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A NOISY DUEL UNDER ARBITRARY MOTION. VIII

1. Introduction. In [20], [21] and in this paper an m versus n bullets noisy duel is considered in which duelists can move at will. It is assumed that Player I has greater maximal speed. The cases $m = 1, 2, 3$, $n = 1, 2, 3$ are solved. Let a be the point in which Player I is at the beginning of the duel, $0 \le a < 1$ (Player II is at 1). In contrast to [14]–[19] where the duels are solved for small a, now we solve the duels for any $0 \le a < 1$.

In this paper we consider the cases $m = 2$, $n = 2$; $m = 1$, $n = 3$; $m = 2$, $n = 3$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys the opponent) if he fires when the distance between the players is $1 - s$. We assume that $P(s)$ is increasing and continuous in $[0, 1]$, has continuous second derivative in $(0, 1)$, and $P(s) = 0$ for $s \le 0$, $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The game is over if at least one player is destroyed or all bullets are shot. In the other case the duel lasts infinitely long and the payoff is zero.

The duel is noisy—each player hears every shot of his opponent.

As will be seen from the sequel, without loss of generality we can assume that Player II is motionless. It is also assumed that the maximal speed of Player I is 1 and that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

We suppose that between successive shots of the same player there has to pass a time $\widehat{\varepsilon} > 0$. We also assume that the reader knows the papers [14]–[19] and remembers the definitions, assumptions and results given there.

For other results in the theory of games of timing see [1]–[13], [22], [23].

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2. The duel $(2,2)$**.** Consider the case where Players I and II have two bullets each and the duel begins when Player I is at the point a, $0 \le a < 1$. Let $Q(s) = 1 - P(s)$.

The duel $(2,2), \langle a\rangle$

Case 1: $Q(a) \ge Q(a_{12}) \cong 0.812085$. We define the following strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I. If Player II has not fired before, reach the point a_{22}, fire a shot at a_{22}^{ε} and play optimally afterwards. If he fired, play optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I has not fired before, fire at $\langle a_{22}\rangle$ and play optimally afterwards. If Player I has not reached the point a_{22}, do not fire.

The a_{22}^{ε} is an absolutely continuous random variable taking values between $\langle a_{22}\rangle$ and $\langle a_{22}\rangle + \alpha(\varepsilon)$ for properly chosen $\alpha(\varepsilon)$. The $\langle a_{22}\rangle$ is the first time when Player I reaches the point a_{22}.

In [15] it is proved that if $Q(a) \ge Q(a_{22}) \cong 0.812085$, then the strategies ξ and η are optimal in limit (i.e. as $\widehat{\varepsilon} \to 0$) and the limit value of the game is $v_{22}^a = -P(a_{22}) + Q(a_{22})\overset{2}{v}{}_{21}^{a} = 0.148461$.

Case 2: $0.781133 \cong Q(\widehat{a}_{22}) \le Q(a) \le Q(a_{22}) \cong 0.812083$. Define ξ and η as follows:

STRATEGY OF PLAYER I. If Player II has not fired before, fire at a_{22}^{ε} and play optimally the resulting duel.

STRATEGY OF PLAYER II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$v_{22}^a = -P(a) + Q(a)\overset{2}{v}{}_{21}^{a} = -1 + (1 + v_{21})Q(a),$$

where $v_{21} = \sqrt{2} - 1 \cong 0.414214$.

Suppose that Player I fires at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi},\eta) \le Q^2(a)v_{11}^a + k(\widehat{\varepsilon}) \le -1 + (1 + v_{21})Q(a) + k(\widehat{\varepsilon})$$

if

$$Q^2(a)v_{11} - (1 + v_{21})Q(a) + 1 \le 0,$$

i.e. if

(1) $$Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133.$$

On the other hand, suppose that Player II fires after $\langle a_{22}\rangle+\alpha(\varepsilon)$ or does not fire at all. For such a strategy $\widehat{\eta}$ we obtain

$$
\begin{aligned}
(2)\qquad K(\xi,\widehat{\eta}) &\ge P(a)+Q(a)\overset{1}{v}{}^{a}_{12}-k(\widehat{\varepsilon})\\
&=1-Q(a)+(-1+(1+v_{11})Q(a))Q(a)-k(\widehat{\varepsilon})\\
&=1-2Q(a)+(1+v_{11})Q^2(a)-k(\widehat{\varepsilon})\\
&\ge -1+(1+v_{21})Q(a)+k(\widehat{\varepsilon})
\end{aligned}
$$

provided that

$$0.853553\cong Q(a_{12})\ge Q(a)\ge Q(a_{11})\cong 0.585787.$$

The inequality (2) holds if

$$Q(a)\le Q(a_{22})\cong 0.812085.$$

From (1) and (2) it follows that the strategies ξ and η are optimal in limit and the limit value of the game is $v^a_{22}=-1+(1+v_{21})Q(a)$ if

$$0.781133\cong Q(\widehat{a}_{22})\le Q(a)\le Q(a_{22})\cong 0.812085.$$

Case 3: $Q(a)\le Q(\widehat{a}_{22})\cong 0.781133$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Fire at $\langle a\rangle$ and play optimally the resulting duel.

