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COMPUTER SIMULATION OF A NONLINEAR MODEL FOR ELECTRICAL CIRCUITS WITH α-STABLE NOISE

Abstract. The aim of this paper is to apply the appropriate numerical, statistical and computer techniques to the construction of approximate solutions to nonlinear 2nd order stochastic differential equations modeling some engineering systems subject to large random external disturbances. This provides us with quantitative results on their asymptotic behavior.

1. Introduction. The stable distributions have already found applications in various fields of engineering, such as signal processing (see, e.g. Shao and Nikias (1993)) or impulsive noise modeling and communication (Mandelbrot and van Ness (1968) or Stuck and Kleiner (1974)). In such cases, and generally when noises deviate from the ideal Gaussian models, the non-gaussian statistical methods are involved. The methodology of studying models which are described by stochastic differential equations with α-stable noise is presented in Janicki and Weron (1994a). In this paper, on the basis of appropriate computer experiments, we want to investigate the differences between a 2nd order nonlinear Gaussian model of electrical circuit and its α-stable counterpart, which in fact are not so big "on average" when a stationary solution acting as a strong attractor appears in a model. So, this paper should be regarded as a complement to Weron's (1995), where a linear stochastic model was chosen as a tutorial example.

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2. A stochastic differential equation of 2nd order. It is obvious that the 2nd order ODE of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \tag{2.1}$$

can be rewritten as an autonomous system of 2 ODE's of the 1st order

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x,y). \tag{2.2}$$

Adding the additive α-stable noise to (2.1) we get analogously the system of stochastic differential equations driven by an α-stable Lévy motion, which can be expressed in the following integral form:

$$\begin{aligned} X(t) &= X_0 + \int_0^t Y(s-)\,ds, \\ Y(t) &= Y_0 + \int_0^t f(X(s-),Y(s-))\,ds + c\int_0^t dL_\alpha(s) \quad \text{for } t\in[0,\infty), \end{aligned} \tag{2.3}$$

where $X(0) = X_0$ and $Y(0) = Y_0$ are given α-stable or discrete random variables.

This system (called in the sequel *SDE of the* 2*nd order*) includes as a special case the equation which will serve as model for an electrical circuit discussed below.

3. Simulation of SDE's of the 2nd order. Now we describe briefly a method of approximate computer simulation of a bivariate stochastic process $\{(X(t),Y(t)) : t\in[0,T]\}$ with independent increments. The method is based on the construction of a discrete time process of the form $\{(X^\tau_{t_i}, Y^\tau_{t_i})\}_{i=0}^I$. It is enough to define the set $\{t_i = i\tau : i = 0,1,\ldots,I\}$, $\tau = T/I$, describing a fixed mesh on the interval $[0,T]$, and a sequence of i.i.d. random variables $\Delta L^\tau_{\alpha,i}$ playing the role of random α-stable measures of the intervals $[t_{i-1},t_i)$, i.e. α-stable random variables defined by

$$\Delta L^\tau_{\alpha,i} = L_\alpha([t_{i-1},t_i)) \sim S_\alpha(\tau^{1/\alpha},0,0); \tag{3.1}$$

to choose $X^\tau_0 = X_0 \sim S_\alpha(\sigma,0,\mu)$ or $X^\tau_0 = x_0$, $Y^\tau_0 = Y_0 \sim S_\alpha(\sigma,0,\mu)$ or $Y^\tau_0 = y_0$, and to compute

$$\begin{aligned} X^\tau_{t_i} &= X^\tau_{t_{i-1}} + Y^\tau_{t_{i-1}}\tau, \\ Y^\tau_{t_i} &= Y^\tau_{t_{i-1}} + f(X^\tau_{t_{i-1}}, Y^\tau_{t_{i-1}})\tau + c\Delta L^\tau_{\alpha,i}, \end{aligned} \tag{3.2}$$

for $i = 1,\ldots,I$.

The theorem on convergence of this method, based on some properties of measures on the space $\mathbb{D}([0,T],\mathbb{R}^2)$ of so-called cadlag trajectories, and justifying the method can be found in Janicki, Michna and Weron (1994).

