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COMPUTER-AIDED MODELING AND SIMULATION OF ELECTRICAL CIRCUITS WITH α-STABLE NOISE

Abstract. The aim of this paper is to demonstrate how the appropriate numerical, statistical and computer techniques can be successfully applied to the construction of approximate solutions of stochastic differential equations modeling some engineering systems subject to large disturbances. In particular, the evolution in time of densities of stochastic processes solving such problems is discussed.

1. Introduction. The past few years have witnessed an explosive growth in interest in physical and engineering systems that could be studied using stochastic and chaotic methods; see Berliner (1992), Chatterjee and Yilmaz (1992), and Shao and Nikias (1993). "Stochastic" and "chaotic" refer to nature's two paths to unpredictability, or uncertainty. To scientists and engineers the surprise was that chaos (making a very small change in the universe can lead to a very large change at some later time) is unrelated to randomness. Things are unpredictable if you look at the individual events; however, one can say a lot about averaged-out quantities. This is where the stochastic stuff comes in. Stochastic processes are recognized to play an important role in a wide range of problems encountered in mathematics, physics and engineering. Recent developments show that in many practical applications leading to appropriate stochastic models a particular class of Lévy α-stable processes is involved. While the attempt at mathematical understanding of these processes leads to severe analytical difficulties, there exist very useful approximate numerical and statistical techniques (see Janicki and Weron (1994a)). Also non-Gaussian statistical methods in impulsive

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noise modeling are important when noises deviate from the ideal Gaussian model. Stable distributions are among the most important non-Gaussian models. They share defining characteristics with the Gaussian distribution, such as the stability property and central limit theorems, and include in fact the Gaussian distributions as a special case. To help the interested reader better understand the stable models and necessary methodologies we discuss here a tutorial example of the resistive-inductive electrical circuit subject to large external disturbances.

Computer methods of constructing stochastic processes involve at least two kinds of discretization techniques: the discretization of the continuous time problem involving discrete time parameter and the approximate representation of random variates with the aid of artificially produced finite statistical samples. So, we are interested in statistical methods of data analysis such as quantiles or kernel probability densities estimates, etc. Applying computer graphics, we attempt to explain to what extent they can provide results good enough to be applied to solve approximately quite complicated problems involving α-stable random variates (the discussion of the 2nd order nonlinear stochastic electric circuit model is presented in Janicki (1995)).

2. Computer generation of α-stable distributions. The most common and convenient way to introduce α-stable random variables is to define their *characteristic function*, which involves four parameters: α — the index of stability, β — the skewness parameter, σ — the scale parameter and μ — the shift. This function is given by

$$\log\phi(\theta) = \begin{cases} -\sigma^\alpha|\theta|^\alpha\{1 - i\beta(\operatorname{sgn}\theta)\tan(\alpha\pi/2)\} + i\mu\theta & \text{if } \alpha \neq 1, \\ -\sigma|\theta|\{1 + i\beta\frac{\pi}{2}(\operatorname{sgn}\theta)\ln|\theta|\} + i\mu\theta & \text{if } \alpha = 1, \end{cases} \tag{2.1}$$

where $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma \in \mathbb{R}_+$, $\mu \in \mathbb{R}$.

For a random variable X distributed according to the above described rule we will use the notation $X \sim S_\alpha(\sigma,\beta,\mu)$. Notice that $S_2(\sigma,0,\mu)$ and $S_1(\sigma,0,\mu)$ give the Gaussian distribution $\mathcal{N}(\mu,2\sigma^2)$ and the Cauchy distribution, respectively.

When we start working with α-stable distributions, the main problem is that, except for a few values of four parameters describing the characteristic function, their density functions are not known explicitly. The best method of computer simulation of a very important class of symmetric α-stable random variables $X \sim S_\alpha(1,0,0)$ for $\alpha \in (0,2]$ consists in the following:

- generate a random variable V uniformly distributed on $(-\pi/2,\pi/2)$ and an exponential random variable W with mean 1;

• compute

$$X = \frac{\sin(\alpha V)}{\{\cos(V)\}^{1/\alpha}} \times \left\{ \frac{\cos(V - \alpha V)}{W} \right\}^{(1-\alpha)/\alpha}. \tag{2.2}$$

The formula (2.2) is generalized below by (2.3).