STRATEGY OF PLAYER II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$
(3)\quad v^a_{22}=Q^2(a)v^a_{11}=\begin{cases} v_{11}Q^2(a) & \text{if } Q(a_{11})\le Q(a)\le Q(\widehat{a}_{22}),\\ Q^2(a)(2Q(a)-1) & \text{if } 1/2\le Q(a)\le Q(a_{11}),\\ 0 & \text{if } Q(a)\le 1/2.\end{cases}
$$

Suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we obtain

$$
\begin{aligned}
K(\xi,\widehat{\eta}) &\ge P(a)+Q(a)\overset{1}{v}{}^{a}_{12}-k(\widehat{\varepsilon})\\
&=\begin{cases} 1-2Q(a)+(1+v_{11})Q^2(a)-k(\widehat{\varepsilon})\\ \qquad\qquad \text{if } Q(a_{11})\le Q(a)\le Q(a_{12})\cong 0.853553,\\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+Q^4(a)-k(\widehat{\varepsilon})\\ \qquad\qquad \text{if } Q(a)\le Q(a_{11})\cong 0.585787.\end{cases}
\end{aligned}
$$

(a) Let $Q(a_{11})\le Q(a)\le Q(a_{12})$. We obtain

$$1-2Q(a)+(1+v_{11})Q^2(a)\ge v_{11}Q^2(a),$$

which is always satisfied.

(b) Let $1/2 \le Q(a) \le Q(a_{11})$. We have

$$1-2Q(a)+2Q^2(a)-2Q^3(a)+Q^4(a) \ge 2Q^3(a)-Q^2(a). \tag{4}$$

Let $S(Q)$ be the difference between the left and right sides:

$$S(Q)=1-2Q(a)+3Q^2(a)-4Q^3(a)+Q^4(a).$$

The function $S(Q)$ is decreasing for $Q>1/2$ and $S(Q(a_{11})) \cong S(0.585787) > 0$. Thus for the case (b) the inequality (4) holds.

(c) $Q(a) \le 1/2$. Now we have to prove that

$$S(Q)=Q^4(a)-2Q^3(a)+2Q^2(a)-2Q(a)+1 \ge 0.$$

But

$$S(Q)=(Q^2(a)+1)(Q(a)-1)^2$$

and it is always nonnegative.

On the other hand, suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$\begin{aligned}K(\widehat{\xi},\eta) &\le -P(a)+Q(a)\overset{2}{v}{}^{a}_{21}+k(\widehat{\varepsilon})\\ &= \begin{cases} -1+(1+v_{21})Q(a)+k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a) \ge Q(a_{21}) \cong 0.707107, \\ -1+2Q(a)-(1-v_{11})Q^2(a)+k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787, \\ -1+2Q(a)-2Q^2(a)+2Q^3(a)+k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a) \le Q(a_{11}). \end{cases}\end{aligned}$$

Consider the following cases:

(a) $Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133$. In this case we obtain

$$-1+(1+v_{21})Q(a) \le v_{11}Q^2(a),$$

which is satisfied if $Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133$.

(b) $Q(\widehat{a}_{22}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787$. In this case we obtain

$$-1+2Q(a)-(1-v_{11})Q^2(a) \le v_{11}Q^2(a),$$

which is always satisfied.

(c) $Q(a_{11}) \ge Q(a) \ge 1/2$. In this case we have

$$-1+2Q(a)-2Q^2(a)+2Q^3(a) \le 2Q^3(a)-Q^2(a),$$

which is also always satisfied.

(d) In the last case $Q(a) \le 1/2$ we obtain

$$S(Q)=-1+2Q(a)-2Q^2(a)+2Q^3(a) \le 0.$$

The function $S(Q)$ is increasing for $Q(a) \le 1/2$ and $S(1/2)<0$. Thus the inequality holds.

From the above it follows that if $Q(a) \le Q(\widehat{a}_{22})$ then the strategies ξ and η are optimal in limit and the limit value of the game is given by (3).

