example. In this section we consider a particular control system of the form

$$x_{t+1} = (x_t + a_t \eta_t - \xi_t)^+, \quad t = 0, 1, \ldots, \ x_0 = x \text{ given}, \tag{5.1}$$

with state space $X = [0, \infty)$, and give conditions under which all the hypotheses of Theorem 2.6 hold true.

The model (5.1) appears in several application areas. For instance, in inventory theory (cf. [1, 2, 9]), $\eta_t = 1$ for all t, and x_t denotes the stock level at time t; the control variable a_t is the amount ordered (or produced) in the interval $[t, t+1)$, and ξ_t denotes the demand in $[t, t+1)$. The model also appears in a single server queueing system of general type GI/GI/1 with controllable service rates. In this case, which is the particular application we have in mind, x_t and η_t denote, respectively, the waiting time and a "base" service time of the tth customer ($t = 0, 1, \ldots$), whereas ξ_t denotes the interarrival time between the tth and $(t+1)$th customers; a_t stands for the reciprocal of the service rate u_t (i.e., $u_t := 1/a_t$) for the tth customer.

Throughout the following we suppose:

ASSUMPTION 5.1. (a) $\{\eta_t\}$ and $\{\xi_t\}$ are independent sequences of nonnegative i.i.d. (independent and identically distributed) random variables;

(b) $A(x) = A$ for all $x \in X$ where A is a compact subset of the interval $(0, \theta]$ for some (finite) number θ;

(c) the random variable $\zeta = \theta\eta - \xi$, where η and ξ denote generic random variables distributed as η_0 and ξ_0 respectively, satisfies:

$$\text{(5.2)} \qquad \text{(i) } E(\zeta) < 0 \quad \text{and} \quad \text{(ii) } Ee^{\overline{q}\zeta} < \infty$$

for some number $\overline{q} > 0$;

(d) η and ξ have bounded densities ϱ_1 and ϱ_2 respectively, continuous on $[0, \infty)$.

Observe that (5.2) implies

$$\text{(5.3)} \qquad \alpha := Ee^{q\zeta} < 1 \quad \text{for some } 0 < q \le \overline{q},$$

since the moment generating function $g(z) := Ee^{z\zeta}$ is such that $g(0) = 1$ and $g'(0) = E(\zeta) < 0$. The number α in (5.3) can be explicitly computed in some specific cases. For instance, if η and ξ are exponentially distributed with mean values $E(\eta) = 1/\overline{\eta}$ and $E(\xi) = 1/\overline{\xi}$, then

$$\text{(5.4)} \qquad \alpha = \left[\frac{\overline{\eta}}{\overline{\eta} - q\theta}\right]\left[\frac{\overline{\xi}}{\overline{\xi} + q}\right]$$

and $\alpha < 1$ if $q < \overline{\eta}/\theta - \overline{\xi}$.

On the other hand, by Assumption 5.1(b) and the Remark following Assumption 2.4, we may restrict ourselves to verifying Assumption 2.4′ (instead of 2.4). Thus we shall suppose:

ASSUMPTION 5.2. The one-stage cost c is a nonnegative measurable function such that, for every $x \in X$, $c(x, \cdot)$ is l.s.c. on A and, moreover,

$$\text{(5.5)} \qquad \sup_A c(x, a) \le v(x) \quad \text{with} \quad v(x) = \overline{c}e^{qx},$$

where q is the number in (5.3) and $\overline{c}$ is some positive constant. In addition, c satisfies Assumption 2.4′(i).

We will now proceed to verify Assumptions 2.1–2.3 and 2.4′. We begin with the following.

PROPOSITION 5.3. *Assumptions* 5.1(a), (b) *and* (c)(i) *imply Assumption* 2.2.

Proof. In (5.1) let $a_t = \theta$ for all t and call the corresponding Markov process $\{x_t^\theta\}$, i.e., $x_{t+1}^\theta = (x_t^\theta + \zeta)^+$, $t = 0, 1, \ldots$, with ζ as in Assumption 5.1(c). Then the condition (i) in (5.2) implies that $\{x_t^\theta\}$ is positive Harris-recurrent (see e.g. [12, Example 5.2]). The latter, in turn, implies that $E(\tau^\theta) < \infty$, where τ^θ denotes the time of first return to $x = 0$ given the initial state $x_0 = 0$.

