APPLICATIONES MATHEMATICAE
23,2 (1995), pp. 191–198

K. MOSZYŃSKI (Warszawa)

CONCERNING DECOMPOSITION OF A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Introduction. Let us consider a system of linear algebraic equations

$$Ax = b, \tag{1}$$

where A is an $N \times N$ real, invertible matrix. In [1] a method of decomposition of (1) was proposed. The purpose of such a decomposition is to enable parallelization of the algorithm, and if possible to make the problem better conditioned.

Let $R = UA - AU$. The general idea of the method mentioned above is based on the following observation: if an $N \times N$ matrix U of rank $r < N$ *commutes sufficiently well with* A, i.e. R *is sufficiently small*, then U defines an approximate decomposition of (1).

Let $U = QF$, where Q is an $N \times r$ matrix and F is an $r \times N$ matrix, both of rank r. In [1] it is proposed to replace (1) by one of following systems, which can be solved by iteration:

$$\begin{aligned} Q^T AQy_{n+1} + Q^T RQy_n + Q^T RSz_n &= Q^T Ub, \\ S^T ASz_{n+1} - S^T RQy_n - S^T RSz_n &= G(I - U)b, \end{aligned} \tag{2}$$

or

$$\begin{aligned} AQy_{n+1} + FRQy_n + FRSz_n &= FUb, \\ GASz_{n+1} - GRQy_n - GRSz_n &= G(I - U)b, \end{aligned} \tag{3}$$

where $I - U = SG$ with an $N \times s$ matrix S and an $s \times N$ matrix G, and, in general, $N - r \le s \le N$. Moreover, $x_n = Qy_n + Sz_n$ converges to the solution $x = A^{-1}b$ of the system (1).

We may easily transform (2) and (3) to a more convenient form not containing R (see [1]):

1991 *Mathematics Subject Classification*: 65F10, 65F35.
Key words and phrases: decomposition of a matrix, nearly commuting matrices, parallelization.

(4) $$Q^T AQv_{n+1} = Q^T Ur_n, \quad S^T ASw_{n+1} = S^T (I - U)r_n,$$

or

(5) $$FAQv_{n+1} = FUr_n, \quad GASw_{n+1} = G(I - U)r_n,$$

where

$$v_{n+1} = y_{n+1} - Fx_n, \quad x_n = Qy_n + Sz_n,$$
$$w_{n+1} = z_{n+1} - Gx_n, \quad r_n = b - Ax_n.$$

If U is a *projector*, i.e. if $U^2 = U$, each of the systems (4) and (5) contains exactly N equations ($s = N - r$), hence such a choice is preferable.

This paper concerns the following problem:

Given $U = QF$, *where* Q *and* F *are* $N \times r$ *and* $r \times N$ *matrices respectively*, *both of rank* $r \le N$, *we have to construct an* $N \times N$ *matrix* V, *satisfying the following conditions*:

1. $\mathrm{rank}(V) = \mathrm{rank}(U) = r$;
2. $V^2 = V$;
3. $\mathrm{Im}(V) = \mathrm{Im}(U)$;
4. *If at least one of the processes* (4) *and* (5) *converges and* R *is sufficiently small*, *then after replacing* U *by* V, *at least one of* (4) *and* (5) *will converge as well.*

The matrix V

LEMMA 1. *Let* U *be an* $N \times N$ *matrix of rank* $r \le N$. *Assume that there are* r *linearly independent columns* $u_{p_1}, \ldots, u_{p_r}$ *of* U *and* r *linearly independent columns* $w_{p'_1}, \ldots, w_{p'_r}$ *of* U^T *such that* $(w_{p'_i}, u_{p_i}) \neq 0$ *for* $i = 1, \ldots, r$. *Then there exist four matrices* Q, Q', F, F' *of dimensions* $N \times r$, $N \times r$, $r \times N$, $r \times N$ *respectively*, *such that*

(6) $$U = QF, \quad U^T = Q'F'$$

and

(7) $$Q^T Q' = Q'^T Q = I_r.$$

Proof. Observe that in this case a kind of Gram–Schmidt process of *biorthogonalization* can be applied to the double system of vectors $u_{p_1}, \ldots$ $\ldots, u_{p_r}$, $w_{p'_1}, \ldots, w_{p'_r}$.

