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ON MONOTONE DEPENDENCE FUNCTIONS OF THE QUANTILE TYPE

Abstract. We introduce the concept of monotone dependence function of bivariate distributions without moment conditions. Our concept gives, among other things, a characterization of independent and positively (negatively) quadrant dependent random variables.

1. Introduction. The concept of monotone dependence function for two random variables having continuous distributions has been introduced in [4]. An extension of that concept to a larger class of continuous and discrete distributions has been given in [3]. The properties of that quantity and up-dated references on function-valued parameters of dependence are summarized in the book [5]. The existence of expectation is an essential assumption in those papers. We propose a concept of monotone dependence function for two random variables without any assumptions on moments of the variables (cf. [6], [9], [10]). These new measures of dependence will be called the quantile monotone dependence functions. First we need to recall the definition of the monotone dependence function $\mu_{X,Y}(p)$, $p \in (0,1)$, of a random variable X on a random variable Y (cf. [3] and [4]).

For (X,Y) with continuous marginals the monotone dependence functions $\mu^{(k)}_{X,Y}(p)$, $k = 1,2$, are defined by the following equivalent formulae: for any $p \in (0,1)$,

$$(1.1)\quad \mu^{(1)}_{X,Y}(p) = \begin{cases} \dfrac{E(X \mid Y > y_p) - EX}{E(X \mid X > x_p) - EX} & \text{if } E(X \mid Y > y_p) - EX \ge 0, \\ \dfrac{E(X \mid Y > y_p) - EX}{EX - E(X \mid X < x_{1-p})} & \text{otherwise;} \end{cases}$$

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$$(1.2)\qquad \mu^{(2)}_{X,Y}(p)=\begin{cases}\dfrac{EX-E(X\mid Y<y_p)}{EX-E(X\mid X<x_p)} & \text{if } EX-E(X\mid Y<y_p)\ge 0,\\[2ex] \dfrac{EX-E(X\mid Y<y_p)}{E(X\mid Y>y_p)-EX\ge 0} & \text{otherwise,}\end{cases}$$

where y_p denotes the pth quantile of Y.

For (X,Y) with any nondegenerate distribution functions F_X, F_Y the following monotone dependence functions are used:

$$(1.3)\qquad \mu^{(1)}_{X,Y}(p)$$
$$=\begin{cases}\dfrac{EXI[Y>y_p]+(1-p-P[Y>y_p])E(X\mid Y=y_p)-(1-p)EX}{EXI[X>x_p]+(1-p-P[X>x_p])x_p-(1-p)EX}\\ \hfill \text{if } \alpha^{(1)}_{X,Y}(p)\ge 0,\\ \dfrac{EXI[Y>y_p]+(1-p-P[Y>y_p])E(X\mid Y=y_p)-(1-p)EX}{(1-p)EX-EXI[X<x_{1-p}]-(1-p-P[X<x_{1-p}])x_{1-p}}\\ \hfill \text{if } \alpha^{(1)}_{X,Y}(p)<0;\end{cases}$$

$$(1.4)\qquad \mu^{(2)}_{X,Y}(p)$$
$$=\begin{cases}\dfrac{pEX-EXI[Y<y_p]-(p-P[Y<y_p])E(X\mid Y=y_p)}{pEX-EXI[X<x_p]-(p-P[X<x_p])x_p}\\ \hfill \text{if } \alpha^{(2)}_{X,Y}(p)\ge 0,\\ \dfrac{pEX-EXI[Y<y_p]-(p-P[Y<y_p])E(X\mid Y=y_p)}{EXI[X>x_{1-p}]+(p-P[X>x_{1-p}])x_{1-p}-pEX}\\ \hfill \text{if } \alpha^{(2)}_{X,Y}(p)<0,\end{cases}$$

where $I[\cdot]$ denotes the indicator function and

$$\alpha^{(1)}_{X,Y}(p):=EXI[Y>y_p]$$
$$\qquad +(1-p-P[Y>y_p])E(X\mid Y=y_p)-(1-p)EX,$$
$$\alpha^{(2)}_{X,Y}(p):=pEX-EXI[Y<y_p]-(p-P[Y<y_p])E(X\mid Y=y_p).$$

2. Notations. For any $p\in(0,1)$, y_p stands for a pth quantile of Y, i.e. y_p is any real number satisfying $P[Y<y_p]\le p\le P[Y\le y_p]$. The qth quantiles $Q_q(X\mid Y>y_p)$, $Q_q(X\mid Y<y_p)$ and $Q_q(X\mid Y=y_p)$ of the distribution functions $P[X<x\mid Y>y_p]$, $P[X<x\mid Y<y_p]$ and $P[X<x\mid Y=y_p]$, respectively, are denoted by $x^{(1)}_{q|p}$, $x^{(2)}_{q|p}$ and $\overline{\overline{x}}_{q|p}$, respectively. Put

$$\overline{\overline{\alpha}}^{(1)}_{X,Y}(q,p):=P[Y>y_p]x^{(1)}_{q|p}+(1-p-P[Y>y_p])\overline{\overline{x}}_{q|p}-(1-p)x_q,$$
$$\beta^{(1,1)}_{X,X}(q,p):=P[X>x_p]x_{q+(1-q)(1-P[X>x_p])}$$

$$+ (1-p-P[X > x_p])x_p - (1-p)x_q,$$

$$\beta^{(1,2)}_{X,X}(q,p) := (1-p)x_q - P[X < x_{1-p}]x_{qP[X<x_{1-p}]} - (1-p-P[X < x_{1-p}])x_{1-p},$$

$$\overline{\overline{\alpha}}^{(2)}_{X,Y}(q,p) := px_q - P[Y < y_p]x^{(2)}_{q|p} - (p - P[Y < y_p])\overline{\overline{x}}_{q|p},$$

$$\beta^{(2,1)}_{X,X}(q,p) := px_q - P[X < x_p]x_{qP[X<x_p]} - (p - P[X < x_p])x_p,$$

$$\beta^{(2,2)}_{X,X}(q,p) := P[X > x_{1-p}]x_{1-(1-q)P[X>x_{1-p}]} + (p - P[X > x_{1-p}])x_{1-p} - px_q.$$

Write

$$F^{(1)}_p(x) := P[X < x, Y > y_p]/(1-p) + (1-p-P[Y > y_p])P[X < x \mid Y = y_p]/(1-p),$$

$$F^{(2)}_p(x) := P[X < x, Y < y_p]/p + (p - P[Y < y_p])P[X < x \mid Y = y_p]/p,$$

$$F^{(3)}_p(x) := \{(P[X < x] - p)/(1-p)\}I_{(x_p,\infty)}(x),$$

$$F^{(4)}_p(x) := \{P[X < x]/p\}I_{(-\infty,x_p)}(x) + I_{[x_p,\infty)}(x).$$

Note that $F^{(1)}_p(x)$ is the mixture of the distribution functions $P[X < x \mid Y > y_p]$ and $P[X < x \mid Y = y_p]$ with the coefficients $\alpha^{(1)} := P[Y > y_p] \times (1-p)^{-1}$ and $1-\alpha^{(1)}$, while $F^{(2)}_p(x)$ is the mixture of the distribution functions $P[X < x \mid Y < y_p]$ and $P[X < x \mid Y = y_p]$ with the coefficients $\alpha^{(2)} := P[Y < y_p]/p$ and $1-\alpha^{(2)}$.