Now let $f \in \Delta_0$ be an arbitrary stationary policy and denote by $\{x_t^f\}$ the corresponding Markov process given by (5.1) when $a_t = f(x_t)$ for all t. Let τ^f be the time of first return of $\{x_t^f\}$ to $x = 0$, given $x_0^f = 0$. By Assumption 5.1(b), $f(x) \le \theta$ for all x in X and, therefore, $x_t^f \le x_t^\theta$ for $t = 0, 1, \ldots$ Hence

$$E(\tau^f) \le E(\tau^\theta) < \infty,$$

so that (by Corollary 5.3 of [12]) $\{x_t^f\}$ is positive Harris-recurrent. As f in Δ_0 was arbitrary, we obtain Assumption 2.2. ∎

To verify Assumption 2.3 let us, first, note that

$$(5.6)\qquad Q((-\infty, y] \mid x, a) := P(x_{t+1} \le y \mid x_t = x, a_t = a) = P(x + a\eta - \xi \le y).$$

Hence

$$Q((-\infty, 0] \mid x, a) = P(x + a\eta - \xi \le 0)$$

and

$$Q(B \mid x, a) = P(x + a\eta - \xi \in B)$$

if $B \in \mathbb{B}(0, \infty)$. Now, if $f \in \Delta_0$, let

$$(5.7)\qquad h_f(x) := P(x + f(x)\eta - \xi \le 0), \quad x \in X, \quad \text{and} \quad \nu(\cdot) := p_0,$$

where p_0 is the Dirac measure concentrated at $x = 0$, and let α and $v(\cdot)$ be as in (5.3) and (5.5) respectively.

PROPOSITION 5.4. *Assumptions* 5.1(a), (b), (c) *imply Assumption* 2.3.

Proof. Assumption 2.3(a) follows from (5.6)–(5.7), while 2.3(c) is obtained from (5.7) and Assumptions 5.1(a), (b), (c)(i):

$$\inf_{\Delta_0} \int_X h_f \, d\nu = \inf_{\Delta_0} P(f(x)\eta - \xi \le 0) \ge P(\theta\eta - \xi \le 0) = P(\zeta \le 0) > 0.$$

To verify Assumption 2.3(b) note that (with ν and v as in (5.7) and (5.5) respectively)

$$(5.8)\qquad \|\nu\|_v = v(0) = \overline{c}.$$

On the other hand, from (5.6)–(5.7), for any stationary policy f,

$$(5.9)\qquad \int_X v(y)\, Q_f(dy \mid x) = \overline{c} h_f(x) + \overline{c} \int_0^\infty e^{qy} \, dF_{x+f(x)\eta-\xi}(y),$$

where $F_{x+f(x)\eta-\xi}(y) := P(x + f(x)\eta - \xi \le y)$. Hence

$$\begin{aligned}\int_X v(y)\,Q_f(dy \mid x) - h_f(x)\|\nu\|_v &= \overline{c}\int_{-x}^{\infty} e^{q(s+x)}\,dF_{f(x)\eta-\xi}(s)\\ &\le \overline{c}e^{qx}Ee^{[q(f(x)\eta-\xi)]}\\ &\le v(x)Ee^{q\zeta} \quad \text{[see (5.2)–(5.3)]}\\ &= \alpha v(x),\end{aligned}$$

which yields Assumption 2.3(b). ■

Observe that if in (5.9) we replace $f(x)$ by an arbitrary $a \in A$, we obtain (2.2b). Consequently, to verify Assumption 2.1 it only remains to prove part (c) and (2.2c). To prove this, let ϱ_1 and ϱ_2 be as in Assumption 5.1(d) and, for every $a \in A$, let ϱ_a be the density of $a\eta-\xi$. Then, for every real number y,

$$\varrho_a(y) = \frac{1}{a}\int_0^{\infty} \varrho_1\left(\frac{y+s}{a}\right)\varrho_2(s)\,ds = \int_{y/a}^{\infty} \varrho_1(s)\varrho_2(as-y)\,ds, \tag{5.10}$$

and, therefore, by Assumptions 5.1(a), (d) and the Bounded Convergence Theorem, the mapping $a \mapsto \varrho_a(y)$ is continuous on A. Observe also that $\varrho_a(y)$ is bounded, since

$$0 \le \varrho_a(y) \le M \quad \forall y \in \mathbb{R} \text{ and } a \in A, \tag{5.11}$$

where M is an upper bound for ϱ_2. Moreover, for any bounded measurable function u on X, (5.1), (5.6) and (5.10) yield

$$\begin{aligned}\int_X u(y)\,Q(dy \mid x,a) &= E[u(x_{t+1}) \mid x_t = x, a_t = a]\\ &= \int_{-\infty}^{\infty} u[(x+y)^+]\varrho_a(y)\,dy\\ &= u(0)\int_{-\infty}^{-x} \varrho_a(y)\,dy + \int_0^{\infty} u(y)\varrho_a(y-x)\,dy.\end{aligned} \tag{5.12}$$

