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GROWTH AND ACCRETION OF MASS IN AN ASTROPHYSICAL MODEL

Abstract. We study asymptotic behavior of radial solutions of a nonlocal Fokker–Planck equation describing the evolution of self-attracting particles. In particular, we consider stationary solutions in balls and in the whole space, self-similar solutions defined globally in time, blowing up self-similar solutions, and singularities of solutions that blow up in a finite time.

1. Introduction. The parabolic-elliptic system of partial differential equations in a domain Ω of $\mathbb{R}^n$,

$$u_t = \Delta u + \nabla \cdot (u \nabla \varphi), \tag{1}$$

$$\Delta \varphi = u, \tag{2}$$

supplemented with appropriate boundary and/or growth conditions (when $\Omega = \mathbb{R}^n$), was studied in papers [1], [6], [4], [2], [9], [10–11]. This system can be interpreted as a nonlocal Fokker–Planck equation when the drift coefficient $\nabla \varphi$ is reconstructed from the relation (2). Physical interpretations of the system (1)–(2), described in more detail in [6], [10–11], include the evolution version of the Chandrasekhar equation for the gravitational equilibrium of polytropic stars, and an evolution equation for self-interacting clusters of particles obtained from a kinetic model of Vlasov–Poisson–Boltzmann type.

Results on local-in-time solvability of the initial(-boundary) value problem for (1)–(2) have been proved even in more general situations ([5] based on techniques in [3], and [1]). Questions related to the existence of solutions defined globally in time versus finite time blow up of solutions are far more delicate, and some results can be found in [6], [4], [2], [9]. The ex-

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istence of solutions that cannot be continued to global-in-time ones (hence describing gravitational collapse phenomena) is intimately connected with nonexistence of stationary solutions to (1)–(2); see [6], [4], [2], [9]. Concerning the existence, multiplicity or nonexistence of stationary solutions we cite also [7–8]. Roughly speaking, besides strong dependence of qualitative properties of solutions on the topology of the domain Ω, there is a parameter, whose dimension is mass $\times$(length)$^{2-n}$, which can be called *concentration*. This parameter measures how stationary states may be complicated, and controls when solutions of the evolution problem may cease to exist. Mathematically, criteria for the occurrence of these phenomena can be expressed in terms of the Morrey space norm of exponent $n/2$; see [6], [2], [1]. Except for [4], [9], all the above-mentioned papers deal with general solutions to the problem (1)–(2). Even in [4], [9], devoted entirely to radially symmetric solutions, we avoided—for simplicity sake—discussions of some fine properties of solutions.

This paper deals with the equation (cf. [4, (6)])

$$Q_t = Q_{rr} - (n-1)r^{-1}Q_r + \sigma_n^{-1} r^{1-n} Q Q_r, \tag{3}$$

where σ_n is the area of the unit sphere in $\mathbb{R}^n$, satisfied by $Q(r,t) = \int_{B_r} u(x,t)\,dx$ defined for a radial nonnegative solution u of (1)–(2) in either a ball $\Omega = B_R$, or the whole space $\Omega = \mathbb{R}^n$. Hence the integrated density Q is a nondecreasing positive function defined on either the interval $[0,R]$ or the half-line $[0,\infty)$. Since the conservation of the total mass $M = \int_\Omega u(x,t)\,dx$ is a natural requirement (cf. [6, (3)]), we add the condition

$$Q(0,t) = 0, \qquad Q(R,t) = M, \tag{4}$$

with an admissible value $M = \infty$ if $R = \infty$. For the initial data we put

$$Q(r,0) = Q_0(r) \tag{5}$$

with a positive nondecreasing function Q_0 satisfying the obvious compatibility condition $Q_0(0) = 0$, $Q_0(R) = M$.