For $p, q \in (0,1)$ we write $\widehat{x}^{(1)}_{q|p}$, $\widehat{x}^{(2)}_{q|p}$, $\widehat{x}^{(3)}_{q|p}$ and $\widehat{x}^{(4)}_{q|p}$ for the qth quantiles of the distribution functions $F^{(1)}_p(x)$, $F^{(2)}_p(x)$, $F^{(3)}_p(x)$ and $F^{(4)}_p(x)$, respectively.

3. Definitions. The formulae (1.1)–(1.4) for the monotone dependence functions of X on Y inspire introducing a nonparametric measure of dependence.

Definition 1. Let $q \in (0,1)$. The *quantile monotone dependence functions* $\widehat{\mu}^{(k)}_{X,Y}(q,\cdot)$, $k = 1,2$, of X on Y with any nondegenerate distribution functions are defined by the following formulae: for $p \in (0,1)$,

$$\widehat{\mu}^{(1)}_{X,Y}(q,p) = \begin{cases} \dfrac{\widehat{x}^{(1)}_{q|p} - x_q}{x_{q+(1-q)p} - x_q} & \text{if } \widehat{x}^{(1)}_{q|p} - x_q \geq 0, \\ \dfrac{\widehat{x}^{(1)}_{q|p} - x_q}{x_q - x_{q(1-p)}} & \text{otherwise;} \end{cases} \tag{3.1}$$

$$(3.2)\qquad \widehat{\mu}^{(2)}_{X,Y}(q,p)=\begin{cases}\dfrac{x_q-\widehat{x}^{(2)}_{q|p}}{x_q-x_{qp}} & \text{if } x_q-\widehat{x}^{(2)}_{q|p}\ge 0,\\[2ex] \dfrac{x_q-\widehat{x}^{(2)}_{q|p}}{x_{q+(1-q)(1-p)}-x_q} & \text{otherwise.}\end{cases}$$

By convention we set $0/0=0$. One can see that the numerators in the above fractions are zero whenever the corresponding denominators are zero.

For continuous distribution functions Definition 1 can be written as follows.

Definition 2. Let $q\in(0,1)$. The *quantile monotone dependence functions* $\mu^{(k)}_{X,Y}(q,\cdot)$, $k=1,2$, of X on Y with continuous strictly increasing distribution functions are defined by the following formulae: for $p\in(0,1)$,

$$(3.3)\qquad \mu^{(1)}_{X,Y}(q,p)=\begin{cases}\dfrac{x^{(1)}_{q|p}-x_q}{x_{q+(1-q)p}-x_q} & \text{if } x^{(1)}_{q|p}-x_q\ge 0,\\[2ex] \dfrac{x^{(1)}_{q|p}-x_q}{x_q-x_{q(1-p)}} & \text{otherwise;}\end{cases}$$

$$(3.4)\qquad \mu^{(2)}_{X,Y}(q,p)=\begin{cases}\dfrac{x_q-x^{(2)}_{q|p}}{x_q-x_{qp}} & \text{if } x_q-x^{(2)}_{q|p}\ge 0,\\[2ex] \dfrac{x_q-x^{(2)}_{q|p}}{x_{q+(1-q)(1-p)}-x_q} & \text{otherwise.}\end{cases}$$

Note that for $q=1/2$ these functions are expressed in terms of medians as follows: for $p\in(0,1)$,

$$\mu^{(1)}_{X,Y}(1/2,p)=\begin{cases}\dfrac{\operatorname{med}(X\mid Y>y_p)-\operatorname{med}(X)}{x_{(1+p)/2}-\operatorname{med}(X)} & \text{if } \operatorname{med}(X\mid Y>y_p)-\operatorname{med}(X)\ge 0,\\[2ex] \dfrac{\operatorname{med}(X\mid Y>y_p)-\operatorname{med}(X)}{\operatorname{med}(X)-x_{1-(1+p)/2}} & \text{otherwise;}\end{cases}$$

$$\mu^{(2)}_{X,Y}(1/2,p)=\begin{cases}\dfrac{\operatorname{med}(X)-\operatorname{med}(X\mid Y<y_p)}{\operatorname{med}(X)-x_{p/2}} & \text{if } \operatorname{med}(X)-\operatorname{med}(X\mid Y<y_p)\ge 0,\\[2ex] \dfrac{\operatorname{med}(X)-\operatorname{med}(X\mid Y<y_p)}{x_{1-p/2}-\operatorname{med}(X)} & \text{otherwise.}\end{cases}$$

The formulae (3.1) and (3.2) give the quantile monotone dependence functions in terms of quantile characteristics of mixtures of distribution functions. Looking at (1.3) and (1.4) we can give general formulae which use the quantiles of F_X, $P[X < x \mid Y > y_p]$, $P[X < x \mid Y = y_p]$ or F_X, $P[X < x \mid Y < y_p]$, $P[X < x \mid Y = y_p]$, respectively.

DEFINITION 3. Let $q \in (0,1)$. The *quantile monotone dependence functions* $\overline{\overline{\mu}}^{(k)}_{X,Y}(q,\cdot)$, $k = 1,2$, of X on Y with any bivariate nondegenerate distribution functions such that $\beta^{(j,k)}_{X,X}(q,p) > 0$, $j,k = 1,2$, are defined by the following formulae: for $p \in (0,1)$,

$$\overline{\overline{\mu}}^{(1)}_{X,Y}(q,p) = \begin{cases} \overline{\overline{\alpha}}^{(1)}_{X,Y}(q,p)/\beta^{(1,1)}_{X,X}(q,p) & \text{if } \overline{\overline{\alpha}}^{(1)}_{X,Y}(q,p) \ge 0, \\ \overline{\overline{\alpha}}^{(1)}_{X,Y}(q,p)/\beta^{(1,2)}_{X,X}(q,p) & \text{if } \overline{\overline{\alpha}}^{(1)}_{X,Y}(q,p) < 0; \end{cases} \tag{3.5}$$

$$\overline{\overline{\mu}}^{(2)}_{X,Y}(q,p) = \begin{cases} \overline{\overline{\alpha}}^{(2)}_{X,Y}(q,p)/\beta^{(2,1)}_{X,X}(q,p) & \text{if } \overline{\overline{\alpha}}^{(2)}_{X,Y}(q,p) \ge 0, \\ \overline{\overline{\alpha}}^{(2)}_{X,Y}(q,p)/\beta^{(2,2)}_{X,X}(q,p) & \text{if } \overline{\overline{\alpha}}^{(2)}_{X,Y}(q,p) < 0. \end{cases} \tag{3.6}$$

We set $\overline{\overline{\mu}}^{(k)}_{X,Y}(q,p) = 0$, $k = 1,2$, whenever $\alpha^{(k)}_{X,Y}(q,p) = \beta^{(k,1)}_{X,X}(q,p) = 0$, $k = 1,2$.