In computer calculations each of the random variables $X^\tau_{t_i}$ and $Y^\tau_{t_i}$ defined by (3.2) is represented by its N independent realizations, i.e. random samples $\{X^\tau_i(n)\}_{n=1}^N$ and $\{Y^\tau_i(n)\}_{n=1}^N$. So, let us fix $N \in \mathbb{N}$ large enough. The algorithm consists in the following:

1. Simulate random samples $\{X^\tau_0(n)\}_{n=1}^N$ and $\{Y^\tau_0(n)\}_{n=1}^N$ for X^τ_0 and Y^τ_0.

2. For $i = 1, \ldots, I$ simulate a random sample $\{\Delta L^\tau_{\alpha,i}(n)\}_{n=1}^N$ for an α-stable random variable $\Delta L^\tau_{\alpha,i} \sim S_\alpha(\tau^{1/\alpha}, 0, 0)$.

3. For $i = 1, \ldots, I$, in accordance with (3.2), compute the random samples

$$
\begin{aligned}
X^\tau_i(n) &= X^\tau_{i-1}(n) + Y^\tau_i(n)\tau, \\
Y^\tau_i(n) &= Y^\tau_{i-1}(n) + f(X^\tau_{i-1}(n), Y^\tau_i(n))\tau + c\Delta L^\tau_{\alpha,i}(n),
\end{aligned}
\tag{3.3}
$$

$n = 1, \ldots, N$.

4. Construct kernel density estimators $f_{X,i} = f^{I,N}_{X,i} = f^{I,N}_{X,i}(x)$ and $f_{Y,i} = f^{I,N}_{Y,i} = f^{I,N}_{Y,i}(x)$ of the densities of $X(t_i)$ and $Y(t_i)$, using for example the optimal version of the Rosenblatt–Parzen method.

5. Construct two-dimensional kernel density estimators of the joint densities for the vectors $(X(t_i), Y(t_i))$, using the corresponding version of the Rosenblatt–Parzen method or, at least, scatterplots of $(X(t_i), Y(t_i))$ for some t_i.

Observe that we have produced N finite time series of the form $\{X^\tau_i(n)\}_{i=0}^I$ for $n = 1, \ldots, N$ and another N finite time series of the form $\{Y^\tau_i(n)\}_{i=0}^I$ for $n = 1, \ldots, N$. We regard them as "good" approximations of the trajectories of the coordinate processes $\{X(t) : t \in [0, T]\}$ and $\{Y(t) : t \in [0, T]\}$.

4. An example of electrical circuit. Here we present an example of a nonlinear stochastic differential equation of the 2nd order, involving stochastic integrals with stationary α-stable increments, i.e., an example of (2.3).

Let us start with the observation that the closed electrical contour with nonlinear elements presented in the figure below can be described by the following 2nd order ODE:

$$
L\frac{d^2x}{dt^2}(t) + R\frac{dx}{dt}(t) + \frac{1}{C}x(t) + g\left(x(t), \frac{dx}{dt}(t)\right) = 0.
\tag{4.1}
$$

Here $x, i = \frac{dx}{dt}$, R, L, C denote, respectively, electrical charge, electric force, resistance, induction, and capacity.

We have chosen the function g to be defined as $g(x, y) = \text{Const} * (x^3 + y^3)$, which should be equal to 0 at $(0, 0)$ and approach this value at least as fast as

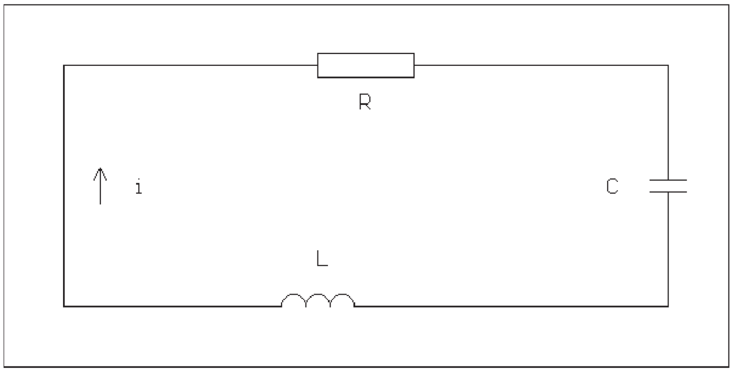

Fig. 4.1. Electric circuit corresponding to equation (4.1)