We start with

$$u_{p_1} = \gamma_{1,1} q_1, \quad w_{p'_1} = \gamma'_{1,1} q'_1, \quad \gamma_{1,1}\gamma'_{1,1} = (w_{p'_1}, u_{p_1}) \neq 0,$$

and then we proceed with the formulas

$$u_{p_k} = \sum_{j=1}^{k} \gamma_{k,j} q_j, \quad \gamma_{k,j} = (q'_j, u_{p_k}), \quad j = 1, \ldots, k-1,$$

$$w_{p'_k} = \sum_{j=1}^{k} \gamma'_{k,j} q'_j, \quad \gamma'_{k,j} = (q_j, w_{p'_k}), \quad j = 1, \ldots, k-1,$$

$$\gamma_{k,k}\gamma'_{k,k} = (w_{p'_k}, u_{p_k}) - \sum_{j=1}^{k-1} \gamma_{k,j}\gamma'_{k,j}$$

for $k = 1, \ldots, r$. In this way we get

$$u_s = \sum_{j=1}^{r} \gamma_{s,j} q_j, \qquad \gamma_{s,j} = (u_s, q'_j),$$

$$w_s = \sum_{j=1}^{r} \gamma'_{s,j} q'_j, \qquad \gamma'_{s,j} = (w_s, q_j),$$

for all $s = 1, \ldots, N$, $j = 1, \ldots, N$. The last formulas can be written in the form

$$U = QF \quad \text{and} \quad U^T = Q'F',$$

where Q and Q' are the $N \times r$ matrices with columns q_j and q'_j respectively, and F and F' are the $r \times N$ matrices of the coefficients $\gamma_{i,j}$ and $\gamma'_{i,j}$, respectively. ∎

Assume now that the decompositions from Lemma 1: $U = QF$ and $U^T = Q'F'$ are possible, and are given. Define

$$V = QQ'^T \quad \text{and} \quad R' = VA - AV.$$

PROPOSITION 1. *V is a projector.*

Proof. $VV = QQ'^T QQ'^T = QI_r Q'^T = V$. ∎

Since V is a projector of rank r, $I - V$ is a projector of rank $N - r$. Hence $I - V = S'G'$, where S' and G' are $N \times (N-r)$ and $(N-r) \times N$ matrices respectively. This decomposition may be obtained for example by usual Gram–Schmidt orthogonalization, applied to the columns of $I - V$.

PROPOSITION 2. $U = UV = VU$.

Proof. We have $VU = QQ'^T QF = QI_r F = QF = U$. Moreover, $UV = (Q'F')^T QQ'^T = F'^T Q'^T QQ'^T = F'^T I_r Q'^T = (Q'F')^T = U$. ∎

Proposition 3.

$$R(I-V)=UR',\qquad (I-V)R=R'U.$$

P r o o f. By Proposition 2 it follows that $R=UA-AU=UVA-AUV$; since $AU=UA-R$, we have

$$R=UVA-UAV+RV=UR'+RV$$

and so $UR'=R(I-V)$. Similarly, $R=UA-AU=VUA-AVU$, and $UA=AU+R$, hence

$$R=VAU+VR-AVU=R'U+VR,$$

whence $R'U=(I-V)R$. ∎

Proposition 4. $Q'^TR'Q=0$.

P r o o f. Observe that $R'=VA-AV=A(I-V)-(I-V)A$. This yields

$$VR'V=VA(I-V)V-V(I-V)AV=0,$$

because V is a projector and $V(I-V)=(I-V)V=0$. On the other hand, $0=VR'V=QQ'^TR'QQ'^T$ and $0=Q^TVR'VQ'=Q^TQQ'^TR'QQ'^TQ'$. Now, Q^TQ and Q'^TQ' are the Gram matrices of the bases $q_1,\ldots,q_r$ and $q_1',\ldots,q_r'$, and hence are invertible. Finally, we deduce that $Q'^TR'Q=0$. ∎

Proposition 5. $G'R'S'=0$.