It is not difficult to see that for X and Y with continuous strictly increasing distribution functions we have $\widehat{\mu}^{(k)}_{X,Y}(q,p) = \mu^{(k)}_{X,Y}(q,p) = \overline{\overline{\mu}}^{(k)}_{X,Y}(q,p)$, $(q,p) \in (0,1) \times (0,1)$, $k = 1,2$.

4. Properties of $\widehat{\mu}^{(k)}_{X,Y}$, $\mu^{(k)}_{X,Y}$, and $\overline{\overline{\mu}}^{(k)}_{X,Y}$. To establish the properties of $\widehat{\mu}^{(1)}_{X,Y}(q,\cdot)$ and $\widehat{\mu}^{(2)}_{X,Y}(q,\cdot)$ we need the following lemmas.

LEMMA 1. *For all $x \in \mathbb{R}$ and $p \in (0,1)$,*

(i) $F^{(3)}_p(x) \le F^{(1)}_p(x) \le F^{(4)}_{1-p}(x)$,

(ii) $F^{(3)}_{1-p}(x) \le F^{(2)}_p(x) \le F^{(4)}_p(x)$.

Proof. For $x \le x_p$ the left hand inequality of (i) is trivial as then $F^{(3)}_p(x) = 0$. If $x > x_p$ then

$$\begin{aligned} P[X < x] - P[X < x, Y < y_p] - (p - P[Y < y_p])P[X < x \mid Y = y_p] \\ \ge P[X < x] - P[Y < y_p] - (p - P[Y < y_p]) = P[X < x] - p, \end{aligned}$$

which implies that $F^{(3)}_p(x) \le F^{(1)}_p(x)$. Now we note that trivially $F^{(1)}_p(x) \le F^{(4)}_{1-p}(x)$ for $x \ge x_{1-p}$ as $F^{(4)}_{1-p}(x) = 1$, and that for $x < x_{1-p}$,

$$\begin{aligned} P[X < x] - P[X < x, Y < y_p] - (p - P[Y < y_p])P[X < x \mid Y = y_p] \\ \le P[X < x]. \end{aligned}$$

This implies the right hand inequality of (i).

A similar argument establishes (ii).

LEMMA 2. *For all $q, p \in (0,1)$,*

(i) $\widehat{x}^{(3)}_{q|p} = x_{q+(1-q)p} = x_{p+(1-p)q}$,

(ii) $\widehat{x}^{(4)}_{q|p} = x_{pq}$.

P r o o f. The estimate $F^{(3)}_p(x) \le q \le F^{(3)}_p(x+0)$, i.e.

$$(P[X < x] - p)/(1-p) \le q \le (P[X \le x] - p)/(1-p),$$

is equivalent to

$$P[X < x] \le q(1-p) + p \le P[X \le x];$$

this gives (i). Similarly, $F^{(4)}_p(x) \le q \le F^{(4)}_p(x+0)$, i.e.

$$P[X < x]/p \le q \le P[X \le x]/p$$

is equivalent to

$$P[X < x] \le qp \le P[X \le x],$$

which implies (ii).

The following theorem gives the properties of $\widehat{\mu}^{(k)}_{X,Y}(q,p)$, $k = 1, 2$.

THEOREM 1. (1) *For all* $q, p \in (0,1)$,

$$-1 \le \widehat{\mu}^{(k)}_{X,Y}(q,p) \le 1, \qquad k = 1, 2.$$

(2) *For any fixed* $q \in (0,1)$,

$\widehat{\mu}^{(1)}_{X,Y}(q,p) = 1\ \forall p \in (0,1)$

iff $\widehat{x}^{(1)}_{q|p} = x_{q+(1-q)p}\ (= \widehat{x}^{(3)}_{q|p})\ \forall p \in (0,1)$

iff $P[X < \widehat{x}^{(1)}_{q|p}] = q + (1-q)p\ \forall p \in (0,1) : P[X = \widehat{x}^{(1)}_{q|p}] = 0$,

$\widehat{\mu}^{(1)}_{X,Y}(q,p) = -1\ \forall p \in (0,1)$

iff $\widehat{x}^{(1)}_{q|p} = x_{q(1-p)}\ (= \widehat{x}^{(4)}_{q|(1-p)})\ \forall p \in (0,1)$

iff $P[X < \widehat{x}^{(1)}_{q|p}] = q(1-p)\ \forall p \in (0,1) : P[X = \widehat{x}^{(1)}_{q|p}] = 0$,

and also

$\widehat{\mu}^{(2)}_{X,Y}(q,p) = 1\ \forall p \in (0,1)$

iff $\widehat{x}^{(2)}_{q|p} = x_{qp}\ (= \widehat{x}^{(4)}_{q|p})\ \forall p \in (0,1)$

iff $P[X < \widehat{x}^{(2)}_{q|p}] = qp\ \forall p \in (0,1) : P[X = \widehat{x}^{(2)}_{q|p}] = 0$;

$\widehat{\mu}^{(2)}_{X,Y}(q,p) = -1\ \forall p \in (0,1)$

iff $\widehat{x}^{(2)}_{q|p} = x_{q+(1-q)(1-p)}\ (= \widehat{x}^{(3)}_{q|(1-p)})\ \forall p \in (0,1)$

iff $P[X < \widehat{x}^{(2)}_{q|p}] = q + (1-q)(1-p)\ \forall p \in (0,1) : P[X = \widehat{x}^{(2)}_{q|p}] = 0$.

(3) *For any fixed* $q \in (0,1)$ *and* $k = 1, 2$,

$$\widehat{\mu}^{(k)}_{X,Y}(q,p) = 0 \ \forall p \in (0,1) \quad \textit{iff} \quad \widehat{x}^{(k)}_{q|p} = x_q \ \forall p \in (0,1) \quad \textit{iff}$$

$$\begin{cases} P[X < x_q \mid Y \ge y] \le q \le P[X \le x_q \mid Y \ge y] \\ P[X < x_q \mid Y > y] \le q \le P[X \le x_q \mid Y > y] \end{cases} \quad \forall y \in \mathbb{R} \ (\textit{for } k = 1),$$

$$\begin{cases} P[X < x_q \mid Y \le y] \le q \le P[X \le x_q \mid Y \le y] \\ P[X < x_q \mid Y < y] \le q \le P[X \le x_q \mid Y < y] \end{cases} \quad \forall y \in \mathbb{R} \ (\textit{for } k = 2).$$

(4) *Random variables* X *and* Y *are independent iff* $\widehat{\mu}^{(1)}_{X,Y}(\cdot,\cdot) = 0$ *iff* $\widehat{\mu}^{(2)}_{X,Y}(\cdot,\cdot) = 0$.