Note that under the radial symmetry assumption the nonlocal problem (1)–(2) is equivalent to a (local!) differential equation for Q. Moreover, natural but nonlinear boundary conditions for (1)–(2) become a simpler Dirichlet condition (4). Finally, the formulation (3)–(4) permits us to deal with less regular densities u (e.g. including measures, not only L^1 functions) than those considered in [6].

We will study in this paper: asymptotic growth of stationary solutions to (3) when $n \ge 3$, $R = \infty$, conditions for the uniqueness of steady states in a ball (pertinent to the smallness assumption of Th. 2 in [4] on global-in-time solutions), growth of self-similar solutions defined globally in time, asymptotic properties of blowing up self-similar solutions, and minimal singularities of solutions that blow up in a finite time.

The methods used in this note are connected with elementary comparison techniques for ordinary and (parabolic) partial differential equations, hence they are much simpler than those in the cited papers.

2. Steady states. Time independent solutions of (3)–(4) satisfy

$$Q_{rr} - (n-1)r^{-1}Q_r + \sigma_n^{-1} r^{1-n} Q Q_r = 0, \qquad Q(0) = 0, \qquad Q(R) = M. \tag{6}$$

Due to scaling properties of (6) it is sufficient to consider the cases $R = 1$ (a finite ball) and $R = \infty$ (the whole space $\mathbb{R}^n$). Indeed, if Q solves (6) on $[0, R]$, $R < \infty$, then $\widetilde{Q}(r) = R^{2-n} Q(Rr)$ solves (6) on $[0, 1]$ with the total mass condition $\widetilde{Q}(1) = M R^{2-n}$.

PROPOSITION 1. *If* $n \ge 3$, $R = \infty$, $Q \not\equiv 0$, *then necessarily* $M = \infty$, *and* $\lim_{r\to\infty} r^{2-n} Q(r)$ *exists for each solution to* (6).

The two-dimensional case was studied in detail in [4] and [9], see also [7–8]. Note that if $n = 2$, then $M = 8\pi$ is the unique value of $M > 0$ for which solutions of (6) with $R = \infty$ do exist; cf. [9, Section 2]. Moreover, for $n = 3$, the asymptotically linear growth of $Q(r)$ was proved in [9, Section 2].

P r o o f o f P r o p o s i t i o n 1. First, it is very easy to obtain an upper bound for Q. Indeed, multiplying (6) by r^{n-1} and integrating on $[0, \varrho]$, after some integrations by parts, we arrive at

$$r^{n-1} Q_r \big|_0^\varrho - 2(n-1) r^{n-2} Q \big|_0^\varrho + 2(n-1)(n-2) \int_0^\varrho r^{n-3} Q(r)\, dr + (2\sigma_n)^{-1} Q^2(\varrho) = 0$$

for each $\varrho \ge 0$. Together with monotonicity properties of Q this leads to

$$Q^2(\varrho) \le 2\sigma_n\, 2(n-1) \varrho^{n-2} Q(\varrho), \quad \text{i.e.} \quad Q(\varrho) \le 2\sigma_n\, 2(n-1) \varrho^{n-2}.$$

For a proof of the asymptotic growth property in Proposition 1 we recall a useful reformulation of the problem (6) (involving the phase plane method) studied in [4, (9)]. Namely, if $s = \log r$, $v(s) = \sigma_n^{-1} r^{3-n} Q_r(r)$, $w(s) = \sigma_n^{-1} r^{2-n} Q(r)$, then v and w satisfy the (autonomous!) system

$$v' = (2-w)v, \qquad w' = v - (n-2)w, \qquad {}' = \frac{d}{ds}, \tag{7}$$

together with the boundary conditions

$$w(-\infty) = 0, \qquad w(\log R) = \sigma_n^{-1} M \tag{8}$$

($R = 1$, $R = \infty$ being of interest).

