Introduction. Let us consider a system of linear algebraic equations
\[ Ax = b, \]
where \( A \) is an \( N \times N \) real, invertible matrix. In [1] a method of decomposition of (1) was proposed. The purpose of such a decomposition is to enable parallelization of the algorithm, and if possible to make the problem better conditioned.

Let \( R = UA - AU \). The general idea of the method mentioned above is based on the following observation: if an \( N \times N \) matrix \( U \) of rank \( r < N \) commutes sufficiently well with \( A \), i.e. \( R \) is sufficiently small, then \( U \) defines an approximate decomposition of (1).

Let \( U = QF \), where \( Q \) is an \( N \times r \) matrix and \( F \) is an \( r \times N \) matrix, both of rank \( r \). In [1] it is proposed to replace (1) by one of following systems, which can be solved by iteration:

\[
\begin{align*}
    Q^T AQy_{n+1} + Q^T RQy_n + Q^T RSz_n &= Q^T Ub, \\
    S^T ASz_{n+1} - S^T RQy_n - S^T RSz_n &= G(I - U)b,
\end{align*}
\]

or

\[
\begin{align*}
    AQy_{n+1} + FRQy_n + FRSz_n &= FUb, \\
    GASz_{n+1} - GRQy_n - GRSz_n &= G(I - U)b,
\end{align*}
\]

where \( I - U = SG \) with an \( N \times s \) matrix \( S \) and an \( s \times N \) matrix \( G \), and, in general, \( N - r \leq s \leq N \). Moreover, \( x_n = Qy_n + Sz_n \) converges to the solution \( x = A^{-1}b \) of the system (1).

We may easily transform (2) and (3) to a more convenient form not containing \( R \) (see [1]):

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(4) \[ Q^T AQ v_{n+1} = Q^T U r_n, \quad S^T AS w_{n+1} = S^T (I - U) r_n, \]

or

(5) \[ FAQ v_{n+1} = FU r_n, \quad GASw_{n+1} = G(I - U) r_n, \]

where

\[ v_{n+1} = y_{n+1} - Fx_n, \quad x_n = Q y_n + Sz_n, \]
\[ w_{n+1} = z_{n+1} - Gx_n, \quad r_n = b - Ax_n. \]

If \( U \) is a projector, i.e. if \( U^2 = U \), each of the systems (4) and (5) contains exactly \( N \) equations \((s = N - r)\), hence such a choice is preferable.

This paper concerns the following problem:

Given \( U = QF \), where \( Q \) and \( F \) are \( N \times r \) and \( r \times N \) matrices respectively, both of rank \( r \leq N \), we have to construct an \( N \times N \) matrix \( V \), satisfying the following conditions:

1. \( \text{rank}(V) = \text{rank}(U) = r; \)
2. \( V^2 = V; \)
3. \( \text{Im}(V) = \text{Im}(U); \)
4. If at least one of the processes (4) and (5) converges and \( R \) is sufficiently small, then after replacing \( U \) by \( V \), at least one of (4) and (5) will converge as well.

The matrix \( V \)

**Lemma 1.** Let \( U \) be an \( N \times N \) matrix of rank \( r \leq N \). Assume that there are \( r \) linearly independent columns \( u_{p_1}, \ldots, u_{p_r} \) of \( U \) and \( r \) linearly independent columns \( w'_{p'_1}, \ldots, w'_{p'_r} \) of \( U^T \) such that \((w'_{p'_i}, u_{p_i}) \neq 0\) for \( i = 1, \ldots, r \). Then there exist four matrices \( Q, Q', F, F' \) of dimensions \( N \times r, N \times r, r \times N, r \times N \) respectively, such that

\[ U = QF, \quad U^T = Q'F' \]

and

\[ Q^T Q' = Q' Q = I_r. \]

**Proof.** Observe that in this case a kind of Gram–Schmidt process of biorthogonalization can be applied to the double system of vectors \( u_{p_1}, \ldots, u_{p_r}, w'_{p'_1}, \ldots, w'_{p'_r} \).