(5) *Suppose* $F_X = F_{X'}$ *and* $F_Y = F_{Y'}$. *Then for any fixed* $q \in (0,1)$,

$$\widehat{\mu}^{(1)}_{X,Y}(q,p) \le \widehat{\mu}^{(1)}_{X',Y'}(q,p) \ \forall p \in (0,1) \quad \textit{iff} \quad \widehat{x}^{(1)}_{q|p} \le \widehat{x}'^{(1)}_{q|p} \ \forall p \in (0,1),$$

where $\widehat{x}'^{(1)}_{q|p}$ *is the* q*-th quantile of*

$$F'^{(1)}_p(x) = \frac{P[X' < x, Y' > y_p]}{1-p} + \frac{1 - p - P[Y' > y_p]}{1-p} P[X' < x \mid Y' = y_p],$$

and we have

$$F'^{(1)}_p(x) \le q \le F^{(1)}_p(x+0) \quad \textit{whenever} \quad x = \widehat{x}^{(1)}_{q|p}, \widehat{x}'^{(1)}_{q|p}.$$

Moreover,

$$\widehat{\mu}^{(2)}_{X,Y}(q,p) \le \widehat{\mu}^{(2)}_{X',Y'}(q,p) \ \forall p \in (0,1) \quad \textit{iff} \quad \widehat{x}^{(2)}_{q|p} \ge \widehat{x}'^{(2)}_{q|p} \ \forall p \in (0,1),$$

where $\widehat{x}'^{(2)}_{q|p}$ *is the* q*-th quantile of*

$$F'^{(2)}_p(x) = \frac{P[X' < x, Y' < y_p]}{p} + \frac{p - P[Y' < y_p]}{p} P[X' < x \mid Y' = y_p],$$

and we have

$$F^{(2)}_p(x) \le q \le F'^{(2)}_p(x+0) \quad \textit{whenever} \quad x = \widehat{x}^{(2)}_{q|p}, \widehat{x}'^{(2)}_{q|p}.$$

(6) *Suppose* $F_X = F_{X'}$ *and* $F_Y = F_{Y'}$. *Then*

$$\begin{aligned} \widehat{\mu}^{(1)}_{X,Y}(\cdot,\cdot) \le \widehat{\mu}^{(1)}_{X',Y'}(\cdot,\cdot) \quad &\textit{iff} \quad \widehat{\mu}^{(2)}_{X,Y}(\cdot,\cdot) \le \widehat{\mu}^{(2)}_{X',Y'}(\cdot,\cdot) \\ &\textit{iff} \quad F_{X,Y}(\cdot,\cdot) \le F_{X',Y'}(\cdot,\cdot). \end{aligned}$$

(7) *A random variable* X *is positively* (*negatively*) *quadrant dependent on a random variable* Y *iff* $\widehat{\mu}^{(1)}_{X,Y}(\cdot,\cdot) \ge 0$ (≤ 0) *iff* $\widehat{\mu}^{(2)}_{X,Y}(\cdot,\cdot) \ge 0$ (≤ 0).

(8) *For all* $q, p \in (0,1)$ *and* $k = 1, 2$,

$$\widehat{\mu}^{(k)}_{aX+b,f(Y)}(q,p) = \begin{cases} \widehat{\mu}^{(k)}_{X,Y}(q,p), & a > 0,\ f\uparrow, \\ -\widehat{\mu}^{(k)}_{X,Y}(1-q,p), & a < 0,\ f\uparrow, \\ -\widehat{\mu}^{(3-k)}_{X,Y}(q,1-p), & a > 0,\ f\downarrow, \\ \widehat{\mu}^{(3-k)}_{X,Y}(1-q,1-p), & a < 0,\ f\downarrow, \end{cases}$$

where f *is a strictly increasing or decreasing function.*

Proof. To prove (1) it is enough to see that

$$x_{q(1-p)} \le \widehat{x}^{(1)}_{q|p} \le x_{q+(1-q)p} \quad \text{and} \quad x_{qp} \le \widehat{x}^{(2)}_{q|p} \le x_{q+(1-q)(1-p)}.$$

By Lemma 2 the above inequalities are equivalent to

$$\widehat{x}^{(4)}_{q|(1-p)} \le \widehat{x}^{(1)}_{q|p} \le \widehat{x}^{(3)}_{q|p}, \qquad \widehat{x}^{(4)}_{q|p} \le \widehat{x}^{(2)}_{q|p} \le \widehat{x}^{(3)}_{q|(1-p)},$$

respectively. But these inequalities are consequences of Lemma 1.

The statements of (2) follow from Definition 1 of $\widehat{\mu}^{(k)}_{X,Y}(q,p)$, $k = 1, 2$, and Lemma 2.

The first equivalence in (3) is obvious by Definition 1. Assume now that $\widehat{x}^{(1)}_{q|p} = x_q$. Using the definition inequality for $\widehat{x}^{(1)}_{q|p}$ we have

$$P[X < \widehat{x}^{(1)}_{q|p},\ Y > y_p] + (1 - p - P[Y > y_p])P[X < \widehat{x}^{(1)}_{q|p} \mid Y = y_p] \le q(1-p)$$
$$\le P[X \le \widehat{x}^{(1)}_{q|p},\ Y > y_p] + (1 - p - P[Y > y_p])P[X \le \widehat{x}^{(1)}_{q|p} \mid Y = y_p].$$

Replacing now $\widehat{x}^{(1)}_{q|p}$ by x_q we get

$$P[X < x_q,\ Y > y_p] + (1 - p - P[Y > y_p])P[X < x_q \mid Y = y_p] \le q(1-p)$$
$$\le P[X \le x_q,\ Y > y_p] + (1 - p - P[Y > y_p])P[X \le x_q \mid Y = y_p].$$

Hence, for p such that $P[Y \ge y_p] = 1 - p$ we have

$$P[X < x_q,\ Y \ge y_p] \le qP[Y \ge y_p] \le P[X \le x_q,\ Y \ge y_p],$$

while for p such that $P[Y > y_p] = 1 - p$ we get

$$P[X < x_q,\ Y > y_p] \le qP[Y > y_p] \le P[X \le x_q,\ Y > y_p].$$

Taking into account that the above inequalities depend only on the values y_p we see that for $p \in (0,1)$,

$$\begin{aligned} &P[X < x_q \mid Y \ge y_p] \le q \le P[X \le x_q \mid Y \ge y_p], \\ &P[X < x_q \mid Y > y_p] \le q \le P[X \le x_q \mid Y > y_p], \end{aligned} \tag{4.1}$$

which give the stated inequalities for any given $q \in (0,1)$ and all $y \in \mathbb{R}$,

$$\begin{aligned} &P[X < x_q \mid Y \ge y] \le q \le P[X \le x_q \mid Y \ge y], \\ &P[X < x_q \mid Y > y] \le q \le P[X \le x_q \mid Y > y]. \end{aligned} \tag{4.2}$$

If (4.2) holds true then (4.1) is satisfied. Therefore, by the definition of $F_p^{(1)}(x)$ we trivially get the equality $\widehat{x}_{q|p}^{(1)} = x_q$ for p such that $P[Y = y_p] = 0$. Suppose now $P[Y = y_p] > 0$. Then using (4.1) and (4.2) we have

$$\begin{aligned}
F_p^{(1)}(x_q) &= \frac{P[Y \geq y_p] + p - 1}{P[Y = y_p](1-p)} P[X < x_q, Y > y_p] \\
&\quad + \frac{1 - p - P[Y > y_p]}{P[Y = y_p](1-p)} P[X < x_q,\ Y \geq y_p] \\
&\leq \frac{P[Y \geq y_p] + p - 1}{P[Y = y_p](1-p)} qP[Y > y_p] + \frac{1 - p - P[Y > y_p]}{P[Y = y_p](1-p)} qP[Y \geq y_p] \\
&= q \\
&\leq \frac{P[Y \geq y_p] + p - 1}{P[Y = y_p](1-p)} P[X \leq x_q,\ Y > y_p] \\
&\quad + \frac{1 - p - P[Y > y_p]}{P[Y = y_p](1-p)} P[X \leq x_q,\ Y \geq y_p] \\
&= F_p^{(1)}(x_q + 0),
\end{aligned}$$

which completes the proof of (3) for $\widehat{\mu}_{X,Y}^{(1)}(q,p)$.