Let us recall from [4] that $L(v, w) = (w-2)^2/2 + (v - 2(n-2)) - 2(n-2)\log(v/(2(n-2)))$ is a Lyapunov function for the dynamical system

(7) in the positive quadrant $v > 0$, $w > 0$. Note that the (forward) invariance of this quadrant with respect to the flow of (7) means that each solution of the equation (6) with $Q(0) \geq 0$, $Q'(0) \geq 0$ is positive and nondecreasing (this justifies the use of monotonicity of Q in the first part of the proof).

Using this Lyapunov function we can prove (as in [9] for $n = 3$) that the separatrix joining the stationary points $(0,0)$ and $(2(n-2), 2)$ is the unique trajectory with $v(s) > 0$ for all $s \in \mathbb{R}$. Now $\lim_{s\to\infty} w(s) = 2$ is equivalent to the relation $\lim_{r\to\infty} r^{2-n}Q(r) = 2\sigma_n$. Moreover, $\lim_{r\to\infty} r^{3-n}Q_r(r) = 2(n-2)\sigma_n$, hence the asymptotic growth of Q and Q_r is established. Observe that if $3 \leq n \leq 9$ then $r^{2-n}Q(r)$ is not monotone, since the separatrix turns around the point $(2(n-2), 2)$ infinitely many times.

An analysis in [4, Section 2] shows that for $R = 1$ and $M > M^*(n)$ with some $M^*(n) > 0$ (e.g. $M^*(n) = 2\sigma_n$ for $n \geq 10$) there are no solutions to (7)–(8), if $3 \leq n \leq 9$ then solutions are not unique in some range of M's: $M \in (M_*(n), M^*(n))$, $0 < M_*(n) < M^*(n)$, and in all the remaining cases the uniqueness of solutions holds (cf. [4, Th. 2]).

PROPOSITION 2. *If $R = 1$, $5 \leq n \leq 9$, then $M_*(n) > 2\sigma_n(n-4)/(n-2)$, i.e. for each $M \in (0, 2\sigma_n(n-4)/(n-2)]$ solutions of* (7)–(8) *are unique.*

This result is of interest for questions of the continuation in time of solutions to (3)–(5). Namely, Th. 2 in [4] states that $Q_0(r) \leq ((n-4)/(n-2)) \times 2\sigma_n r^{n-2}$, $0 \leq r \leq 1$, is a sufficient condition for global-in-time existence of solutions and their convergence to a stationary solution. Proposition 2 assures that in this range of mass M there exists *a unique* stationary solution.

Proof of Proposition 2. First note that local maxima and local minima of the w-coordinate on the spiral trajectory connecting the origin with the point $(2(n-2), 2)$ are situated on the line $v = (n-2)w$ (where $w' = 0$). The second observation is that the Lyapunov function L enjoys the property $L(2(n-2), w) = (w-2)^2/2$, so this is monotone for $w \in [0, 2]$ and $w \in [2, \infty)$. Denoting by w^* (w_* resp.) the first local maximum (resp. minimum) of w on the separatrix, $v^* = (n-2)w^*$, $v_* = (n-2)w_*$, and by $\underline{w} \in (0, 2)$ the w-coordinate of the first point where $v = 2(n-2)$, we have

$$L(v^*, w^*) < L(2(n-2), \underline{w}) < 2 = L(2(n-2), 0),$$
$$L(v_*, w_*) < (w^* - 2)^2/2 = L(2(n-2), w^*).$$

Monotonicity properties of L and a particularly simple form of the function

$$L((n-2)w, w) = \frac{1}{2}(w-2)^2 + (n-2)(w-2) - 2(n-2)\log\frac{w}{2} \tag{9}$$

allow us to estimate w^* from above, and then w_* from below. This will lead to some bounds for $M^*(n) = \sigma_n w^*$ and $M_*(n) = \sigma_n w_*$. Reasonable precision can be obtained with a calculator, but for our purposes it suffices to estimate (9) crudely using the inequalities

$$\log(1+\varepsilon) < \varepsilon - \varepsilon^2/2 + \varepsilon^3/3, \qquad \log(1-\varepsilon) < -\varepsilon - \varepsilon^2/2 - \varepsilon^3/3,$$

valid for each $\varepsilon > 0$, applied to $w = 2(1 \pm \varepsilon)$. We skip the details of calculations noting that for the proof of Proposition 2 we obtain e.g.