$x^2 + y^2$ in order to give rise to an asymptotically stable stationary solution to (4.1) at this point. Fixing values of all constants we have chosen the following example:

$$\frac{d^2x}{dt^2}(t) + \frac{1}{2}\frac{dx}{dt}(t) + x(t) + \frac{1}{10}\left((x(t))^3 + \left(\frac{dx}{dt}(t)\right)^3\right) = 0, \tag{4.2}$$

and, finally, we get the 2nd order SDE

$$\begin{aligned} X(t) &= X_0 + \int_0^t Y(s-)\,ds, \\ Y(t) &= Y_0 + \int_0^t \left(-X(s-) - \frac{1}{2}Y(s-) - \frac{1}{10}(X^3(s-) + Y^3(s-))\right) ds, \\ &\quad + \frac{1}{10}\int_0^t dL_\alpha(s), \end{aligned} \tag{4.3}$$

for $t \in [0, T]$. The following starting values of the solution were chosen for computer experiments: $X(0) \sim S_\alpha(1, 0, 5)$, $Y(0) = 5$ a.s.

Of course, $Y(t) = i(t)$, so in what follows we focus our attention on this coordinate of the solution.

5. Results of computer experiments. Figures 5.1–5.11, obtained from computer experiments concerning the 2nd order SDE (4.3), visualize some significant features of the solution $\{(X(t), Y(t)) : t \geq \infty\}$.

Figure 5.1 is intended to show the existence of an asymptotically stable stationary solution $(x(t) = 0,\ y(t) = 0)$ to the deterministic equation (4.2), which acts as a strong attractor in the phase space. Figures 5.2–5.3 illustrate the appearance of a stochastic attractor in (4.3). The solution $\{(X(t), Y(t)) : t \geq 0\}$ converges at infinity (rather quickly) to a bivariate random vector

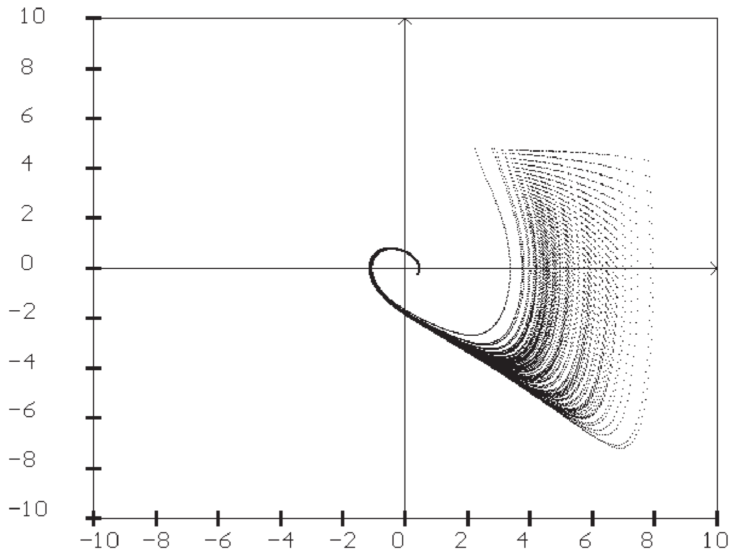

Fig. 5.1. 100 trajectories of the solution to the deterministic electric circuit equation with the random vector $(S_2(1,0,5),5)$ as starting value of the solution, for $t \in [0,7]$