Now we define the duels $(m,n), \langle 1, a\wedge c, a\rangle$ and $(m,n), \langle 2, a, a\wedge c\rangle$. We have supposed that a time $\widehat{\varepsilon}$ has to elapse between successive shots of the same player. Let

$$(m,n), \langle 2, a, a\wedge c\rangle, \qquad 0 < c \le \widehat{\varepsilon},$$

be the duel in which Player I has m bullets, Player II has n bullets but if $c < \widehat{\varepsilon}$, Player I can fire his bullets from time $\langle a\rangle$ on, and Player II from time $\langle a\rangle + c$ on. If $c = \widehat{\varepsilon}$ the rule is the same with the only exception that Player I is not allowed to fire at time $\langle a\rangle$.

Similarly we define the duel $(m,n), \langle 1, a\wedge c, a\rangle$.

The duel $(2,2), \langle 1, a\wedge c, a\rangle$

Case 1: $Q(a) \ge Q(a_{22}) \cong 0.812085$. The strategies optimal in limit are the same as in the duel $(2,2), \langle a\rangle$, Case 1.

Case 2: $Q(a) \le Q(a_{22})$. Let $\rangle t\langle$ be the point in which Player I has been at time t. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally the resulting duel.

Strategy of Player II. Fire at t, $\langle a\rangle < t < \langle a\rangle + c$, and play optimally afterwards.

We now have

$$(5)\qquad \overset{1}{v}{}^{a}_{22} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{21}$$
$$= \begin{cases} -1 + (1+v_{21})Q(a) \\ \qquad\qquad \text{if } 0.812085 \cong Q(a_{22}) \ge Q(a) \ge Q(a_{21}) = \sqrt{2}/2, \\ -1 + 2Q(a) - (1-v_{11})Q^2(a) \\ \qquad\qquad \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787, \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) \quad \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

Player II always assures these values.

On the other hand, suppose that Player II fires before or at $\langle a\rangle + c$. We obtain

$$K(\xi, \widehat{\eta}) \ge -P(a) + Q(a)\overset{2}{v}{}^{a}_{21} - k(\widehat{\varepsilon}) = \overset{1}{v}{}^{a}_{22} - k(\widehat{\varepsilon}).$$

Finally, suppose that Player II does not fire before $\langle a\rangle + c + \alpha(\varepsilon)$. We obtain

$$K(\xi, \widehat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}^{a}_{12} - k(\widehat{\varepsilon})$$
$$= \begin{cases} 1 - 2Q(a) + (1+v_{11})Q^2(a) - k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a_{12}) \ge Q(a) \ge Q(a_{11}), \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^4(a) - k(\widehat{\varepsilon}) \\ \qquad\qquad \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

Consider the following cases:

(a) $0.707107 \cong \sqrt{2}/2 = Q(a_{21}) \le Q(a) \le Q(a_{12}) \cong 0.853553$. In this case we have

$$1 - 2Q(a) + (1 + v_{11})Q^2(a) \ge -1 + (1 + v_{21})Q(a).$$

This inequality is satisfied if $Q(a) \le Q(a_{22}) \cong 0.812085$.

(b) $0.585787 \cong Q(a_{11}) \le Q(a) \le Q(a_{21})$. We obtain

$$1 - 2Q(a) + (1 + v_{11})Q^2(a) \ge -1 + 2Q(a) - (1 - v_{11})Q^2(a).$$

This inequality always holds.

(c) $Q(a) \le Q(a_{11})$. In this case we have

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^4(a) \ge -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a).$$

We can rewrite this inequality in the form

$$2(1 - Q(a))^2 + Q^2(a)(2 - 4Q(a) + Q^2(a)) \ge 0.$$

Both expressions on the left hand side are nonnegative for $Q(a) \ge Q(a_{11})$, which ends the proof of the inequality.

Thus if $Q(a) \le Q(a_{22})$, then the strategies ξ and η are optimal in limit and the limit value of the game is given by (5).

The duel $(2, 2), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \ge Q(a_{22}) \cong 0.812085$. In this case the strategies optimal in limit are the same as in the duel $(2, 2), \langle a\rangle$, Case 1.

Case 2: $Q(a) \le Q(a_{22}) \cong 0.812085$. Define ξ and η as follows:

Strategy of Player I. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$, and play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally afterwards.