The case of $\widehat{\mu}_{X,Y}^{(2)}(q,p)$ can be proved in a similar way.

The equality $\widehat{\mu}_{X,Y}^{(1)}(\cdot,\cdot) = 0$ in (4) for X and Y being independent random variables follows directly from the definitions of $F_p^{(1)}(x)$ and $\widehat{\mu}_{X,Y}^{(1)}(q,p)$. Conversely if $\widehat{\mu}_{X,Y}^{(1)}(\cdot,\cdot) = 0$ then by (3), for all $q \in (0,1)$ and $y \in \mathbb{R}$,

$$P[X < x_q \mid Y \geq y] \leq q \leq P[X \leq x_q \mid Y \geq y],$$

and, by the definition of x_q,

$$P[X < x_q,\ Y \geq y] \leq qP[Y \geq y] \leq P[X \leq x_q,\ Y \geq y].$$

Hence, we get

$$P[X < x_q,\ Y \geq y] \leq qP[Y \geq y] \leq P[X \leq x_q]P[Y \geq y],$$

and

$$P[X < x_q]P[Y \geq y] \leq qP[Y \geq y] \leq P[X \leq x_q,\ Y \geq y].$$

Therefore, for all $q \in (0,1)$ such that $P[X = x_q] = 0$ and all $y \in \mathbb{R}$,

$$P[X < x_q,\ Y \geq y] = P[X < x_q]P[Y \geq y].$$

Thus we see that for all $x \in \mathbb{R}$ such that $P[X = x] = 0$ and all $y \in \mathbb{R}$,

$$P[X < x,\ Y < y] = P[X < x]P[Y < y],$$

which implies that X and Y are independent random variables.

Similar considerations lead to the same statement for $\widehat{\mu}_{X,Y}^{(2)}(q,p)$ and end the proof of (4).

The first equivalence in (5) follows from Definition 1, and the inequalities

$$0 \le \frac{\widehat{x}_{q|p} - x_q}{x_{q+(1-q)p} - x_q} \le \frac{\widehat{x}'^{(1)}_{q|p} - x_q}{x_{q+(1-q)p} - x_q},$$
$$\frac{\widehat{x}^{(1)}_{q|p} - x_q}{x_q - x_{q(1-p)}} < 0 \le \frac{\widehat{x}'^{(1)}_{q|p} - x_q}{x_{q+(1-q)p} - x_q},$$

and

$$\frac{\widehat{x}^{(1)}_{q|p} - x_q}{x_q - x_{q(1-p)}} \le \frac{\widehat{x}'^{(1)}_{q|p} - x_q}{x_q - x_{q(1-p)}} < 0.$$

Assuming now that $\widehat{x}^{(1)}_{q|p} \le \widehat{x}'^{(1)}_{q|p}$ we see that

$$F'^{(1)}_p(\widehat{x}^{(1)}_{q|p}) \le F'^{(1)}_p(\widehat{x}'^{(1)}_{q|p}) \le q \le F^{(1)}_p(\widehat{x}^{(1)}_{q|p} + 0)$$

and

$$F'^{(1)}_p(\widehat{x}'^{(1)}_{q|p}) \le q \le F^{(1)}_p(\widehat{x}^{(1)}_{q|p} + 0) \le F^{(1)}_p(\widehat{x}'^{(1)}_{q|p} + 0).$$

A similar argument can be used for the statements on $\widehat{\mu}^{(2)}_{X,Y}(q,p)$ in (5).

Now assuming that the first inequality in (6) holds true we see by (5) that

$$\widehat{x}^{(1)}_{q|p} \le \widehat{x}'^{(1)}_{q|p}, \qquad \forall p, q \in (0,1).$$

Hence, using again (5) we get

$$P[X' < \widehat{x}^{(1)}_{q|p},\ Y' \ge y_p] \le P[X \le \widehat{x}^{(1)}_{q|p},\ Y \ge y_p]$$

for all $q \in (0,1)$ and all $p \in (0,1)$ such that $P[Y = y_p] = 0$.

Therefore, for every x such that $P[X = x] = 0$ and every y such that $P[Y = y] = 0$ we have

$$P[X' < x,\ Y' \ge y] \le P[X < x,\ Y \ge y],$$

which implies that $F_{X,Y}(x,y) \le F_{X',Y'}(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Now assume that the last inequality holds true. Then for all $(x,y) \in \mathbb{R}^2$,

$$P[X < x,\ Y \ge y] \ge P[X' < x,\ Y' \ge y],$$
$$P[X < x,\ Y > y] \ge P[X' < x,\ Y' > y].$$

Hence for $p \in (0,1)$ such that $P[Y = y_p] > 0$ we have

$$\begin{aligned} F'^{(1)}_p(x) &= \frac{1 - p - P[Y > y_p]}{(1-p)P[Y = y_p]} P[X' < x,\ Y' \ge y_p] \\ &\quad + \frac{P[Y = y_p] - 1 + p + P[Y > y_p]}{(1-p)P[Y = y_p]} P[X' < x,\ Y' > y_p] \\ &\le \frac{1 - p - P[Y > y_p]}{(1-p)P[Y = y_p]} P[X < x,\ Y \ge y_p] \end{aligned}$$

$$+\frac{P[Y=y_p]-1+p+P[Y>y_p]}{(1-p)P[Y=y_p]}P[X<x,\ Y>y_p]$$
$$=F_p^{(1)}(x).$$

Moreover, whenever $P[Y=y_p]=0$, then

$$F_p'^{(1)}(x)=\frac{P[X'<x,\ Y'>y_p]}{1-p}\le\frac{P[X<x,\ Y>y_p]}{1-p}=F_p^{(1)}(x).$$

Therefore, $F_p'^{(1)}(x)\le F_p^{(1)}(x)$ for all $x\in\mathbb{R}$ and $p\in(0,1)$, which by (5) completes the proof of (6) for $\widehat{\mu}^{(1)}_{X,Y}(q,p)$. The statements for $\widehat{\mu}^{(2)}_{X,Y}(q,p)$ are proved similarly.