The algorithm providing skewed stable random variables $Y \sim S_\alpha(1, \beta, 0)$ with $\alpha \in (0,1) \cup (1,2)$ and $\beta \in [-1,1]$ consists in the following:

• generate a random variable V uniformly distributed on $(-\pi/2, \pi/2)$ and an exponential random variable W with mean 1;

• compute

$$Y = D_{\alpha,\beta} \times \frac{\sin(\alpha(V + C_{\alpha\beta}))}{\{\cos(V)\}^{1/\alpha}} \times \left\{ \frac{\cos(V - \alpha(V + C_{\alpha,\beta}))}{W} \right\}^{(1-\alpha)/\alpha} \tag{2.3}$$

and

$$C_{\alpha,\beta} = \frac{\arctan(\beta \tan(\pi\alpha/2))}{\alpha},$$
$$D_{\alpha,\beta} = [\cos(\arctan(\beta \tan(\pi\alpha/2)))]^{-1/\alpha}.$$

(In the case of $\alpha = 2$ or $\alpha = 1$ the only reasonable choice of β is $\beta = 0$, so (2.2) is applicable. Notice also that if $X \sim S_\alpha(1, \beta, 0)$, then $\sigma X + \mu \sim S_\alpha(\sigma, \beta, \mu)$.)

Generalizing the result of Kanter (1975) or slightly modifying the algorithm of Chambers, Mallows and Stuck (1976), one can see that Y belongs to the class of $S_\alpha(1, \beta, 0)$ random variables. For more details see Janicki and Weron (1994a).

We regard the method defined by (2.2) and (2.3) as a good technique of computer simulation of α-stable random variables, stochastic measures and processes of different kinds. Of course, it has its own limitations in applicability as any computer technique has.

3. Simulation of stable stochastic processes. Now we describe a rather general technique of approximate computer simulation of univariate α-stable stochastic processes $\{X(t) : t \in [0,T]\}$ with independent increments, which is based on the construction of a discrete time process of the form $\{X^\tau_{t_i}\}_{i=0}^I$, defined by the formula

$$X^\tau_{t_i} = X^\tau_{t_{i-1}} + \mathcal{F}(t_{i-1}, X^\tau_{t_{i-1}}) + Y^\tau_i, \tag{3.1}$$

with a given X^τ_0, and where Y^τ_i's form a sequence of i.i.d. α-stable random variables.

In computer calculations each random variable $X^\tau_{t_i}$ defined by (3.1) is represented by its N independent realizations, i.e. a random sample $\{X^\tau_i(n)\}_{n=1}^N$. So, let us fix $N \in \mathcal{N}$ large enough. The algorithm consists in the following:

1. Simulate a random sample $\{X_0^\tau(n)\}_{n=1}^N$ for X_0^τ.

2. For $i = 1, \dots, I$ simulate a random sample $\{Y_i^\tau(n)\}_{n=1}^N$ for an α-stable random variable $Y_i^\tau \sim S_\alpha(\tau^{1/\alpha}, 0, 0)$, with appropriately chosen τ.

3. For $i = 1, \dots, I$, in accordance with (3.1), compute the random sample $X_i^\tau(n) = X_{i-1}^\tau(n) + \mathcal{F}(t_{i-1}, X_{i-1}^\tau(n)) + Y_i^\tau(n)$, $n = 1, \dots, N$.

4. Construct kernel density estimators $f_i = f_i^{I,N} = f_i^{I,N}(x)$ of the densities of $X(t_i)$, using for example the optimal version of the Rosenblatt–Parzen method.

Observe that we have produced N finite time series of the form $\{X_i^\tau(n)\}_{i=0}^I$ for $n = 1, \dots, N$. We regard them as "good" approximations of the trajectories of the process $\{X(t) : t \in [0, T]\}$.

In particular, the above algorithm can be successfully applied to the construction of approximate solutions to the following linear stochastic differential equation driven by an α-stable Lévy motion:

$$X(t) = X_0 + \int_0^t (a(s) + b(s)X(s-))\,ds + \int_0^t c(s)\,dL_\alpha(s) \quad \text{for } t \in [0, \infty), \tag{3.2}$$

with $X(0) = X_0$ a given α-stable or discrete random variable.

Let us notice that this linear stochastic equation is of independent interest because, as is easily seen, the general solution belongs to the class of α-stable processes. It may be expressed in the form

$$X(t) = \Phi(t, 0)X_0 + \int_0^t \Phi(t, s)a(s)\,ds + \int_0^t \Phi(t, s)c(s)\,dL_\alpha(s),$$

where $\Phi(t, s) = \exp\{\int_s^t b(u)\,du\}$.