We now have

$$\overset{2a}{v}_{22} = P(a) + Q(a)\overset{1a}{v}_{12} = \begin{cases} 1 - 2Q(a) + (1 + v_{11})Q^2(a) \\ \qquad\qquad \text{if } Q(a_{11}) \le Q(a) \le Q(a_{22}) \cong 0.812085, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^4(a) \\ \qquad\qquad \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

Player I always assures these values.

On the other hand, suppose that Player I fires before or at $\langle a\rangle + c$. We obtain

$$K(\widehat{\xi}, \eta) \le P(a) + Q(a)\overset{1a}{v}_{12} + k(\widehat{\varepsilon}) = \overset{2a}{v}_{22} + k(\widehat{\varepsilon}).$$

Finally, suppose that Player I does not fire before $\langle a\rangle + c + \alpha(\varepsilon)$. We obtain

$$K(\widehat{\xi},\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{21} + k(\widehat{\varepsilon}) = \begin{cases} -1 + (1+v_{21})Q(a) + k(\widehat{\varepsilon}) & \text{if } Q(a) \ge Q(a_{21}) = \sqrt{2}/2, \\ -1 + 2Q(a) - (1-v_{11})Q^2(a) + k(\widehat{\varepsilon}) & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}), \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) + k(\widehat{\varepsilon}) & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

Consider the following cases:

(a) $Q(a_{22}) \ge Q(a) \ge Q(a_{21}) = \sqrt{2}/2$. Then

$$-1 + (1+v_{21})Q(a) \le 1 - 2Q(a) + (1+v_{11})Q^2(a).$$

This inequality is satisfied if $Q(a) \le Q(a_{22}) \cong 0.812085$ (see case 2(a) of the duel $(2,2)$, $\langle 1, a\wedge c, a\rangle$).

(b) $Q(a_{21}) \ge Q(a) \ge Q(a_{11})$. Now we have

$$-1 + 2Q(a) - (1-v_{11})Q^2(a) \le 1 - 2Q(a) + (1+v_{11})Q^2(a),$$

which is always satisfied.

(c) $Q(a) \le Q(a_{11})$. In this case we obtain

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) \le 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^4(a)$$

or

$$S(Q) = 2 - 4Q(a) + 4Q^2(a) - 4Q^3(a) + Q^4(a) \ge 0.$$

This function is decreasing for $Q \le Q(a_{11})$ and $S(Q(a_{11})) \cong S(0.585787) > 0$. Thus the inequality holds for $Q(a) \le Q(a_{11})$.

Thus if $Q(a) \le Q(a_{22}) \cong 0.812085$, then the strategies ξ and η are optimal in limit.

3. Results for the duel $(2,2)$**.** We have

$$\overset{1}{v}{}^{a}_{22} = \begin{cases} v_{22} = 0.148461 & \text{if } Q(a) \ge Q(a_{22}) \cong 0.812085, \\ -1 + (1+v_{21})Q(a) & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{21}) = \sqrt{2}/2, \\ -1 + 2Q(a) - (1-v_{11})Q^2(a) & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787, \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) & \text{if } Q(a) \le Q(a_{11}); \end{cases}$$

$$v^{a}_{22} = \begin{cases} 0.148461 & \text{if } Q(a) \ge Q(a_{22}), \\ -1 + (1+v_{21})Q(a) & \text{if } Q(a_{22}) \ge Q(a) \ge Q(\widehat{a}_{22}) \cong 0.781133, \\ v_{11}Q^2(a) & \text{if } Q(\widehat{a}_{22}) \ge Q(a) \ge Q(a_{11}), \\ 2Q^3(a) - Q^2(a) & \text{if } Q(a_{11}) \ge Q(a) \ge 1/2, \\ 0 & \text{if } Q(a) \le 1/2; \end{cases}$$

$$\overset{2}{v}{}^a_{22} = \begin{cases} 0.148461 & \text{if } Q(a) \ge Q(a_{22}), \\ 1 - 2Q(a) + (1 + v_{11})Q^2(a) & \text{if } Q(a_{22}) \ge Q(a) \ge Q(a_{11}), \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^4(a) & \\ & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

4. The duel $(1,3)$

The duel $(1,3), \langle a\rangle$

Case 1: $Q(a) \ge Q(a_{13}) \cong 0.814115$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Escape if Player II has not fired yet. If he fired (say at a'), play optimally the resulting duel $(1,2), \langle 2, a', a' \wedge \widehat{\varepsilon}\rangle$.

STRATEGY OF PLAYER II. Fire at $\langle a\rangle$ and play optimally afterwards.