To prove (7) we recall that by definition (cf. [1], [7]), a random variable X is positively (negatively) quadrant dependent on a random variable Y if

$$F_{X,Y}(x,y)\ge\text{(resp. }\le)\ F_X(x)F_Y(y)\qquad\text{for all }(x,y)\in\mathbb{R}^2. \tag{4.3}$$

Suppose that X is positively quadrant dependent on Y and that X' and Y' are independent random variables such that $F_X=F_{X'}$ and $F_Y=F_{Y'}$. Then by (4.3),

$$\begin{aligned}F_{X,Y}(x,y)&\ge F_X(x)F_Y(y)\\&=F_{X'}(x)F_{Y'}(y)=F_{X'Y'}(x,y)\qquad\text{for all }(x,y)\in\mathbb{R}^2.\end{aligned}$$

Hence using (6) and (4) we deduce that $\widehat{\mu}^{(1)}_{X,Y}(\cdot,\cdot)\ge0$; conversely, the latter inequality together with (4) and (6) gives (4.3), which completes the proof of (7).

In proving (8) we only consider the case with $k=1$, $a<0$ and f increasing. Write

$$\begin{aligned}F_p^{(1)}(x;a,b,f)=\ &P[aX+b<x,\ f(Y)>Q_p(f(Y))]/(1-p)\\&+(1-p-P[f(Y)>Q_p(f(Y))])\\&\times P[aX+b<x\mid f(Y)=Q_p(f(Y))]/(1-p),\end{aligned}$$

where $Q_p(f(Y))$ denotes the pth quantile of $f(Y)$, and let $\widehat{x}^{(1)}_{q|p}(a,b,f)$ stand for the qth quantile of $F_p^{(1)}(x;a,b,f)$.

Note that the qth quantile $Q_q(aX+b)$ of $aX+b$ is given by

$$Q_q(aX+b)=ax_{1-q}+b,$$

and

$$\widehat{x}^{(1)}_{q|p}(a,b,f)=\widehat{x}^{(1)}_{q|p}(a,b,g)=a\widehat{x}^{(1)}_{(1-q)|p}(1,0,g)+b=a\widehat{x}^{(1)}_{(1-q)|p}+b,$$

with $g(x) := x$, $x \in \mathbb{R}$. Hence

$$\widehat{\mu}^{(1)}_{aX+b,f(Y)}(q,p) = \frac{\widehat{x}^{(1)}_{q|p}(a,b,f) - Q_q(aX+b)}{Q_{q+(1-q)p}(aX+b) - Q_q(aX+b)}$$
$$= -\frac{\widehat{x}^{(1)}_{(1-q)|p} - x_{1-q}}{x_{1-q} - x_{(1-q)(1-p)}} = -\widehat{\mu}^{(1)}_{X,Y}(1-q,p).$$

It is obvious that the properties of $\mu^{(k)}_{X,Y}(\cdot,\cdot)$, $k = 1,2$, can be deduced from Theorem 1. However, some of them can be expressed in simpler forms. For instance, in the case of continuous and strictly increasing marginal and conditional distribution functions, (3) and (5) read as follows:

(3′) $\exists q \in (0,1)\ \forall p \in (0,1)\ \mu^{(1)}_{X,Y}(q,p) = 0$

$$\text{iff} \quad \exists q \in (0,1)\ \forall p \in (0,1)\ \mu^{(2)}_{X,Y}(q,p) = 0$$
$$\text{iff} \quad \exists q \in (0,1)\ \forall p \in (0,1)\ x^{(1)}_{q|p} = x_q$$
$$\text{iff} \quad \exists q \in (0,1)\ \forall p \in (0,1)\ x^{(2)}_{q|p} = x_q$$
$$\text{iff} \quad \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ F_{X,Y}(x,y) = F_X(x)\cdot F_Y(y).$$

(5′) Suppose $F_X = F_{X'}$ and $F_Y = F_{Y'}$. Then for any fixed $q \in (0,1)$,

$$\mu^{(1)}_{X,Y}(q,p) \le \mu^{(1)}_{X',Y'}(q,p)\ \forall p \in (0,1)$$
$$\text{iff} \quad x^{(1)}_{q|p} \le x'^{(1)}_{q|p}\ \forall p \in (0,1)$$
$$\text{iff} \quad P[X' < x^{(1)}_{q|p} \mid Y' > y_p] \le P[X < x^{(1)}_{q|p} \mid Y > y_p]\ \forall p \in (0,1)$$
$$\text{iff} \quad P[X' < x'^{(1)}_{q|p} \mid Y' > y_p] \le P[X < x'^{(1)}_{q|p} \mid Y > y_p]\ \forall p \in (0,1),$$

where $x'^{(1)}_{q|p} = Q_q(X' \mid Y' > y_p)$, and

$$\mu^{(2)}_{X,Y}(q,p) \le \mu^{(2)}_{X',Y'}(q,p)\ \forall p \in (0,1)$$
$$\text{iff} \quad x^{(2)}_{q|p} \ge x'^{(2)}_{q|p}\ \forall p \in (0,1)$$
$$\text{iff} \quad P[X' < x^{(2)}_{q|p} \mid Y' < y_p] \ge P[X < x^{(2)}_{q|p} \mid Y < y_p]\ \forall p \in (0,1)$$
$$\text{iff} \quad P[X' < x'^{(2)}_{q|p} \mid Y' < y_p] \ge P[X < x'^{(2)}_{q|p} \mid Y < y_p]\ \forall p \in (0,1),$$

where $x'^{(2)}_{q|p} = Q_q(X' \mid Y' < y_p)$.

Now we are going to give some properties for $\overline{\overline{\mu}}^{(k)}_{X,Y}(q,p)$, $k = 1,2$. In what follows we take $q \in (0,1)$ such that the used quantities are well defined.

THEOREM 2. (1) *For any fixed* $q \in (0,1)$,

$$\overline{\overline{\mu}}^{(k)}_{X,Y}(q,p) = 0 \ \forall p \in (0,1) : P[Y = y_p] = 0$$
$$\text{iff} \quad x^{(k)}_{q|p} = x_q \ \forall p \in (0,1) : P[Y = y_p] = 0$$
$$\text{iff} \quad P[X < x_q \mid Y > y] \le q \le P[X \le x_q \mid Y > y] \ \forall y \in \mathbb{R} \ (\text{for } k = 1),$$
$$P[X < x_q \mid Y < y] \le q \le P[X \le x_q \mid Y < y] \ \forall y \in \mathbb{R} \ (\text{for } k = 2).$$

(2) *If random variables* X *and* Y *are independent then*

$$\overline{\overline{\mu}}^{(1)}_{X,Y}(\cdot,\cdot) \equiv 0 \quad \text{and} \quad \overline{\overline{\mu}}^{(2)}_{X,Y}(\cdot,\cdot) \equiv 0.$$