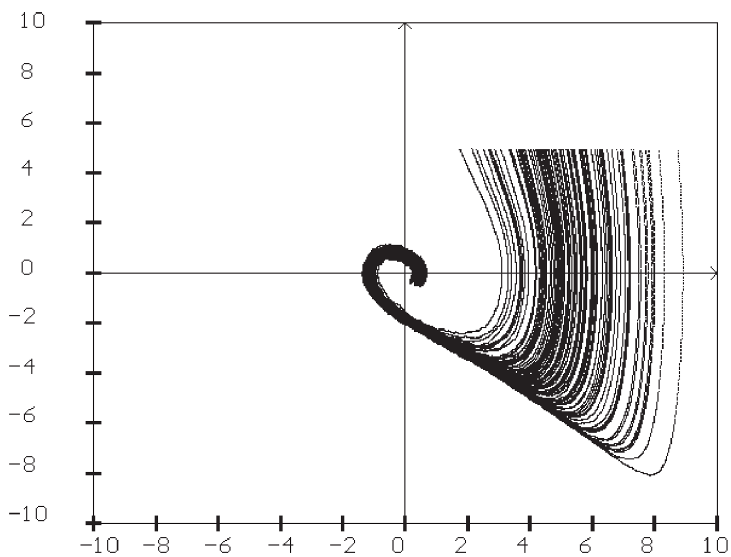

Fig. 5.2. 100 trajectories of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 2.0$, with the random vector $(S_2(1,0,5),5)$ as starting value of the solution, for $t \in [0,7]$

$(X(\infty), Y(\infty))$. The scatterplots visible in Figs. 5.10–5.11 are intended to

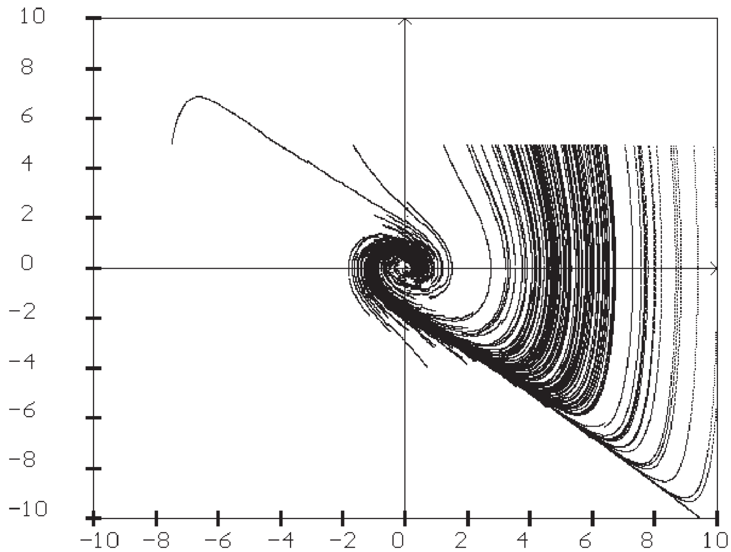

Fig. 5.3. 100 trajectories of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 1.2$, with the random vector $(S_{1.2}(1, 0, 5), 5)$ as starting value of the solution, for $t \in [0, 7]$

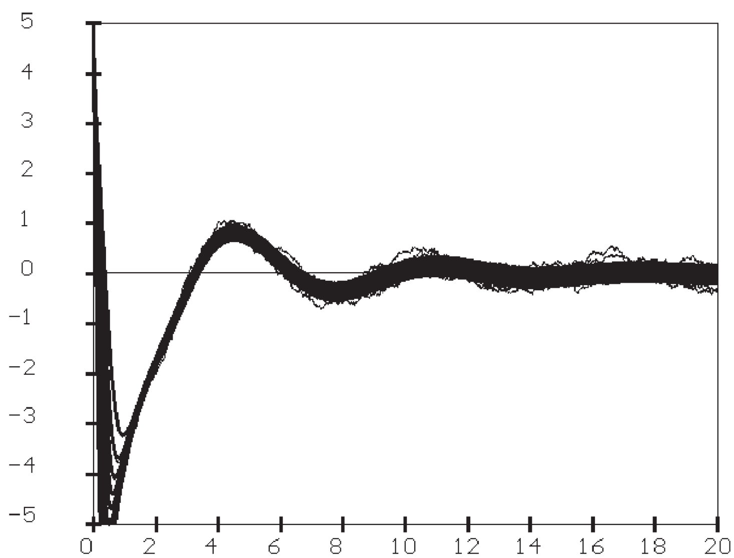

Fig. 5.4. Visualization of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 2.0$, with the random vector $(S_2(1, 0, 5), 5)$ as starting value of the solution