In [15] it is proved that for $Q(a) \ge Q(a_{13})$ the strategies ξ and η are optimal in limit and

$$v^a_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1 + v_{11})Q^2(a) & \text{if } Q(a_{13}) \le Q(a) \le Q(a_{12}) \cong 0.853553. \end{cases}$$

Case 2: $Q(a) \le Q(a_{13})$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Fire at $\langle a\rangle$ and escape.

STRATEGY OF PLAYER II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$v^a_{13} = -Q^2(a) + Q^4(a).$$

Suppose Player II does not fire at $\langle a\rangle$. It is assumed that he fires immediately after the shot of Player I. Thus we have

$$\begin{aligned} K(\xi, \widehat{\eta}) &\ge P(a) - Q(a)(1 - Q^3(a)) - k(\widehat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^4(a) - k(\widehat{\varepsilon}) \ge -Q^2(a) + Q^4(a) - k(\widehat{\varepsilon}). \end{aligned}$$

On the other hand, suppose Player I does not fire at $\langle a\rangle$. We have

$$\begin{aligned} K(\widehat{\xi}, \eta) &\le -P(a) + Q(a)\overset{2}{v}{}^a_{12} + k(\widehat{\varepsilon}) \\ &= \begin{cases} -1 + (1 + v_{11})Q^2(a) + k(\widehat{\varepsilon}) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}) \cong 0.780539, \\ 1 - 2Q(a) + Q^3(a) + k(\widehat{\varepsilon}) & \text{if } Q(a) \le Q(\check{a}_{12}). \end{cases} \end{aligned}$$

Consider the following cases:

(a) $0.853553 \cong Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}) \cong 0.780539$. In this case we obtain the inequality

$$Q^4(a) - (2 + v_{11})Q^2(a) + 1 \ge 0,$$

which is satisfied if $Q(a) \le Q(a_{13})$.

(b) Let $Q(a) \le Q(\check{a}_{12})$. In this case we obtain

$$-1 + 2Q(a) - 2Q^2(a) + Q^4(a) \le -Q^2(a) + Q^4(a),$$

which is always satisfied.

The duel $(1,3), \langle 1, a \wedge c, a \rangle$

Case 1: $Q(a) \ge Q(\check{a}_{12}) \cong 0.780539$.

STRATEGY OF PLAYER I. Escape if Player II has not fired. If he fired, play optimally the resulting duel.

STRATEGY OF PLAYER II. Fire before $\langle a \rangle + c$ and play optimally afterwards.

In [15] it is proved that the above strategies are optimal in limit if $Q(a) \ge Q(\check{a}_{12})$ and the limit value of the game is

$$\overset{1}{v}{}^{a}_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1 + v_{11})Q^2(a) & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}). \end{cases}$$

Case 2: $Q(a) \le Q(\check{a}_{12})$. Define ξ and η as follows:

STRATEGY OF PLAYER I. If Player II has not fired before, fire at $\rangle\langle a \rangle + c\langle^{\varepsilon}$ and play optimally afterwards. If he fired, play optimally the resulting duel.

STRATEGY OF PLAYER II. Fire before $\langle a \rangle + c$ and play optimally afterwards.

We now have

$$\overset{1}{v}{}^{a}_{13} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{12} = -1 + 2Q(a) - 2Q^2(a) + Q^4(a)$$

if $Q(a) \le Q(\check{a}_{12})$.

Player II always assures this value.

On the other hand, suppose that Player II fires before or at $\langle a \rangle + c$. For such a strategy $\widehat{\eta}$ we obtain

$$K(\xi, \widehat{\eta}) \ge -P(a) + Q(a)\overset{2}{v}{}^{a}_{12} - k(\widehat{\varepsilon}) = \overset{1}{v}{}^{a}_{13} - k(\widehat{\varepsilon}).$$

Finally, suppose that Player II has not fired before $\langle a \rangle + c + \alpha(\varepsilon)$. For such a strategy $\widehat{\eta}$ we obtain

$$\begin{aligned} K(\xi, \widehat{\eta}) &\ge P(a) - Q(a)(1 - Q^3(a)) - k(\widehat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^4(a) - k(\widehat{\varepsilon}) \\ &\ge -1 + 2Q(a) - 2Q^2(a) + Q^4(a) - k(\widehat{\varepsilon}), \end{aligned}$$

which is always satisfied. Thus if $Q(a) \le Q(\check{a}_{13}) \cong 0.780539$, then the strategies ξ and η are optimal in limit.

The duel $(1,3), \langle 2, a, a \wedge c \rangle$

Case 1: $Q(a) \ge Q(\widehat{a}_{13}) \cong 0.834554$.