We start with

\[ u_{p_1} = \gamma_{1,1} q_1, \quad w'_{p'_1} = \gamma'_{1,1} q'_1, \quad \gamma_{1,1} \gamma'_{1,1} = (w'_{p'_1}, u_{p_1}) \neq 0, \]
and then we proceed with the formulas

\[ u_{pk} = \sum_{j=1}^{k} \gamma_{k,j} q_j, \quad \gamma_{k,j} = (q'_j, u_{pk}), \quad j = 1, \ldots, k-1, \]

\[ w_{pk}' = \sum_{j=1}^{k} \gamma'_{k,j} q'_j, \quad \gamma'_{k,j} = (q_j, w_{pk}'), \quad j = 1, \ldots, k-1, \]

\[ \gamma_{k,k} \gamma'_{k,k} = (w_{pk}', u_{pk}) - \sum_{j=1}^{k-1} \gamma_{k,j} \gamma'_{k,j} \]

for \( k = 1, \ldots, r \). In this way we get

\[ u_s = \sum_{j=1}^{r} \gamma_{s,j} q_j, \quad \gamma_{s,j} = (u_s, q'_j), \]

\[ w_s' = \sum_{j=1}^{r} \gamma'_{s,j} q'_j, \quad \gamma'_{s,j} = (w_s, q_j), \]

for all \( s = 1, \ldots, N, \ j = 1, \ldots, N \). The last formulas can be written in the form

\[ U = QF \quad \text{and} \quad U^T = Q'F', \]

where \( Q \) and \( Q' \) are the \( N \times r \) matrices with columns \( q_j \) and \( q'_j \), respectively, and \( F \) and \( F' \) are the \( r \times N \) matrices of the coefficients \( \gamma_{i,j} \) and \( \gamma'_{i,j} \), respectively.

Assume now that the decompositions from Lemma 1: \( U = QF \) and \( U^T = Q'F' \) are possible, and are given. Define

\[ V = QQ^T \quad \text{and} \quad R' = VA - AV. \]

**Proposition 1.** \( V \) is a projector.

**Proof.** \( VV = QQ^T QQ^T = QI, Q^T = V. \)

Since \( V \) is a projector of rank \( r \), \( I - V \) is a projector of rank \( N - r \). Hence \( I - V = S'G' \), where \( S' \) and \( G' \) are \( N \times (N - r) \) and \( (N - r) \times N \) matrices respectively. This decomposition may be obtained for example by usual Gram-Schmidt orthogonalization, applied to the columns of \( I - V \).

**Proposition 2.** \( U = UV = VU. \)

**Proof.** We have \( VU = QQ^T QF = QIF = QF = U \). Moreover, \( UV = (Q'F')^T QQ^T = F'^T Q^T QQ^T = F'^T IF^T Q^T = (Q'F')^T = U. \)
Proposition 3.
\[ R(I - V) = UR', \quad (I - V)R = R'U. \]

Proof. By Proposition 2 it follows that \( R = UA - AU = UVA - AVU \); since \( AU = UA - R \), we have
\[ R = UVA - UAV + RV = UR' + RV \]
and so \( UR' = R(I - V) \). Similarly, \( R = UA - AU = VUA - AVU \), and \( UA = AU + R \), hence
\[ R = VAU + VR - AVU = R'U + VR, \]
whence \( R'U = (I - V)R \).

Proposition 4. \( Q^T R' Q = 0 \).

Proof. Observe that \( R' = VA - AV = A(I - V) - (I - V)A \). This yields
\[ VR'V = VA(I - V)V - V(I - V)AV = 0, \]
because \( V \) is a projector and \( V(I - V) = (I - V)V = 0 \). On the other hand, \( 0 = VR'V = QQ^T R'QQ^T \) and \( 0 = Q^T VR' Q' = Q^T QQ^T R' QQ^T Q' \).
Now, \( Q^T Q \) and \( Q^T Q' \) are the Gram matrices of the bases \( q_1, \ldots, q_r \), and \( q'_1, \ldots, q'_r \), and hence are invertible. Finally, we deduce that \( Q^T R' Q = 0 \).

Proposition 5. \( G' R' S' = 0 \).

Proof. Since \( V \) is a projector, we have
\[ (I - V)R'(I - V) = (I - V)(VA - AV)(I - V) = 0, \]
because \( (I - V)V = V(I - V) = 0 \). Therefore
\[ S' G' R' S' G' = 0 \]
and
\[ S'^T S' G' R' S' G'^T = 0. \]
We conclude that \( G' R' S' = 0 \), the Gram matrices \( S'^T S' \) and \( G'^T G'^T \) being invertible.