for $n = 5$: $w^* < 3.6$, $w_* > 0.8$,
for $n = 6$: $w^* < 3.5$, $w_* > 1.2$,
for $n = 7$: $w^* < 3.38$, $w_* > 1.2$,
for $n = 8$: $w^* < 10/3$, $w_* > 4/3$,
for $n = 9$: $w^* < 3.2$, $w_* > 10/7$.

Of course, these bounds can be substantially improved by tracing approximations of the separatrix using any O.D.E. solver, but here we controlled accuracy in a quite simple way. In principle, successive local maxima and minima of w (that determine the number of steady states corresponding to a given mass) can be estimated by the same method.

3. Self-similar solutions. Homogeneity properties of the system (1)–(2) imply that if u solves this system, then the rescaled function $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$, $\lambda > 0$, is also a solution. It is natural to consider solutions which satisfy the scaling property $u_\lambda \equiv u$, i.e. *forward* self-similar solutions. They are defined globally in time, and it is expected that they describe large time behavior of general solutions to (1)–(2). For a discussion of these particular solutions (existence, singularities, etc.) without radial symmetry assumption, we refer to [1]. If $u_\lambda \equiv u$, then $u(x,t) = t^{-1}U(xt^{-1/2})$ for a function U (whose singularity at the origin resembles that of a function homogeneous of degree -2; for a precise meaning of this see (23) in [1] and [1, Section 3]). Here we would like to point out some properties of integrated densities associated with radial self-similar u's. For $u \equiv u_\lambda$ we have

$$Q(r,t) = \sigma_n t^{n/2-1} \zeta(r^2/t) \tag{10}$$

with a nondecreasing positive function ζ. It can be verified that $\zeta = \zeta(y)$, $y = r^2/t$, satisfies the equation ([1, (32)])

$$\zeta'' + \frac{1}{4}\zeta' - \frac{n-2}{2y}\zeta' - \frac{n-2}{8y}\zeta + \frac{1}{2y^{n/2}}\zeta\zeta' = 0, \qquad ' = \frac{d}{dy}, \qquad \zeta(0) = 0. \tag{11}$$

Below we give a refinement of the result in [1, Prop. 3].

PROPOSITION 3. (i) *For $n = 2$ there exists a nondecreasing solution of* (11) *such that* $\lim_{y\to\infty} \zeta(y) = Z$ *if and only if* $Z \in [0, 4)$.

(ii) *If* $n \ge 3$, *then for each nondecreasing solution* $\zeta \not\equiv 0$ *of* (11),

$$\zeta(y) \le \left(1 - \frac{2}{n}\right) y^{n/2} + 4(n-1) y^{n/2-1},$$

and $\lim_{y\to\infty} y^{1-n/2}\zeta(y)$ *is a finite strictly positive number.*

Proof. (i) For $n = 2$, Proposition 3(i) in [1] gives the existence of self-similar solutions with $Z \in [0, 2(1+e^{-2}))$ and nonexistence when $Z > 4$.

For $n = 2$, (11) reads $\zeta'' + \zeta'/4 + \zeta\zeta'/(2y) = 0$ and the change of variables $s = (\log y)/2$, $v(s) = 2y(d\zeta/dy)(y)$, $w(s) = \zeta(y)$ transforms (11) into the problem

$$v' = (2-w)v - \frac{e^{2s}}{2}v, \quad w' = v, \quad ' = \frac{d}{ds},$$
$$v(-\infty) = 0, \quad w(-\infty) = 0.$$

Evidently, $\lim_{s\to\infty} w(s) < 4$ because the function $(w-2)^2 + 2v$ is strictly decreasing along the phase trajectory of the above system.