Strategy of Player I. Escape if Player II has not fired. If he fired a shot, play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire a shot at $\langle a\rangle + c$ and play optimally afterwards.

In [15] it is proved that the above strategies are optimal in limit for the corresponding a and that

$$\overset{2}{v}{}^{a}_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1 + v_{11})Q^2(a) & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\widehat{a}_{13}) \cong 0.834554. \end{cases}$$

Case 2: $Q(a) \le Q(\widehat{a}_{13})$. Define ξ and η as follows:

Strategy of Player I. Fire before $\langle a\rangle + c$.

Strategy of Player II. If Player I has not fired before, fire a shot at $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and play optimally afterwards. If he fired, play optimally the resulting duel.

We now have

$$\overset{2}{v}{}^{a}_{13} = P(a) - Q(a)(1 - Q^3(a)) = 1 - 2Q(a) + Q^4(a).$$

Suppose that Player I does not fire before $\langle a\rangle + c + \alpha(\varepsilon)$. Then we obtain

$$\begin{aligned} K(\widehat{\xi}, \eta) &\le -P(a) + Q(a)\overset{2}{v}{}^{a}_{12} + k(\widehat{\varepsilon}) \\ &= \begin{cases} -1 + (1 + v_{11})Q^2(a) + k(\widehat{\varepsilon}) \\ \qquad \text{if } 0.853553 \cong Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}) \cong 0.780539, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) + k(\widehat{\varepsilon}) \quad \text{if } Q(a) \le Q(\check{a}_{12}). \end{cases} \end{aligned}$$

If $Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12})$ we should have

$$-1 + (1 + v_{11})Q^2(a) \le 1 - 2Q(a) + Q^4(a)$$

or

$$S(Q) = Q^4(a) - (1 + v_{11})Q^2(a) - 2Q(a) + 2 \ge 0.$$

This function is decreasing in Q and $S(Q(\widehat{a}_{13})) = 0$. Thus $S(Q) > 0$ for $Q(\check{a}_{12}) \le Q(a) < Q(\widehat{a}_{13}) \cong 0.834554$.

If $Q(a) \le Q(\check{a}_{12})$ we obtain

$$-1 + 2Q(a) - 2Q^2(a) + Q^4(a) \le 1 - 2Q(a) + Q^4(a).$$

This inequality is always satisfied.

5. Results for the duel $(1,3)$**.** We have

$$\overset{1}{v}{}^a_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}) \cong 0.853553, \\ -1 + (1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}) \cong 0.780539, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) & \text{if } Q(a) \le Q(\check{a}_{12}) \end{cases}$$

$$v^a_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \ge Q(a) \ge Q(a_{13}) \cong 0.814115, \\ -Q^2(a) + Q^4(a) & \text{if } Q(a) \le Q(a_{13}); \end{cases}$$

$$\overset{2}{v}{}^a_{13} = \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(a_{12}) \cong 0.853553, \\ -1 + (1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\widehat{a}_{13}) \cong 0.834554, \\ 1 - 2Q(a) + Q^4(a) & \text{if } Q(a) \le Q(\widehat{a}_{13}). \end{cases}$$

6. The duel $(2,3)$

The duel $(2,3), \langle a \rangle$

Case 1: $Q(a) \ge Q(a_{23}) \cong 0.882709$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire a shot at a^{ε}_{23} and play optimally the resulting duel. If he fired, play optimally the resulting duel $(2,2)$.

Strategy of Player II. If Player I has not fired before, fire at $\langle a_{23} \rangle$ and play optimally afterwards. If he fired (say at a'), play optimally the resulting duel $(1,3), \langle 1, a' \wedge \widehat{\varepsilon}, a' \rangle$. If Player I has not reached the point a_{23}, do not fire.

We now have

$$v^a_{23} \overset{\text{df}}{=} v_{23} = -P(a_{23}) + Q(a_{23})v_{22} \cong 0.013757,$$

where

$$Q^2(a_{23}) - (3 + v_{23})Q(a_{23}) + 2 = 0, \qquad Q(a_{23}) \cong 0.882709.$$

The proof that ξ and η are optimal in limit is given in [15].

Case 2: $Q(a_{23}) \ge Q(a) \ge Q(\widehat{a}_{23}) \cong 0.870730$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at a^{ε} and play optimally the resulting duel. If he fired, play optimally afterwards.

Strategy of Player II. Fire at $\langle a \rangle$ and play optimally the resulting duel.

Also in [15] it is proved that in this case the strategies ξ and η are optimal in limit and

$$v^a_{23} = -1 + (1 + v_{22})Q(a)$$

is the limit value of the game.