This explains why outliers or heavy tails appear in the constructed approximate solutions $\{X_i^\tau(n)\}_{i=0}^I$, $n = 1, \dots, N$, to (3.2), which can be directly derived as a special case of (3.1). It is enough to define the set $\{t_i = i\tau : i = 0, 1, \dots, I\}$, $\tau = T/I$, describing a fixed mesh on the interval $[0, T]$, and a sequence of i.i.d. random variables $\Delta L_{\alpha,i}^\tau$ playing the role of the random α-stable measures of the intervals $[t_{i-1}, t_i)$, i.e. α-stable random variables defined by

$$\Delta L_{\alpha,i}^\tau = L_\alpha([t_{i-1}, t_i)) \sim S_\alpha(\tau^{1/\alpha}, 0, 0); \tag{3.3}$$

and to choose $X_0^\tau = X_0 \sim S_\alpha(\sigma, 0, \mu)$, computing

$$X_{t_i}^\tau = X_{t_{i-1}}^\tau + (a(t_{i-1}) + b(t_{i-1})X_{t_{i-1}}^\tau)\tau + c(t_{i-1})\Delta L_{\alpha,i}^\tau, \tag{3.4}$$

for $i = 1, \dots, I$.

An appropriate convergence result justifying the method can be found in Janicki, Michna and Weron (1994).

4. Visualization of univariate stochastic processes. In order to obtain a graphical computer presentation of the discrete time stochastic process of the form (3.1), and in particular to get some qualitative and quantitative information on the electrical circuit problem discussed below, we propose two different approaches. The first is based on the following:

1. Fix a rectangle $[0,T] \times [c,d]$ that should include the trajectories of $\{X(t)\}$.

2. For each $n = 1, \ldots, n_{\max}$ (with fixed $n_{\max} \ll N$) draw the line segments determined by the points $(t_{i-1}, X^\tau_{i-1}(n))$ and $(t_i, X^\tau_i(n))$ for $i = 1, \ldots, I$, constructing $n_{\max}$ approximate trajectories of the process X (thin lines in Figs. 4.1–4.3, where $N = 2000$, $I = 1000$, $n_{\max} = 10$).

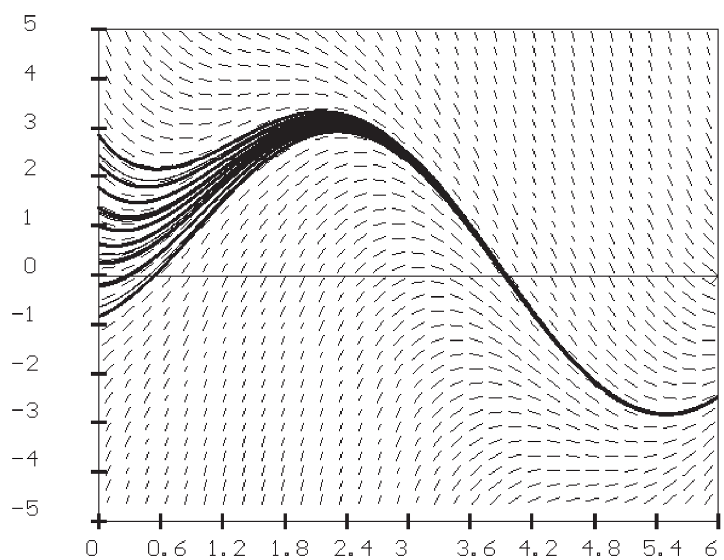

Fig. 4.1. Deterministic electric circuit equation with the random variable $S_2(1,0,1)$ as a starting value of the solution

3. Fixing values of a parameter $p_j \in (0,1)$, $j = 1, \ldots, J$, it is possible do derive from each statistical sample $\{X^\tau_i(n)\}_{n=1}^N$ with fixed $i \in \{0, 1, \ldots, I\}$ estimators of corresponding quantiles $q^{i,j} = F_i^{-1}(p_j)$, where $F_i = F_i(x)$ denotes the unknown density distribution function of the random variable $X^\tau_{t_i}$ represented by the statistical sample $\{X^\tau_i(n)\}_{n=1}^N$. In this way we obtain an approximation of the so-called *quantile lines* (thick lines in Figs. 4.1–4.3,