We also consider an autonomous system

$$\underline{v}' = (2 - \underline{w})\underline{v} - \varepsilon\underline{v}, \quad \underline{w}' = \underline{v},$$

where $\varepsilon > 0$, $\underline{v} = \underline{v}_\varepsilon$, $\underline{w} = \underline{w}_\varepsilon$, with the same condition at $s = -\infty$. A comparison of these vector fields gives the relation $\underline{w}(s) \le w(s)$ for all $s \le s_\varepsilon$ with $e^{2s_\varepsilon} = 2\varepsilon$. Since $\underline{w}(s) = 2(2-\varepsilon)Ae^{(2-\varepsilon)s}(1 + Ae^{(2-\varepsilon)s})^{-1}$ with an arbitrary $A > 0$ is a solution of the auxiliary system, $\underline{w}(s_\varepsilon) = 2(2-\varepsilon)A(2\varepsilon)^{1-\varepsilon/2} \times (1 + A(2\varepsilon)^{1-\varepsilon/2})^{-1}$, so $\sup Z = \sup w(s) \ge \limsup_{\varepsilon\to 0,\, s\le s_\varepsilon,\, A>0} \underline{w}(s) = 4$.

(ii) Note that the Chandrasekhar stationary solution $\widetilde{u}(x) = 2(n-2)|x|^{-2}$ of (1)–(2) corresponds to $\widetilde{\zeta}(y) = 2y^{n/2-1}$, $n \ge 3$.

Multiply (11) by $y^{n/2}$ and integrate on $[0, Y]$. After some elementary calculations we obtain

$$y^{n/2}\zeta'|_0^Y - (n-1)\int_0^Y y^{n/2-1}\zeta'(y)\,dy + \frac{1}{4}y^{n/2}\zeta|_0^Y - \frac{n-1}{4}\int_0^Y y^{n/2-1}\zeta(y)\,dy + \frac{1}{4}\zeta^2(Y) = 0.$$

Taking into account monotonicity of ζ and the initial condition for ζ we write

$$\zeta^2(Y) + Y^{n/2}\zeta(Y) \le (n-1)\int_0^Y y^{n/2-1}\zeta(y)\,dy + 4(n-1)Y^{n/2-1}\zeta(Y)$$
$$\le 2\frac{n-1}{n}Y^{n/2}\zeta(Y) + 4(n-1)Y^{n/2-1}\zeta(Y),$$

which gives the conclusion

$$\zeta(Y) \le \left(2 - 1 - \frac{2}{n}\right) Y^{n/2} + 4(n-1) Y^{n/2-1}.$$

The change of variables in (11) (analogous to that in the proof of Proposition 1), $s = (\log y)/2$, $v(s) = 2y^{2-n/2}(d\zeta/dy)(y)$, $w(s) = y^{1-n/2}\zeta(y)$, now leads to a nonautonomous system in the plane

$$(12) \qquad v' = (2-w)v + \frac{e^{2s}}{2}((n-2)w - v), \qquad w' = v - (n-2)w, \qquad ' = \frac{d}{ds}.$$

If we compare the vector field (7) associated with the steady state problem in the proof of Proposition 1 with that in (12) in the sectors below and above the line $v = (n-2)w$, then we see that forward self-similar solutions are determined by curves which are surrounded with the separatrix of the autonomous problem (7). In other words, spirals of these curves are arranged in a more tight manner than those of the separatrix for (7). Of course, for $n = 2$ and $n \ge 10$, there is a unique scroll of such a curve, since the eigenvalues of the linearization of the vector field at $(2(n-2), 2)$ are real.