Case 3: $Q(a) \le Q(\widehat{a}_{23})$. Define ξ and η as follows:

Strategy of Player I. Fire at $\langle a\rangle$ and if Player II has not fired, play optimally the resulting duel $(1,3), \langle 1, a \wedge \widehat{\varepsilon}, a\rangle$. If he fired, play optimally the resulting duel $(1,2), \langle a_1\rangle$.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally the resulting duel.

We now have

$$v_{23}^a = v_{12}^a Q^2(a) = \begin{cases} 0 & \text{if } Q(\widehat{a}_{23}) \ge Q(a) \ge Q(a_{12}) \cong 0.853553, \\ -Q^2(a) + (1+v_{11})Q^3(a) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\widehat{a}_{12}) \cong 0.730812, \\ Q^5(a) - Q^2(a) & \text{if } Q(a) \le Q(\widehat{a}_{12}). \end{cases}$$

Suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we have

$$K(\xi,\widehat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}_{13}^{a} - k(\widehat{\varepsilon}) = \begin{cases} 1 - 2Q(a) + Q^2(a) - k(\widehat{\varepsilon}) & \text{if } Q(a) \ge Q(a_{12}) \cong 0.853553, \\ 1 - 2Q(a) + (1+v_{11})Q^3(a) - k(\widehat{\varepsilon}) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\check{a}_{12}) \cong 0.780539, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) - k(\widehat{\varepsilon}) & \\ & \text{if } Q(a) \le Q(\check{a}_{12}). \end{cases}$$

Consider the following cases:

(a) $0.870730 \cong Q(\widehat{a}_{23}) \ge Q(a) \ge Q(a_{12}) \cong 0.853553$. In this case the condition $K(\xi,\widehat{\eta}) \ge v_{23}^a - k(\widehat{\varepsilon})$ leads to the inequality

$$1 - 2Q(a) + Q^2(a) \ge 0,$$

which is always satisfied.

(b) $0.780539 \cong Q(\check{a}_{12}) \le Q(a) \le Q(a_{12})$. In this case we obtain

$$1 - 2Q(a) + (1+v_{11})Q^3(a) \ge -Q^2(a) + (1+v_{11})Q^3(a),$$

which is also always satisfied.

(c) $0.730812 \cong Q(\widehat{a}_{12}) \le Q(a) \le Q(\check{a}_{12})$. In this case we should prove that

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) \ge -Q^2(a) + (1+v_{11})Q^3(a).$$

Let $S(Q)$ be the difference of the left and right sides of the above inequality:

$$S(Q) = Q^5(a) - (3+v_{11})Q^3(a) + 3Q^2(a) - 2Q(a) + 1.$$

To prove that $S(Q) \ge 0$, notice that $S'(Q) < 0$ for the considered numbers Q and $S(Q(\check{a}_{12})) \cong S(0.780539) > 0$. Thus the inequality holds.

(d) $Q(a) \le Q(\widehat{a}_{12})$. In this case we obtain

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) \ge Q^5(a) - Q^2(a).$$

This inequality can be rewritten in the form

$$(1 - Q(a))^2 + 2Q^2(a)(1 - Q(a)) \ge 0,$$

which always holds.

The duel $(2,3), \langle 1, a \wedge c, a\rangle$

Case 1: $Q(a) \ge Q(a_{23})$. From the results given at the beginning of this section, it follows that the strategies ξ and η optimal in limit are the same as in the duel $(2,3), \langle a\rangle$, Case 1.

Case 2: $Q(a) \le Q(a_{23})$.

STRATEGY OF PLAYER I. If Player II has not fired before, fire at a random time a_1^{ε}, $a_1 = \rangle\langle a\rangle + c\langle$, and play optimally the resulting duel.

STRATEGY OF PLAYER II. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

We now have

$$\overset{1}{v}{}^{a}_{23} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{22}.$$

The proof is omitted.

The duel $(2,3), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \ge Q(a_{23})$. Also here the strategies optimal in limit for Players I and II are the same as in the duel $(2,3), \langle a\rangle$, Case 1.

Case 2: $Q(a) \le Q(a_{23})$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I has not fired before, fire at random at time a_1, $a_1 = \rangle\langle a\rangle + c\langle$, and play optimally the resulting duel.

Now we have

$$\overset{2}{v}{}^{a}_{23} = P(a) + Q(a)\overset{1}{v}{}^{a}_{13}.$$

The proof that in the case considered the strategies ξ and η are optimal in limit is omitted.