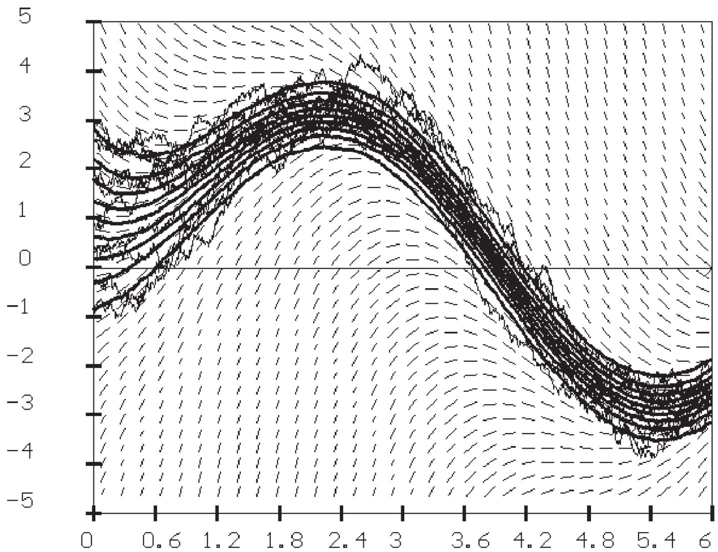

Fig. 4.2. Computer solution to the resistive-inductive electrical circuit equation driven by Lévy motion for $\alpha = 2.0$

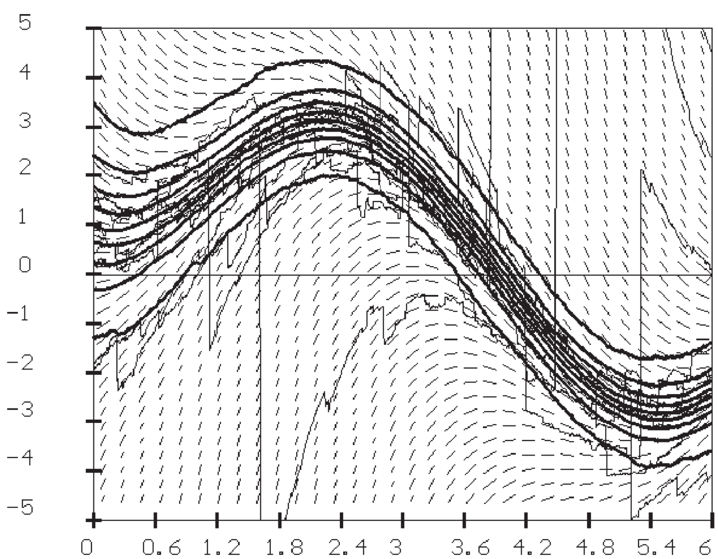

Fig. 4.3. Computer solution to the resistive-inductive electrical circuit equation driven by Lévy motion for $\alpha = 1.2$

where $J = 9$, $p_j \in \{0.1, 0.2, \ldots, 0.9\}$), i.e., the curves $q_j = q_j(t)$ defined by the condition $\mathbb{P}\{X(t) \geq q_j(t)\} = p_j$.

The second idea consists in the construction of kernel density estimators for a finite sequence of random variables $\{X_{t_i}^{\tau}\}$ approximating unknown values $\{X_{t_i}\}$ of the exact solution to (3.2), and represented by artificially produced statistical samples $\{X_i^{\tau}(n)\}_{n=1}^{N}$, for a finite set of equidistant i.

So, let us recall briefly formulas describing kernel density estimators. Suppose that we are interested in a sequence $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ of i.i.d. random variables distributed according to the law described by an unknown density function and we are given a random sample (a sequence of observed values or realizations) $\{\xi^{(1)}, \ldots, \xi^{(n)}\}$. The well known Rosenblatt–Parzen method of construction of a kernel density estimator $f_n = f_n(x)$ is described by the formula

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b_n} K\left(\frac{x - \xi^{(i)}}{b_n}\right),$$

for a univariate density function $f = f(x)$, and where the kernel function $K = K(u)$ should be nonnegative on $\mathbb{R}$ and such that $\int_{\mathbb{R}} K(u)\, du = 1$.