P r o o f. Since V is a projector, we have

$$(I-V)R'(I-V)=(I-V)(VA-AV)(I-V)=0,$$

because $(I-V)V=V(I-V)=0$. Therefore

$$S'G'R'S'G'=0$$

and

$$S'^TS'G'R'S'G'G'^T=0.$$

We conclude that $G'R'S'=0$, the Gram matrices S'^TS' and $G'G'^T$ being invertible. ∎

Proposition 6.

$$R'Q=(I-V)RF^T(FF^T)^{-1}=O(R).$$

P r o o f. From Proposition 3, $(I-V)R=R'QF$, and hence

$$R'Q=(I-V)RF^T(FF^T)^{-1}.\ ∎$$

PROPOSITION 7. *If FQ is invertible* (*that is*, U *is in some sense close to a projector*), *then*

$$Q'^T R' = (FQ)^{-1}(Q^T Q)^{-1} Q^T R(I-V) = O(R).$$

Proof. By Proposition 3, $R(I-V) = UR'$, and by Proposition 2, $U = UV = QFQQ'^T$. Hence $R(I-V) = QFQQ'^T R'$, which implies

$$Q^T R(I-V) = Q^T QFQQ'^T R'.$$

The assertion follows by invertibility of FQ and $Q^T Q$. ∎

THEOREM 1. *Assume that the hypotheses of Lemma* 1 *are satisfied*, *the matrix U depends continuously on R*, *where $R = UA - AU$*, *and FQ is invertible for R small. Then the process* (5), *with U replaced by V*, *converges for R small enough. This process can now be written as follows*:

$$\begin{aligned} Q'^T AQv_{n+1} &= Q'^T r_n, \quad & G'AS'w_{n+1} &= G'(I-V)r_n, \\ v_{n+1} &= y_{n+1} - Q'^T x_n, \quad & x_n &= Qy_n + S'z_n, \\ w_{n+1} &= z_{n+1} - G'x_n, \quad & r_n &= b - Ax_n. \end{aligned} \tag{8}$$

Proof. Let us return to the equation (3), equivalent to (5). Now, if U is replaced by V, in view of Propositions 1–7, the equation (3) admits the following form:

$$\begin{aligned} Q'^T AQy_{n+1} + Q'^T R'S'z_n &= Q'^T b, \\ G'AS'z_{n+1} - G'R'Qy_n &= G'(I-V)b. \end{aligned}$$

By Propositions 1–7, the coefficients of all terms containing y_n and z_n are of order $O(R)$; hence the convergence follows by standard arguments. ∎

Case of A symmetric. Put now $V = QQ^T$, $R = UA - AU$, $R' = VA - AV$, and $U = QF$ with $Q^T Q = I_r$. A decomposition of this kind may be obtained for example by application of the Gram–Schmidt process to the columns of U.

PROPOSITION 8. *V is an orthogonal projector.*

Proof. $VV = QQ^T QQ^T = QI_r Q^T = V$. Moreover, $V^T = (QQ^T)^T = QQ^T = V$. ∎

Since $I - V$ is of rank $N - r$, we may decompose (by the Gram–Schmidt process)

$$I - V = S'G', \quad \text{where} \quad S'^T S' = I_{N-r}.$$

PROPOSITION 9. *If $A = A^T$, then $R'^T = -R'$.*

Proof. We have $R'^T = (VA - AV)^T = A^T V^T - V^T A^T = AV - VA = -R'$. ∎

Proposition 10. $R'Q = (I-V)RF^T(FF^T)^{-1} = O(R)$.

P r o o f. We have $Q^TU = Q^TQF = F$, hence $U = QQ^TU = VU$,

$$R = UA - AU = VUA - AVU = V(AU+R) - AVU = R'U + VR,$$

and so $(I-V)R = R'U = R'QF$. Since FF^T is invertible, we get $R'Q = (I-V)RF^T(FF^T)^{-1}$. ■

Proposition 11. *If* $A = A^T$, *then* $Q^TR' = -(FF^T)^{-1}FR^T(I-V) = O(R)$.

P r o o f. We have

$$R'Q = (I-V)RF^T(FF^T)^{-1},$$

whence by Proposition 9,

$$-Q^TR'^T = Q^TR' = -(FF^T)^{-1}FR^T(I-V). \ ■$$

Proposition 12. *If* $A = A^T$, *then* $Q^TR'Q = 0$.

P r o o f. Observe that $(I-V)Q = Q - QQ^TQ = Q - Q = 0$ and $Q^TR'Q = -(FF^T)^{-1}FR^T(I-V)Q = 0$. ■

Proposition 13. $S'^TR'S' = 0$.