Thus, by Scheffé's Theorem (or Exercise 14 in [14], p. 90), (5.12) defines a continuous function in $a \in A$ for every state x, which implies Assumption 2.1(c)—take $u(\cdot)$ as the indicator function of a Borel set B in X. Finally, to verify (2.2c), in (5.12) replace $u(\cdot)$ by the function $v(\cdot)$ in (5.5) and note that $v(y) \le \overline{c}e^{\overline{q}y}$ with $\overline{q}$ as in (5.2)–(5.3); hence a similar argument yields (2.2c). Summarizing, we have:

PROPOSITION 5.5. *Assumptions* 5.1 *and* 5.2 *imply Assumption* 2.1.

It only remains to verify Assumption 2.4′(ii) and the condition (2.19), which requires additional hypotheses. Let us suppose:

ASSUMPTION 5.6. (a) For every $a \in A$ there exists a positive number $\varepsilon > 0$ such that the function

$$\varrho_{a,\varepsilon}(y) := \sup\{\varrho_{a'}(y) \mid a' \in A \text{ and } |a' - a| \le \varepsilon\}, \quad y \in \mathbb{R},$$

satisfies

$$\int_{-\infty}^{\infty} e^{qy} \varrho_{a,\varepsilon}(y)\, dy < \infty;$$

(b) there is a function ψ in Ψ such that

$$|\varrho_a(z+y) - \varrho_a(z)| \le \psi(|y|) g_a(z) \tag{5.13}$$

for all z, y in $\mathbb{R}$ and $a \in A$, where g_a is a function satisfying

$$\sup_A \int_{-\infty}^{\infty} e^{qz} g_a(z)\, dz < \infty; \tag{5.14}$$

(c) ϱ_1 and ϱ_2 are strictly positive on X.

Remark. By (5.10), Assumption 5.6(a) is satisfied if $\varrho_1(x)$ is a monotone function in $x \ge 0$.

Let us now verify Assumption 2.4′(ii). Let M and $v(\cdot)$ be as in (5.11) and (5.5) respectively. Then, as in (5.12), for any x, x' in X and $a \in A$ we obtain

$$\begin{aligned}
&\|Q(\cdot \mid x, a) - Q(\cdot \mid x', a)\|_v \\
&\quad \le \overline{c}|P(a\eta - \xi \le -x) - P(a\eta - \xi \le -x')| \\
&\qquad + \overline{c} \int_0^{\infty} e^{qy} |\varrho_a(y - x) - \varrho_a(y - x')|\, dy \\
&\quad \le \overline{c} M |x - x'| \\
&\qquad + \overline{c} \int_{-x}^{\infty} e^{q(y+x)} |\varrho_a(y) - \varrho_a(y + x - x')|\, dy \qquad \text{(by (5.13))} \\
&\quad \le \overline{c} M |x - x'| \\
&\qquad + \overline{c}\psi(|x - x'|) e^{qx} \int_{-\infty}^{\infty} e^{qy} g_a(y)\, dy \qquad \text{(by (5.14))} \\
&\quad \le \widehat{\psi}(|x - x'|),
\end{aligned}$$

where $\widehat{\psi} \in \Psi$ is a constant times the function $|x| + \psi(x)$. That is, Assumption 2.4′(ii) is satisfied.

Finally, in (5.12) let $u(\cdot)$ be the indicator function of an arbitrary open set U in X. Then from (5.10) and Assumption 5.6(c) we obtain $Q(U \mid x, a) > 0$ for all x in X and a in A. This implies (2.19) (see the Remark following Theorem 2.6).

In conclusion, Assumptions 5.1, 5.2 and 5.6 imply that the system (5.1) satisfies all the hypotheses of Theorems 2.5 and 2.6.

As a *special case*, let η and ξ be exponentially distributed with mean values $1/\overline{\eta}$ and $1/\overline{\xi}$ respectively (cf. (5.4)). Then the density ϱ_a in (5.10) becomes

$$\varrho_a(x) = \begin{cases} N(a)e^{-\bar{\xi}|x|} & \text{if } x < 0, \\ N(a)e^{-\overline{\eta}x/a} & \text{if } x \geq 0, \end{cases}$$

where $N(a) := (\overline{\eta}\overline{\xi}/a)(\overline{\xi} + \overline{\eta}/a)^{-1}$. Similarly, all of the quantities in this section can be explicity calculated or estimated; in particular, the right-hand side of (5.13) can be found by a direct estimation of $|\varrho_a(z+y) - \varrho_a(z)|$.

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