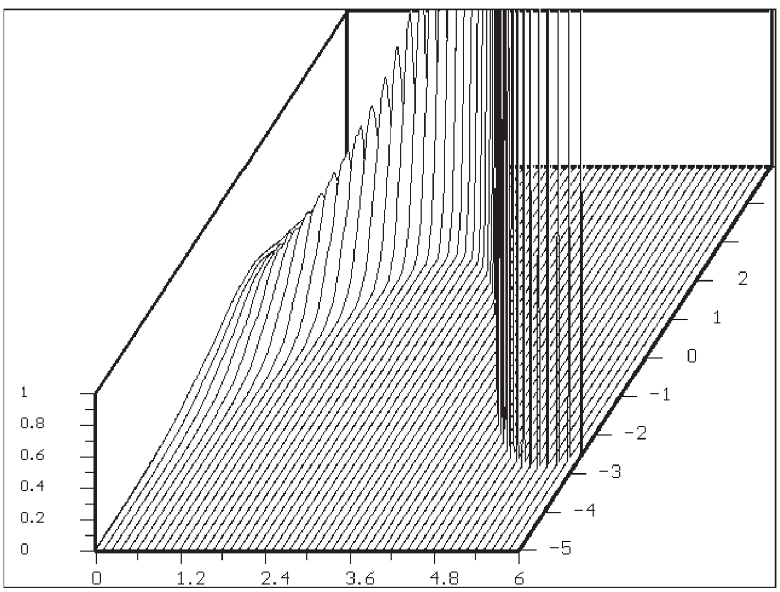

Fig. 4.4. Evolution of densities for deterministic electric circuit equation with the random variable $S_2(1, 0, 1)$ as a starting value of the solution

The crucial problem of optimal selection of the bandwidth parameter b_n was discussed by several authors (see, e.g., Härdle, Hall and Marron (1989) and the references therein). An interesting iterative *self-learning algorithm* leading to the optimal value of b_n is discussed in Gajek and Lenic (1993); however, it seems a little bit too costly in our setting, when statistical samples should be rather large, because of appearance of significant outliers.

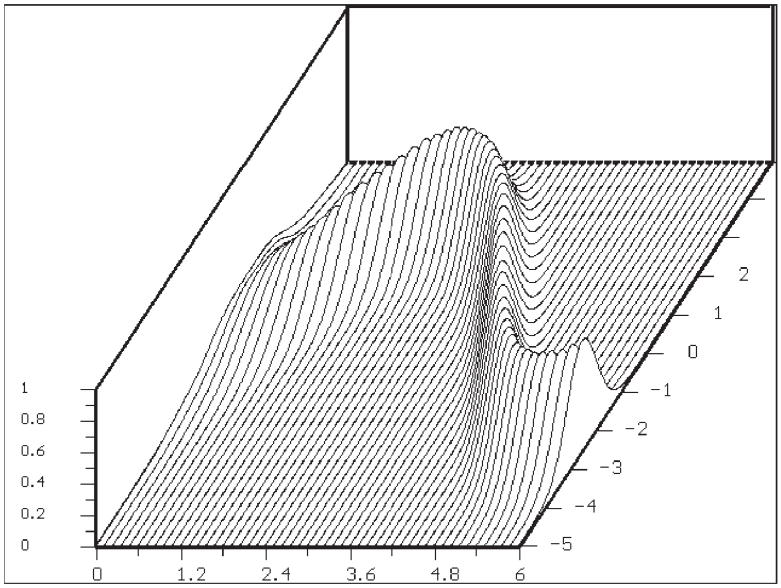

Fig. 4.5. Evolution of densities for resistive-inductive electrical circuit driven by Lévy motion for $\alpha = 2.0$

In computer calculations which provided us with Figs. 4.4–4.6 satisfactory values of this parameter were established experimentally.