(3) *Suppose* $F_X = F_{X'}$ *and* $F_Y = F_{Y'}$. *Then for any fixed* $q \in (0,1)$,

$$\overline{\overline{\mu}}^{(1)}_{X,Y}(q,p) \le \overline{\overline{\mu}}^{(1)}_{X',Y'}(q,p) \ \forall p \in (0,1) : P[Y = y_p] = 0$$
$$\text{iff} \quad x^{(1)}_{q|p} \le x'^{(1)}_{q|p} \ \forall p \in (0,1) : P[Y = y_p] = 0,$$

where $x'^{(1)}_{q|p} := Q_q(X' \mid Y' > y_p)$, *and we have*

$$P[X' < x \mid Y' > y_p] \le q \le P[X \le x \mid Y > y_p] \quad \forall p \in (0,1) : P[Y = y_p] = 0,$$

whenever $x = x^{(1)}_{q|p}, x'^{(1)}_{q|p}$. *Moreover,*

$$\overline{\overline{\mu}}^{(2)}_{X,Y}(q,p) \le \overline{\overline{\mu}}^{(2)}_{X',Y'}(q,p) \ \forall p \in (0,1) : P[Y = y_p] = 0$$
$$\text{iff} \quad x^{(2)}_{q|p} \ge x'^{(2)}_{q|p} \ \forall p \in (0,1) : P[Y = y_p] = 0,$$

where $x'^{(2)}_{q|p} := Q_q(X' \mid Y' < y_p)$, *and we have*

$$P[X < x \mid Y > y_p] \le q \le P[X' \le x \mid Y' < y_p] \quad \forall p \in (0,1) : P[Y = y_p] = 0,$$

whenever $x = x^{(2)}_{q|p}, x'^{(2)}_{q|p}$.

(4) *Suppose* $F_{X,Y}(x,y) \le F_{X',Y'}(x,y)$ *for* $(x,y) \in \mathbb{R}^2$, *with* $F_X = F_{X'}$ *and* $F_Y = F_{Y'}$. *Then*

$$\overline{\overline{\mu}}^{(1)}_{X,Y}(q,p) \le \overline{\overline{\mu}}^{(1)}_{X',Y'}(q,p) \qquad \forall q,p \in (0,1) : P[Y = y_p] = 0,$$
$$\overline{\overline{\mu}}^{(2)}_{X,Y}(q,p) \le \overline{\overline{\mu}}^{(2)}_{X',Y'}(q,p) \qquad \forall q,p \in (0,1) : P[Y = y_p] = 0.$$

(5) *If a random variable* X *is positively (negatively) quadrant dependent on a random variable* Y *then for all* $q,p \in (0,1)$ *with* $P[Y = y_p] = 0$,

$$\overline{\overline{\mu}}^{(1)}_{X,Y}(q,p) \ge 0 \ (\le 0) \quad \text{and} \quad \overline{\overline{\mu}}^{(2)}_{X,Y}(q,p) \ge 0 \ (\le 0).$$

(6) *For all* $q, p \in (0,1)$, $k = 1, 2$,

$$\overline{\overline{\mu}}^{(k)}_{aX+b,f(Y)}(q,p) = \begin{cases} \overline{\overline{\mu}}^{(k)}_{X,Y}(q,p), & a>0,\ f\uparrow, \\ -\overline{\overline{\mu}}^{(k)}_{X,Y}(1-q,p), & a<0,\ f\uparrow, \\ -\overline{\overline{\mu}}^{(3-k)}_{X,Y}(q,1-p), & a>0,\ f\downarrow, \\ \overline{\overline{\mu}}^{(3-k)}_{X,Y}(1-q,1-p), & a<0,\ f\downarrow. \end{cases}$$

Proof. The first equivalence in (1) is obvious by Definition 3. Assume now that $x^{(1)}_{q|p} = x_q$. Using the quantile inequality for $x^{(1)}_{q|p}$ we have

$$P[X < x^{(1)}_{q|p} \mid Y > y_p] \le q \le P[X \le x^{(1)}_{q|p} \mid Y > y_p].$$

Replacing now $x^{(1)}_{q|p}$ by x_q we get

$$P[X < x_q \mid Y > y] \le q \le P[X \le x_q \mid Y > y] \quad \forall y \in \mathbb{R},$$

i.e. the assertion "if" for $\mu^{(1)}_{X,Y}(q,p)$.

Now having the last inequality we see that for all $p \in (0,1)$,

$$P[X < x_q \mid Y > y_p] \le q \le P[X \le x_q \mid Y > y_p],$$

which defines $x^{(1)}_{q|p}$.

The property (2) is obvious.

The first equivalence in (3) follows from Definition 3. Assuming now that $x^{(1)}_{q|p} \le x'^{(1)}_{q|p}$ we see that

$$\begin{aligned} P[X' < x^{(1)}_{q|p} \mid Y' > y_p] &\le P[X' < x'^{(1)}_{q|p} \mid Y' > y_p] \\ &\le q \le P[X \le x^{(1)}_{q|p} \mid Y > y_p] \end{aligned}$$

and

$$\begin{aligned} P[X' < x'^{(1)}_{q|p} \mid Y' > y_p] &\le q \le P[X < x^{(1)}_{q|p} \mid Y' > y_p] \\ &\le P[X \le x'^{(1)}_{q|p} \mid Y > y_p], \end{aligned}$$

which shows (3) for $\overline{\overline{\mu}}^{(1)}_{X,Y}(q,p)$.

Now by the assumption of (4) we have

$$P[X < x \mid Y > y] \ge P[X' < x \mid Y' > y] \quad \forall (x,y) \in \mathbb{R}^2,$$

which implies

$$P[X < x \mid Y > y_p] \ge P[X' < x \mid Y' > y_p] \quad \forall x \in \mathbb{R},\ \forall p \in (0,1),$$

and in consequence we get $x^{(1)}_{q|p} \le x'^{(1)}_{q|p}$, which by Definition 3 gives (4).

The statement (5) follows from (2) and (4).

Finally, the equalities of (6) can be obtained in a similar way to those of Theorem 1.

4. Examples. We consider here distribution functions with marginals having finite or infinite expectations.

1. For an exponential distribution with distribution function

$$F(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+xy)}, \qquad x \ge 0, y \ge 0,$$

the monotone dependence function $\mu_{X,Y}(\cdot)$ has been determined in [3]:

$$\mu_{X,Y}(p) = \frac{(1-p)\ln(1-p)}{p\ln p(\ln(1-p)-1)}.$$

The quantile monotone dependence function $\mu^{(1)}_{X,Y}(q,\cdot)$ can be given explicitly since $x_q = -\ln(1-q)$, $x^{(1)}_{q|p} = -\ln(1-q)/(1-\ln(1-p))$, namely

$$\mu^{(1)}_{X,Y}(q,p) = \frac{\ln(1-q)\ln(1-p)}{(\ln(1-p)-1)\ln\left(1+\frac{pq}{1-q}\right)},$$

while to give $\mu^{(2)}_{X,Y}(q,p)$ we first determine $x^{(2)}_{q|p}$ as the solution of the equation

$$(1-p)^{x+1} + p(1-q)e^{x} - 1 = 0$$

and next put that value in (2.2).