Just as forward self-similar solutions are important for the large time asymptotics of arbitrary global-in-time solutions, *backward* (or *blowing up*) self-similar ones are expected to describe behavior of solutions that cannot be continued beyond a finite time T (i.e. that explode at $t = T$). We mentioned them in [4, Sec. 4]. If u has the self-similar form $u(x,t) = (T-t)^{-1}V(x(T-t)^{-1/2})$ and u is radially symmetric, then the associated integrated density Q equals $Q(r,t) = \sigma_n(T-t)^{n/2-1}\xi(r^2/(T-t))$. Here $y = r^2/(T-t)$ and $\xi = \xi(y)$ is a positive nondecreasing function satisfying

$$(13) \qquad \xi'' - \frac{1}{4}\xi' - \frac{n-2}{2y}\xi' + \frac{n-2}{8y}\xi + \frac{1}{2y^{n/2}}\xi\xi' = 0, \qquad ' = \frac{d}{dy}, \qquad \xi(0) = 0.$$

Note that if $\xi(y) = z(s)$, $y = s^2$, then z solves the equation

$$\ddot{z} - \left(\frac{s}{2} + \frac{n-1}{s}\right)\dot{z} + \left(\frac{n}{2} - 1\right)z + s^{1-n}z\dot{z} = 0, \qquad \dot{} = \frac{d}{ds},$$

considered in [4, (23)].

Besides the stationary Chandrasekhar solution $\widetilde{\xi}(y) = 2y^{n/2-1}$, there is a particular solution $\xi^{\#}(y) = y^{n/2}/n$ of (13) corresponding to the spatially uniform and growing-in-time density $u^{\#}(x,t) = (T-t)^{-1}$. The result below shows that these examples display typical growth of solutions to (13).

Proposition 4. (i) *If $n \ge 2$, then for each nondecreasing solution $\xi \not\equiv 0$ of* (13), $\lim_{y\to\infty} \xi(y) = \infty$.

(ii) *Moreover, ξ is bounded from above by*

$$\xi(y) \le y^{n/2} + 4(n-1)y^{n/2-1}.$$

Proof. The property (i) was observed by A. Krzywicki and justified using a different argument than the following one.

We begin with the case $n = 2$ and the equation

$$\xi'' - \frac{1}{4}\xi' + \frac{1}{2y}\xi\xi' = 0.$$

Suppose, for a contradiction, that $\xi(y) \le A$ for a nondecreasing solution ξ and some constant $A < \infty$. From (13) we obtain the differential inequality $\xi'' - \xi'/4 + (A/(2y))\xi' \ge 0$, and after two integrations,

$$\xi(y) \ge \xi(y_0) + \int\limits_{y_0}^{y} (\xi'(y_0)e^{-y_0/4}y_0^{A/2})e^{s/4}s^{-A/2}\, ds$$

($y_0 > 0$, $\xi'(y_0) > 0$), which tends to $+\infty$ as $y \to \infty$.

The case $n \ge 3$ is only slightly more difficult. Let $\omega = \xi' - ((n-2)/(2y))\xi$; then

$$\omega' - \frac{1}{4}\omega - \frac{n-2}{2y^2}\xi + (2y^{n/2})^{-1}\left(\omega + \frac{n-2}{2y}\xi\right)\xi = 0. \tag{14}$$

Suppose that $\xi(y) \le A < \infty$, so that $(2y^{n/2-1})^{-1}\xi(y) - 1 \le 0$ for large y. The equation (14) then implies $\omega' + (A(2y^{n/2})^{-1} - 1/4)\omega \ge 0$, so after an integration,

$$\omega(y) \ge a_0 e^{y/4} \quad \text{with } a_0 = \omega(y_0)\exp\left(-\frac{y_0}{4} - \frac{A}{n-2}y_0^{1-n/2}\right) > 0,\ y_0 > 0.$$

Finally, we get $(\xi(y)y^{1-n/2})' \ge a_0 e^{y/4}y^{1-n/2}$, and

$$\xi(y) \ge y^{n/2-1}a_0 \int\limits_{y_0}^{y} e^{s/4}s^{1-n/2}\, ds,$$

which tends to $+\infty$ as $y \to \infty$.