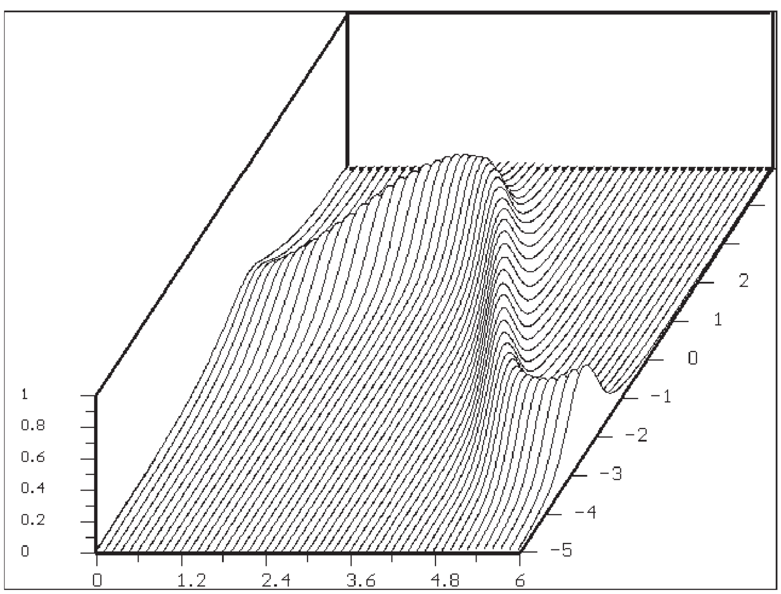

Fig. 4.6. Evolution of densities for resistive-inductive electrical circuit driven by Lévy motion for $\alpha = 1.2$

5. Resistive-inductive electrical circuit. The stable distributions have already found applications in signal processing and communications. For example, Mandelbrot and van Ness (1968) used Gaussian and stable fractional stochastic processes to describe long-range dependence arising in engineering, economics and hydrology. It was also used by Berger and Mandelbrot (1963) to describe the patterns of error clustering in telephone circuits. However, the most important application of the stable distributions is in the area of impulsive noise modeling. It has recently been shown that a general class of man-made and natural impulsive noise is indeed stable under broad conditions, e.g. Stuck and Kleiner (1974) empirically found that the noise over certain telephone lines can be best described by stable laws with the index of stability α close to 2.

Here we present an example of a linear stochastic differential equation involving stochastic integrals with stationary α-stable increments, which has a well known physical interpretation in the deterministic case when the random external noise is absent. This tutorial example allows us to emphasize the role of the α-stable random disturbances and to demonstrate how the solution depends on the parameter α.

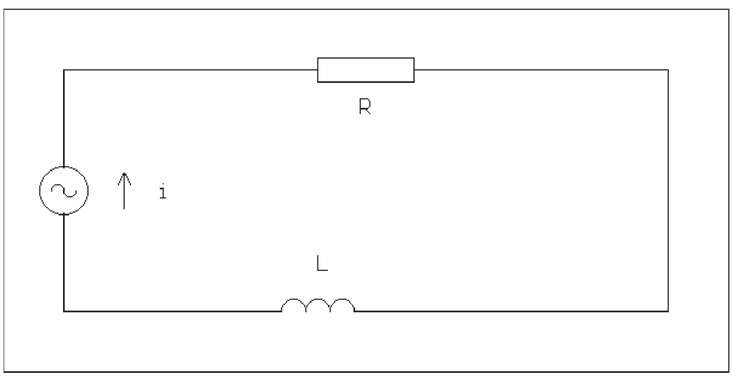

Fig 5.1. Deterministic electric circuit corresponding to equation (5.2)

The deterministic part of the stochastic differential equation

$$dX(t) = (4\sin(t) - X(t))\ dt + \frac{1}{2}dL_\alpha(t) \tag{5.1}$$

can be interpreted as a particular case of the ordinary differential equation

$$\frac{di}{ds} + \frac{R}{L}i = \frac{E}{L}\sin(\gamma s), \tag{5.2}$$

which describes the resistive-inductive electrical circuit, where i, R, L, E and γ denote, respectively, electric force, resistance, induction, electric power and pulsation. (Similar examples can be found in Gardiner (1983).) In order to obtain a realistic model it is enough to choose, for example, $R = 2.5[\mathrm{k}\Omega]$, $L = 0.005[\mathrm{H}]$, $E = 10[\mathrm{V}]$, $\gamma = 500[1/\mathrm{s}]$ and to rescale real time s using the relation $t = \gamma s$.

The simplest Euler type discretization of the equation (5.1) yields a system of the form (3.1). The results of computer simulation and visualization described above for two different values of the parameter $\alpha \in \{2.0, 1.2\}$ are included in Figs. 4.1–4.6. They also contain a field of directions corresponding to the deterministic part of (5.1), i.e., the equation

$$\frac{dx}{dt}(t) = -x(t) + 4\sin(t).$$

This helps us to figure out how the drift acts "against" the diffusion as t tends to infinity.

The discussion of the computer experiments concerning the 2nd order nonlinear stochastic electric circuit model and based on a similar approach is presented in Janicki (1995).

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