7. Results for the duel $(2,3)$**.** We have

$$\overset{1}{v}{}^a_{23}=\begin{cases} v_{23}=0.013757 & \text{if } Q(a)\ge Q(a_{23})\cong 0.882709,\\ -1+(1+v_{22})Q(a) & \text{if } Q(a_{23})\ge Q(a)\ge Q(a_{22})\cong 0.812085,\\ -1+2Q(a)-2Q^2(a)+(1+v_{11})Q^3(a) & \\ & \text{if } Q(a_{22})\ge Q(a)\ge Q(a_{11}),\\ -1+2Q(a)-2Q^2(a)+2Q^3(a)-2Q^4(a)+Q^5(a) & \\ & \text{if } Q(a)\le Q(a_{11});\end{cases}$$

$$v^a_{23}=\begin{cases} v_{23} & \text{if } Q(a)\ge Q(a_{23}),\\ -1+(1+v_{22})Q(a) & \text{if } Q(a_{23})\ge Q(a)\ge Q(\widehat{a}_{23})\cong 0.870730,\\ 0 & \text{if } Q(\widehat{a}_{23})\ge Q(a)\ge Q(a_{12})\cong 0.853553,\\ -Q^2(a)+(1+v_{11})Q^3(a) & \\ & \text{if } Q(a_{12})\ge Q(a)\ge Q(\widehat{a}_{12})\cong 0.730812,\\ Q^5(a)-Q^2(a) & \text{if } Q(a)\le Q(\widehat{a}_{12});\end{cases}$$

$$\overset{2}{v}{}^a_{23}=\begin{cases} v_{23} & \text{if } Q(a)\ge Q(a_{23}),\\ P^2(a) & \text{if } Q(a_{23})\ge Q(a)\ge Q(a_{12}),\\ 1-2Q(a)+(1+v_{11})Q^3(a) & \\ & \text{if } Q(a_{12})\ge Q(a)\ge Q(\check{a}_{12})\cong 0.780539,\\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+Q^5(a) & \text{if } Q(a)\le Q(\check{a}_{12}).\end{cases}$$

For other noisy duels see [3], [8], [11], [14]–[22].

References

[1] A. Cegielski, *Tactical problems involving uncertain actions*, J. Optim. Theory Appl. 49 (1986), 81–105.

[2] —, *Game of timing with uncertain number of shots*, Math. Japon. 31 (1986), 503–532.

[3] M. Fox and G. Kimeldorf, *Noisy duels*, SIAM J. Appl. Math. 17 (1969), 353–361.

[4] S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. 2, Addison-Wesley, Reading, Mass., 1959.

[5] G. Kimeldorf, *Duels: an overview*, in: *Mathematics of Conflict*, North-Holland, 1983, 55–71.

[6] K. Orłowski and T. Radzik, *Non-discrete silent duels with complete counteraction*, Optimization 16 (1985), 257–263.

[7] —, —, *Discrete silent duels with complete counteraction*, ibid., 419–429.

[8] T. Radzik, *General noisy duels*, Math. Japon. 36 (1991), 827–857.

[9] R. Restrepo, *Tactical problems involving several actions*, in: *Contributions to the Theory of Games*, Vol. III, Ann. of Math. Stud. 39, Princeton Univ. Press, 1957, 313–335.

[10] A. Styszyński, *An n-silent-vs.-noisy duel with arbitrary accuracy functions*, Zastos. Mat. 14 (1974), 205–225.

[11] Y. Teraoka, *Noisy duels with uncertain existence of the shot*, Internat. J. Game Theory 5 (1976), 239–250.

[12] Y. Teraoka, *A single bullet duel with uncertain information available to the duelists*, Bull. Math. Statist. 18 (1979), 69–83.

[13] S. Trybuła, *Solution of a silent duel with arbitrary motion and arbitrary accuracy functions*, Optimization 27 (1993), 151–172.

[14]–[19] —, *A noisy duel under arbitrary moving. I–VI*, Zastos. Mat. 20 (1990), 491–495, 497–516, 517–530; Zastos. Mat. 21 (1991), 43–61, 63–81, 83–98.

[20] —, *A noisy duel under arbitrary motion. VII*, Appl. Math. (Warsaw) 23 (1995), 37–49.

[21] —, *A noisy duel under arbitrary motion. IX*, ibid., to appear.

[22] N. N. Vorob'ev, *Foundations of the Theory of Games. Uncoalition Games*, Nauka, Moscow, 1984 (in Russian).

[23] E. B. Yanovskaya, *Duel-type games with continuous firing*, Engrg. Cybernetics 1969 (1), 15–18.

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