(ii) We proceed as in the proof of Proposition 3: (13) multiplied by $y^{n/2}$ and integrated on $[0, Y]$ leads to

$$y^{n/2}\xi'|_0^Y - (n-1)\int\limits_0^Y y^{n/2-1}\xi'(y)\, dy - \frac{1}{4}y^{n/2}\xi|_0^Y + \frac{n-1}{4}\int\limits_0^Y y^{n/2-1}\xi(y)\, dy + \frac{1}{4}\xi^2(Y) = 0.$$

Then we obtain

$$\xi^2(Y) \le Y^{n/2}\xi(Y) + 4(n-1)Y^{n/2-1}\xi(Y),$$

so the upper bound for the growth of ξ is as announced.

4. Blowing up solutions. We recall that the system (3)–(4) on $\Omega = B_R$, with an initial condition (5) of sufficiently large concentration $MR^{2-n} = Q_0(R)R^{2-n} > 2n\sigma_n$, cannot have solutions defined globally in time. The proof of this result ([4, Th. 3]) uses the virial method, i.e. the evolution of moments of the density u is considered (e.g. $v(t) = \int_\Omega |x|^2 u(x,t)\,dx$). Refinements of this nonexistence result can be found in [2], where the system (1)–(2) is considered in general star-shaped domains Ω.

Corollaries of [2] together with an analysis of solvability of the Cauchy problem in [1, Sec. 2] may be interpreted that, loosely speaking, a sufficiently high *local* concentration of u_0 does not allow even to define solutions of the evolution problem.

Here we would like to make a next step in understanding blow up and nonexistence of solutions phenomena by studying a simple case of radial solutions with u_0 having a local singularity at the origin.

Our purpose is to determine what critical singularities of u lead to a blow up, and what kind of singularities can be smoothed out owing to the parabolic regularization effect enjoyed by (1)–(2) (cf. [3, Th. 2(ii)], [6, Th. 2(ii)]).

Below we consider solutions of the problem (3)–(5) on $[0,R]\times[0,T]$ with either $R = 1$ or $R = \infty$ and some $T > 0$.

THEOREM. (i) *If $n \ge 3$ and $Q_0(r) > 2\sigma_n r^{n-2}$ for each $r \in (0,\infty)$, then the solution of* (3)–(5) *cannot be global in time.*

(ii) *If $n = 2$ and $Q(r,t) \le 4\pi\alpha + Nr^2$ for some $\alpha < 1$, $N > 0$ and all $r \in [0,1]$, $t \in [0,T]$, then $Q(r,t) \le Cr^2$ with a constant C independent of time.*

(iii) *If $n \ge 3$ and $Q(r,t) \le 2\sigma_n\alpha r^{n-2} + Nr^n$ for some $\alpha < 1$, $N > 0$ and all $r \in [0,1]$, $t \in [0,T]$, then $Q(r,t) \le Cr^n$ with a constant C independent of time.*

For an interpretation of the Theorem we formulate the following remarks:

• The assumption $Q_0(r) > 2\sigma_n r^{n-2}$, $r \in (0,\infty)$, is satisfied e.g. when $u_0(x) > 2(n-2)|x|^{-2}$. Note that a sufficient condition for blow up in the initial-boundary value problem in terms of $Q_0(1)$ solely is $Q_0(1) > 2n\sigma_n$, see [4, Th. 3].

• (ii) implies that if a finite time blow up of solutions is accompanied by a concentration of mass near the origin, then this mass is greater than 4π.

• (iii) implies that as long as u is strictly below the Chandrasekhar steady state $\widetilde{u}(x) = 2(n-2)|x|^{-2}$ (corresponding to $\widetilde{Q}(r) = 2\sigma_n r^{n-2}$), Q is flat enough at the origin. In other words, the critical singularity of u_0 that prohibits the smoothing of the solution u must be (at least) as strong as $2(n-2)r^{-2}$.