P r o o f. We have

$$\begin{aligned}(I-V)R'(I-V) &= (I-V)(VA-AV)(I-V)\\ &= (I-V)VA(I-V) - (I-V)AV(I-V) = 0\end{aligned}$$

because $V(I-V) = (I-V)V = 0$, where V is an orthogonal projector. Since $I-V$ is symmetric, it follows that $I-V = (I-V)^T = G'^TS'^T$ and $(I-V)R'(I-V) = G'^TS'^TR'S'G'$. Observe that $G'G'^T$ is invertible, whence $G'(I-V)R'(I-V)G'^T = 0$, which completes the proof. ■

Theorem 2. *Assume that* $A = A^T$, *and that* $U = QF$, *where* $Q^TQ = I_r$, *depends continuously on* $R = UA - AU$. *Then the process* (4), *with* U *replaced by* $V = QQ^T$, *which is now of the following form*:

$$\begin{aligned}Q^TAQv_{n+1} &= Q^Tr_n, & S'^TAS'w_{n+1} &= S'^TS'G'r_n,\\ v_{n+1} &= y_{n+1} - Q^Tx_n, & x_n &= Qy_n + S'z_n,\\ w_{n+1} &= z_{n+1} - G'x_n, & r_n &= b - Ax_x,\end{aligned} \tag{9}$$

converges for R *small enough.*

P r o o f. We recall the equation (2), equivalent to (4), which now takes the form

$$\begin{aligned}Q^TAQy_{n+1} + Q^TR'S'z_n &= Q^T(I-V)b,\\ S'^TAS'z_{n+1} - S'^TR'Qy_n &= G'(I-V)b.\end{aligned}$$

From Propositions 8–12 it follows that the terms containing y_n and z_n are of order $O(R)$; hence, for R small the convergence follows by standard arguments. ∎

Example. Assume that an $N \times N$ matrix A and an $M \times M$ matrix C, with $M < N$, are *two finite-dimensional approximations of a certain linear operator*. For simplicity, assume both matrices A and C to be symmetric and invertible.

Let

$$p : \mathbb{R}^M \to \mathbb{R}^N \quad \text{and} \quad r : \mathbb{R}^N \to \mathbb{R}^M$$

be linear *extension* and *restriction operators*, respectively (see [2]). Put

$$U = pCr : \mathbb{R}^N \to \mathbb{R}^N.$$

If p and r are *properly chosen* (see [2]), then we may expect that $R = UA - AU$ will be *small* for sufficiently large N and M, $M < N$. We may also expect (at least in certain situations—see the Laplace operator for example), that in general the matrix C will correspond to a *lower* part of the spectrum of the original operator than the matrix A. This phenomenon may be explained as follows: approximation on a rough grid in general does not allow passing higher frequency oscillations.

We may apply our algorithm (9) to the matrix A and U. Application of the Gram–Schmidt process to the columns of the matrix pC will give $pC = Q\Gamma$ with $Q^TQ = I_M$. Hence we get

$$pCr = QF$$

with $F = \Gamma r$. We can construct in an arbitrary way an $N \times (N - M)$ matrix $\widetilde{Q}$ in order to get an $N \times N$ *orthogonal* matrix

$$[Q|\widetilde{Q}].$$

We have $V = QQ^T$ and

$$I - V = [Q|\widetilde{Q}][Q|\widetilde{Q}]^T - QQ^T = QQ^T + \widetilde{Q}\widetilde{Q}^T - QQ^T = \widetilde{Q}\widetilde{Q}^T.$$

In other words, $S' = \widetilde{Q}$ and $G' = \widetilde{Q}^T$.

Now the system (9) can be written in the following form:

$$\begin{aligned} Q^TAQv_{n+1} &= Q^Tr_n, \quad & \widetilde{Q}^TA\widetilde{Q}z_{n+1} &= \widetilde{Q}^Tr_n, \\ v_{n+1} &= y_{n+1} - Q^Tx_n, \quad & x_n &= Qy_n + \widetilde{Q}z_n, \\ w_{n+1} &= z_{n+1} - \widetilde{Q}^Tx_n, \quad & r_n &= b - Ax_n. \end{aligned}$$

References

[1] K. Moszyński, *On solving linear algebraic equations with an ill-conditioned matrix*, Appl. Math. (Warsaw) 22 (1995), 499–513.
[2] R. Temam, *Numerical Analysis*, Reidel, 1973.

KRZYSZTOF MOSZYŃSKI
DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE AND MECHANICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: KMOSZYNS@MIMUW.EDU.PL

Received on 28.9.1994