Proposition 6. \( R'Q = (I - V)RF^T (FF^T)^{-1} = O(R) \).

Proof. From Proposition 3, \( (I - V)R = R'QF \), and hence
\[ R'Q = (I - V)RF^T (FF^T)^{-1} \].
**Proposition 7.** If $FQ$ is invertible (that is, $U$ is in some sense close to a projector), then
\[ Q^T R' = (FQ)^{-1}(Q^T Q)^{-1}Q^T R(I - V) = O(R). \]

**Proof.** By Proposition 3, $R(I - V) = UR'$, and by Proposition 2, $U = UV = QFQQ^T$. Hence $R(I - V) = QFQQ^T R'$, which implies
\[ Q^T R(I - V) = Q^T QFQQ^T R'. \]
The assertion follows by invertibility of $FQ$ and $Q^T Q$.

**Theorem 1.** Assume that the hypotheses of Lemma 1 are satisfied, the matrix $U$ depends continuously on $R$, where $R = UA - AU$, and $FQ$ is invertible for $R$ small. Then the process (5), with $U$ replaced by $V$, converges for $R$ small enough. This process can now be written as follows:
\begin{align*}
Q^T AQv_{n+1} &= Q^T r_n, \\
G'AS'w_{n+1} &= G'(I - V)r_n, \\
v_{n+1} &= y_{n+1} - Q^T x_n, \\
x_n &= Qy_n + S'z_n, \\
w_{n+1} &= z_{n+1} - G'x_n, \\
r_n &= b - Ax_n.
\end{align*}

**Proof.** Let us return to the equation (3), equivalent to (5). Now, if $U$ is replaced by $V$, in view of Propositions 1–7, the equation (3) admits the following form:
\begin{align*}
Q^T AQy_{n+1} + Q^T R'S'z_n &= Q^T b, \\
G'AS'S_{n+1} - G'R'Qy_{n} &= G'(I - V)b.
\end{align*}
By Propositions 1–7, the coefficients of all terms containing $y_n$ and $z_n$ are of order $O(R)$; hence the convergence follows by standard arguments.

**Case of $A$ symmetric.** Put now $V = QQ^T$, $R = UA - AU$, $R' = VA - AV$, and $U = QF$ with $Q^T Q = I_r$. A decomposition of this kind may be obtained for example by application of the Gram–Schmidt process to the columns of $U$.

**Proposition 8.** $V$ is an orthogonal projector.

**Proof.** $VV = QQ^T QQ^T = QI_r Q^T = V$. Moreover, $V^T = (QQ^T)^T = QQ^T = V$.

Since $I - V$ is of rank $N - r$, we may decompose (by the Gram–Schmidt process)
\[ I - V = S'G', \text{ where } S'^T S' = I_{N-r}. \]

**Proposition 9.** If $A = AT$, then $R^T = -R'$.

**Proof.** We have $R^T = (VA - AV)^T = ATV - VTA = AV - VA = -R'$.  

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PROPONION 10.  \( R'Q = (I - V)RF^T(FF^T)^{-1} = O(R). \)

Proof. We have \( Q^TU = Q^TF = F, \) hence \( U = QQ^TU = VU. \)

\( R = UA - AU = VUA - AV = V(AU + R) - AVU = R'U + VR, \)
and so \( (I - V)R = R'U = R'QF. \) Since \( FF^T \) is invertible, we get \( R'Q = (I - V)RF^T(FF^T)^{-1}. \)

PROPONION 11.  If \( A = A^T, \) then \( Q^TR' = -(FF^T)^{-1}FR^T(I - V) = O(R). \)

Proof. We have \( R'Q = (I - V)RF^T(FF^T)^{-1}, \)
whence by Proposition 9,
\[-Q^TR^T = Q^TR' = -(FF^T)^{-1}FR^T(I - V). \]

PROPONION 12.  If \( A = A^T, \) then \( Q^TRQ = 0. \)

Proof. Observe that \( (I - V)Q = Q - QQ^TQ = Q - Q = 0 \) and \( Q^TRQ = -(FF^T)^{-1}FR^T(I - V)Q = 0. \)