Proof of Theorem. We transform (3)–(5), using a new independent variable $y = r^n$, into the problem

$$Q_t = n^2 y^{2-2/n} Q_{yy} + n\sigma_n^{-1} Q Q_y, \tag{15}$$
$$Q(0,t) = 0, \qquad Q(R^n, t) = M, \qquad Q(y,0) = Q_0(y),$$

exactly as in [4, Sec. 3]. In spite of the singular coefficient $y^{2-2/n}$ the parabolic equation in (15) is slightly simpler to analyze than (3). First we note that the parabolic maximum principle and, more generally, the comparison principle hold for (15). The proof of this, which is rather technical, follows from considerations similar to those in the proof of Theorem 2 in [4], where (15) is approximated by regularized equations with the leading coefficient $(y+\varepsilon)^{2-2/n}$, $\varepsilon > 0$.

(i) We look for a subsolution $\underline{Q}$ of the problem (15) with $R = \infty$, $M = \infty$ in the form $\underline{Q}(y,t) = \beta(t) y^{1-2/n}$ with a function β. An easy calculation shows that $\underline{Q}$ is a subsolution provided

$$\dot\beta \ge (n-2)\beta(\sigma_n^{-1}\beta - 2), \qquad \dot{} = \frac{d}{dt}.$$

This differential inequality leads to $|1 - 2\sigma_n/\beta| \ge ce^{(n/2-1)t}$ for some $c > 0$. For $\beta(0) > 2\sigma_n$, $c < 1$, and $\beta(t) \ge 2\sigma_n(1 - ce^{(n/2-1)t})^{-1}$ blows up in a finite time. Note that on the level of approximating equations the above formal argument works with y replaced by $y + \varepsilon$, $\varepsilon > 0$ small enough.

(ii) Our hypothesis for (15) reads $Q(y,t) \le 4\pi\alpha + Ny$, and this implies

$$Q_t \le 4yQ_{yy} + \pi^{-1} Q_y (4\pi\alpha + Ny).$$

The comparison principle for linear parabolic equations allows us to estimate Q by a supersolution $\overline{Q}$ which solves the problem $\overline{Q}_{yy} + \overline{Q}_y(\alpha/y + N/(4\pi)) = 0$ with the boundary conditions in (15), i.e. $Q(y,t) \le Cy^{1-\alpha}$. Now if $\delta = 1 - \alpha > 0$, then

$$Q_t \le 4yQ_{yy} + \pi^{-1} C y^{\delta} Q_y.$$

Again by the comparison principle we get $Q(y,t) \le Cy$ with a suitable constant C.

(iii) The assumption $Q(y,t) \le 2\sigma_n \alpha y^{1-2/n} + Ny$ implies the differential inequality

$$Q_t \le n^2 y^{2-2/n} Q_{yy} + n\sigma_n^{-1} Q_y (2\sigma_n \alpha y^{1-2/n} + Ny),$$

so $Q \le \overline{Q}$, where $\overline{Q}_{yy} + \overline{Q}_y(2\alpha/(ny) + N(n\sigma_n)^{-1} y^{2/n-1}) = 0$. Explicitly we have $Q(y,t) \le Cy^{1-2\alpha/n}$. Using again the comparison principle, from

$$Q_t \le n^2 y^{2-2/n} Q_{yy} + n\sigma_n^{-1} Q_y (Cy^{1-2\alpha/n})$$

the estimate $Q \le \widehat{Q} \le Cy$ with a suitably large C follows. Here $\widehat{Q}$ is a solution of $\widehat{Q}_{yy} + (n\sigma_n)^{-1}\widehat{Q}_y(Cy^{2/n-1-2\alpha/n}) = 0$.

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