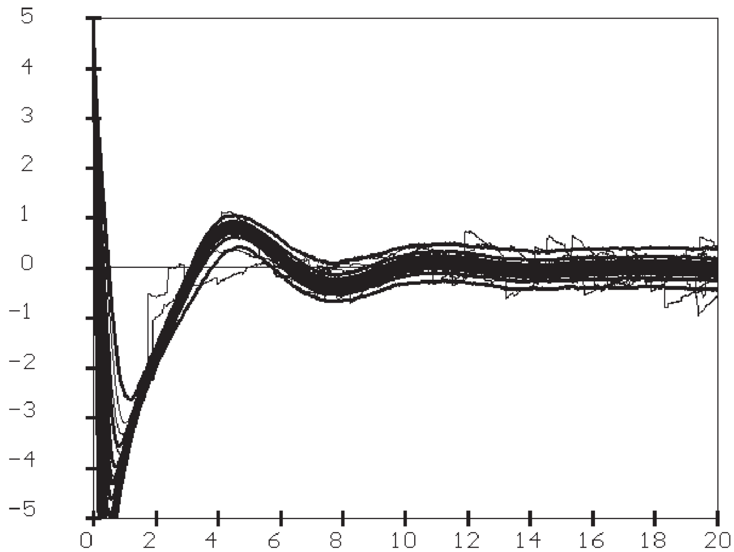

Fig. 5.5. Visualization of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 1.2$, with the random vector $(S_{1.2}(1, 0, 5), 5)$ as starting value of the solution

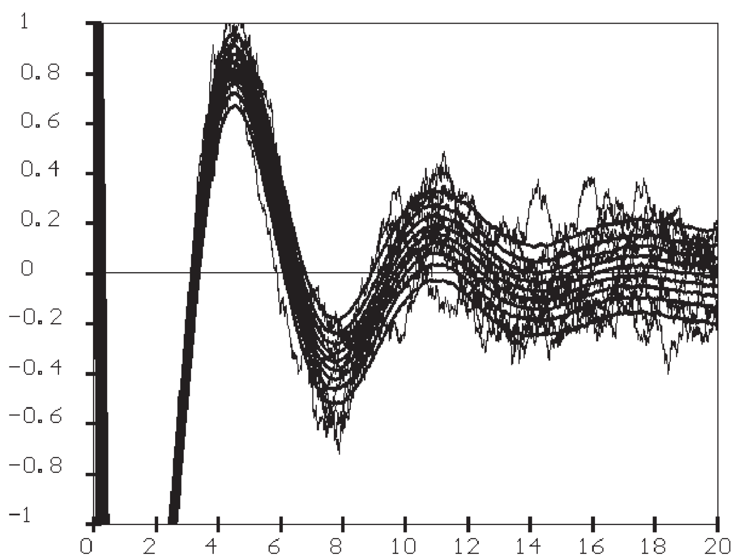

Fig. 5.6. Visualization of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 2.0$, with the random vector $(S_2(1, 0, 5), 5)$ as starting value of the solution