PROPONION 13. \( S^T R'S' = 0. \)

Proof. We have
\[(I - V)R'(I - V) = (I - V)(VA - AV)(I - V)\]
\[= (I - V)VA(I - V) - (I - V)AV(I - V) = 0\]
because \( V(I - V) = (I - V)V = 0, \) where \( V \) is an orthogonal projector.
Since \( I - V \) is symmetric, it follows that \( I - V) = (I - V)^T = G^T S'^T \)
and \( (I - V)R'(I - V) = G^T S'^T R'S'G'. \) Observe that \( G^T G' \) is invertible, whence \( G'(I - V)R'(I - V)G'^T = 0, \) which completes the proof.

THEOREM 2. Assume that \( A = A^T, \) and that \( U = QF, \) where \( Q^TQ = I_r, \) depends continuously on \( R = UA - AU. \) Then the process (4), with \( U \) replaced by \( V = QQ^T, \) which is now of the following form:
\[Q^TAQ_{v_{n+1}} = Q^T r_n, \quad S'^T A S' w_{n+1} = S'^T S'G'r_n, \]
\[v_{n+1} = y_{n+1} = v_{n+1} - Q^T x_n, \quad x_n = Qy_n + S'z_n, \]
\[w_{n+1} = z_{n+1} = z_{n+1} - G'x_n, \quad r_n = b - Ax_n, \]
converges for \( R \) small enough.

Proof. We recall the equation (2), equivalent to (4), which now takes the form
\[Q^TAQ_{y_{n+1}} + Q^T R'S'z_n = Q^T (I - V)b, \]
\[S'^T A S' z_{n+1} - S'^T R'Qy_n = G'(I - V)b, \]
From Propositions 8–12 it follows that the terms containing \( y_n \) and \( z_n \) are of order \( O(R) \); hence, for \( R \) small the convergence follows by standard arguments.

**Example.** Assume that an \( N \times N \) matrix \( A \) and an \( M \times M \) matrix \( C \), with \( M < N \), are two finite-dimensional approximations of a certain linear operator. For simplicity, assume both matrices \( A \) and \( C \) to be symmetric and invertible.

Let

\[
p : \mathbb{R}^M \to \mathbb{R}^N \quad \text{and} \quad r : \mathbb{R}^N \to \mathbb{R}^M
\]

be linear extension and restriction operators, respectively (see [2]). Put

\[
U = pCr : \mathbb{R}^N \to \mathbb{R}^N.
\]

If \( p \) and \( r \) are properly chosen (see [2]), then we may expect that \( R = UA - AU \) will be small for sufficiently large \( N \) and \( M \), \( M < N \). We may also expect (at least in certain situations—see the Laplace operator for example), that in general the matrix \( C \) will correspond to a lower part of the spectrum of the original operator than the matrix \( A \). This phenomenon may be explained as follows: approximation on a rough grid in general does not allow passing higher frequency oscillations.

We may apply our algorithm (9) to the matrix \( A \) and \( U \). Application of the Gram–Schmidt process to the columns of the matrix \( pC \) will give \( pC = Q\Gamma \) with \( Q^TQ = I_M \). Hence we get

\[
pC r = QF
\]

with \( F = \Gamma r \). We can construct in an arbitrary way an \( N \times (N - M) \) matrix \( \tilde{Q} \) in order to get an \( N \times N \) orthogonal matrix

\[
[Q|\tilde{Q}].
\]

We have \( V = QQ^T \) and

\[
I - V = [Q|\tilde{Q}][Q|\tilde{Q}]^T - QQ^T = QQ^T + \tilde{Q}\tilde{Q}^T - QQ^T = \tilde{Q}\tilde{Q}^T.
\]

In other words, \( S' = \tilde{Q} \) and \( G' = \tilde{Q}^T \).

Now the system (9) can be written in the following form:

\[
Q^TAv_{n+1} = Q^Tr_n, \quad \tilde{Q}^TA\tilde{Q}z_{n+1} = \tilde{Q}^Tr_n,
\]
\[
v_{n+1} = y_{n+1} - Q^Tx_n, \quad x_n = Qy_n + \tilde{Q}z_n,
\]
\[
w_{n+1} = z_{n+1} - \tilde{Q}^Tx_n, \quad r_n = b - Ax_n.
\]
References


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