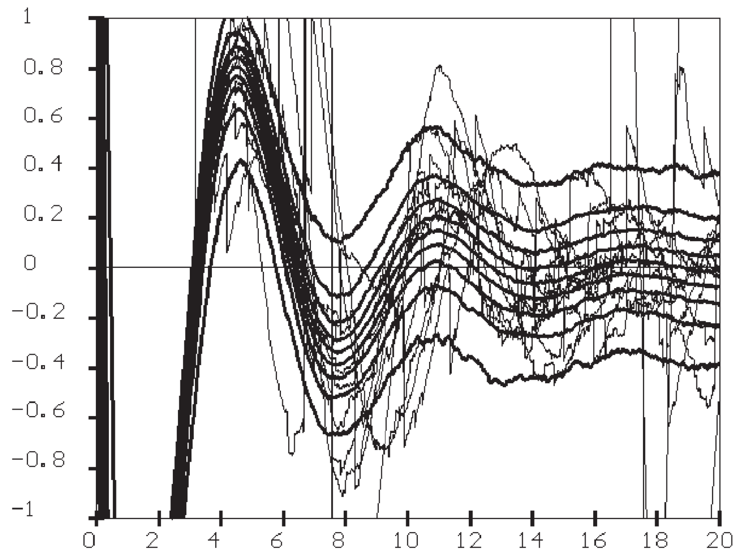

Fig. 5.7. Visualization of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 1.2$, with the random vector $(S_{1.2}(1, 0, 5), 5)$ as starting value of the solution

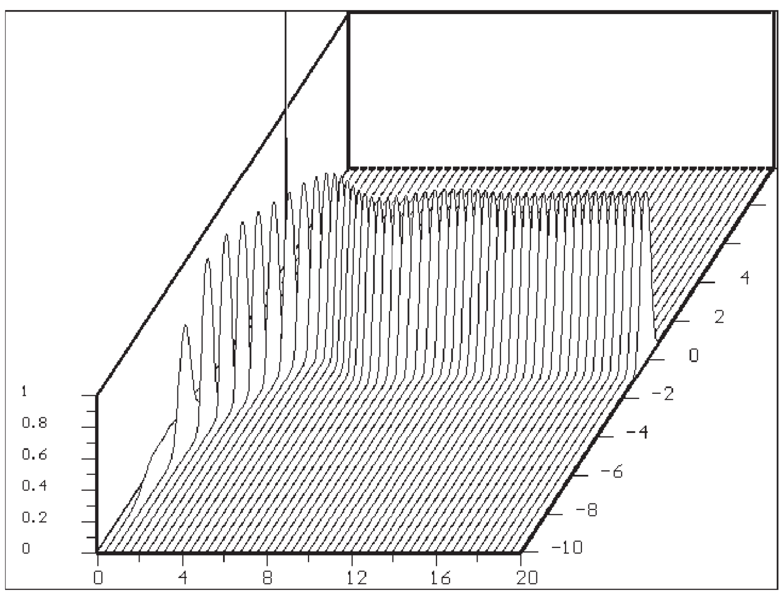

Fig. 5.8. Densities evolution of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 2.0$, with the random vector $(S_2(1, 0, 5), 5)$ as starting value of the solution

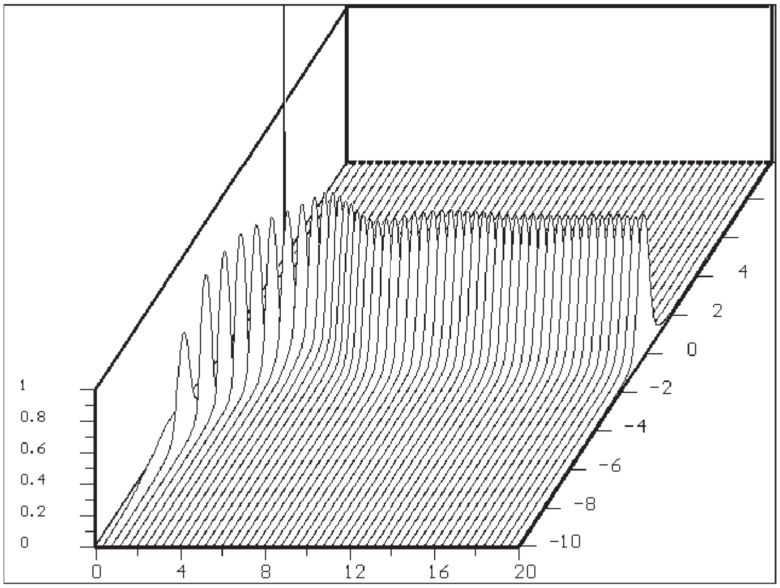

Fig. 5.9. Densities evolution of the second coordinate $\{Y(t) : t \in [0, 20]\}$ of the solution to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 1.2$, with the random vector $(S_{1.2}(1, 0, 5), 5)$ as starting value of the solution

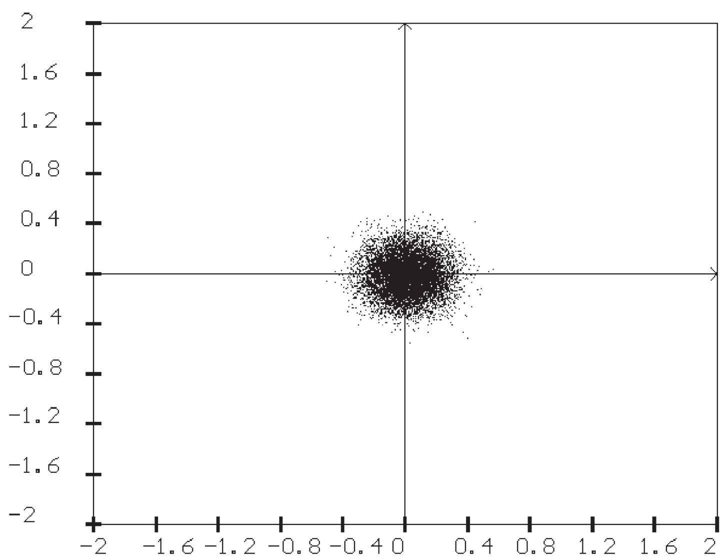

Fig. 5.10. Scatterplot of $(X(20), Y(20))$ of the solution $\{(X(t), Y(t))\}$ to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 2.0$, with the random vector $(S_2(1, 0, 5), 5)$ as starting value of the solution

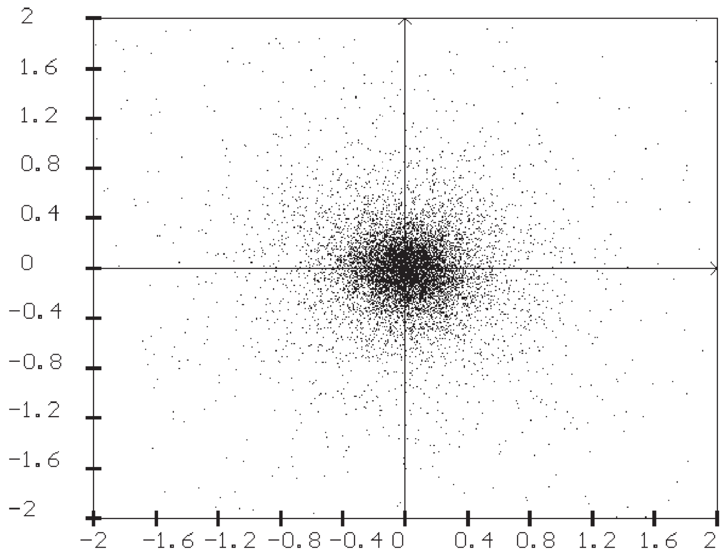

Fig. 5.11. Scatterplot of $(X(20), Y(20))$ of the solution $\{(X(t), Y(t))\}$ to the electrical circuit equation driven by a stable Lévy motion (4.3), for $\alpha = 1.2$, with the random vector $(S_{1.2}(1, 0, 5), 5)$ as starting value of the solution

give an idea of how their joint densities look like, and how they depend on the parameter α. The figures present 10000 final positions of trajectories of the solution at $t = 20$. (For $t > 20$ the results are very similar.)

Figures 5.4–5.9 represent the coordinate $Y(t) = i(t)$ of the solution to (4.3). Applying the methods of visualization of univariate stochastic processes (evolution of quantiles and evolution of densities) described in Weron (1995) we tried to make it evident that for any value of the parameter α the process $\{Y(t)\}$ is asymptotically stationary, in spite of large differences in behavior of trajectories of the process for different values of α (compare Figs. 5.6 and 5.5). Starting from $t = 20$ the process becomes indistinguishable from a stationary process, as far as the accuracy of our approximate method is concerned.

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