2. For a probability law with distribution function

$$F(x,y) = 1 - \frac{1}{x} - \frac{1}{y} + \frac{1}{x^{y}y}, \qquad x \ge 1,\ y \ge 1,$$

we have $x_q = 1/(1-q)$, $x^{(1)}_{q|p} = (1-q)^{p-1}$, and

$$\mu^{(1)}_{X,Y}(q,p) = \frac{1-(1-p)q}{pq}((1-q)^{p}-1).$$

To obtain $\mu^{(2)}_{X,Y}(q,p)$ we have to solve the equation

$$p(1-q)x^{1/(1-p)} - x^{p/(1-p)} + 1 - p = 0.$$

The solution gives $x^{(2)}_{q|p}$ and allows us to find $\mu^{(2)}_{X,Y}(q,p)$.

3. For a Cauchy distribution with distribution function

$$F(x,y) = (1-\alpha)F(x)F(y) + \alpha\min(F(x),F(y)), \qquad 0 \le \alpha \le 1,$$

where

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{t}\right),$$
$$F(y) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{y}{t}\right), \qquad t \in \mathbb{R} - \{0\},$$

we have

$$x_q = t(\tan[(q-1/2)\pi]), \qquad 0<q<1,$$
$$x^{(1)}_{q|p} = \begin{cases} x_{q/(1-\alpha)} & \text{if } q \le p(1-\alpha),\\ x_{(\alpha p+q(1-p))/(\alpha p+1-p)} & \text{if } q > p(1-\alpha);\end{cases}$$
$$x^{(2)}_{q|p} = \begin{cases} x_{qp/(p+\alpha(1-p))} & \text{if } q \le p+\alpha(1-p),\\ x_{(q-\alpha)/(1-\alpha)} & \text{if } q > p+\alpha(1-p).\end{cases}$$

Hence

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} \dfrac{\tan\left[\left(\frac{\alpha q}{1-\alpha}+q-\frac12\right)\pi\right]-\tan\left[\left(q-\frac12\right)\pi\right]}{\tan\left[\left((1-q)p+q-\frac12\right)\pi\right]-\tan\left[\left(q-\frac12\right)\pi\right]} & \text{if } q\le p(1-\alpha),\\ \dfrac{\tan\left[\left(\frac{\alpha p(1-q)}{\alpha p+1-p}+q-\frac12\right)\pi\right]-\tan\left[\left(q-\frac12\right)\pi\right]}{\tan\left[\left((1-q)p+q-\frac12\right)\pi\right]-\tan\left[\left(q-\frac12\right)\pi\right]} & \text{if } q> p(1-\alpha),\end{cases}$$

and

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} \dfrac{\tan\left[\left(q-\frac12\right)\pi\right]-\tan\left[\left(-\frac{\alpha q(1-p)}{p+\alpha(1-p)}+q-\frac12\right)\pi\right]}{\tan\left[\left(q-\frac12\right)\pi\right]-\tan\left[\left(-q(1-p)+q-\frac12\right)\pi\right]} & \text{if } q\le p+\alpha(1-p),\\ \dfrac{\tan\left[\left(q-\frac12\right)\pi\right]-\tan\left[\left(-\frac{\alpha(1-q)}{1-\alpha}+q-\frac12\right)\pi\right]}{\tan\left[\left(q-\frac12\right)\pi\right]-\tan\left[\left(-q(1-p)+q-\frac12\right)\pi\right]} & \text{if } q> p+\alpha(1-p).\end{cases}$$

4. For the distribution function defined by

$$F(x,y) = \begin{cases} 0 \quad \text{if } x\le 0 \text{ or } y\le 0,\\ \dfrac{1}{2\pi\sqrt{1-\varrho^2}}\displaystyle\int_0^x\int_0^y \frac{1}{ut}\exp\left(-\frac{\ln^2 u-2\varrho\ln u\ln t+\ln^2 t}{2(1-\varrho^2)}\right)du\,dt\\ \hfill \text{otherwise,}\end{cases}$$

we see that $x^{(1)}_{q|p}$ is the solution of the equation

$$\Phi(\ln x)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\ln x}\Phi((z_p-\varrho v)/\sqrt{1-\varrho^2})e^{-v^2/2}\,dv = q(1-p),$$

where z_p is the pth quantile of the standard normal distribution function while $x^{(2)}_{q|p}$ is the number satisfying the equation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\ln x}\Phi((z_p+\varrho v)/\sqrt{1-\varrho^2})e^{-v^2/2}\,dv = qp.$$

Hence we have

$$\mu^{(1)}_{X,Y}(q,p)=\begin{cases}\dfrac{x^{(1)}_{q|p}-\exp(z_q)}{\exp(z_{q+(1-q)p})-\exp(z_q)} & \text{if } x^{(1)}_{q|p}-\exp(z_q)\ge 0,\\ \dfrac{x^{(1)}_{q|p}-\exp(z_q)}{\exp(z_q)-\exp(z_{q(1-p)})} & \text{otherwise;}\end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p)=\begin{cases}\dfrac{\exp(z_q)-x^{(2)}_{q|p}}{\exp(z_q)-\exp(z_{qp})} & \text{if } \exp(z_q)-x^{(2)}_{q|p}\ge 0,\\ \dfrac{\exp(z_q)-x^{(2)}_{q|p}}{\exp(z_{q+(1-q)(1-p)})-\exp(z_q)} & \text{otherwise.}\end{cases}$$

5. If

$$F(x,y)=(1-\alpha)F(x)F(y)+\alpha\min(F(x),F(y)),\qquad 0\le\alpha\le 1,$$

where

$$F(x)=\begin{cases}0 & \text{if } x\le 0,\\ \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^x \frac{\exp\{-(\ln u)^2/2\}}{u}\,du & \text{otherwise,}\end{cases}$$

then

$$x^{(1)}_{q|p}=\begin{cases}\exp(z_{q/(1-\alpha)}) & \text{if } q\le p(1-\alpha),\\ \exp(z_{(q-qp+\alpha p)/(1-p+\alpha p)}) & \text{if } q> p(1-\alpha),\end{cases}$$

$$x^{(2)}_{q|p}=\begin{cases}\exp(z_{qp/((1-\alpha)p+\alpha)}) & \text{if } q\le p(1-\alpha)+\alpha,\\ \exp(z_{(q-\alpha)/(1-\alpha)}) & \text{if } q> p(1-\alpha)+\alpha,\end{cases}$$

and we have

$$\mu^{(1)}_{X,Y}(q,p)=\begin{cases}\dfrac{x^{(1)}_{q|p}-\exp(z_q)}{\exp(z_{q+(1-q)p})-\exp(z_q)} & \text{if } x^{(1)}_{q|p}-\exp(z_q)\ge 0,\\ \dfrac{x^{(1)}_{q|p}-\exp(z_q)}{\exp(z_q)-\exp(z_{q(1-p)})} & \text{otherwise, } \forall p\in(0,1);\end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p)=\begin{cases}\dfrac{\exp(z_q)-x^{(2)}_{q|p}}{\exp(z_q)-\exp(z_{qp})} & \text{if } \exp(z_q)-x^{(2)}_{q|p}\ge 0,\\ \dfrac{\exp(z_q)-x^{(2)}_{q|p}}{\exp(z_{q+(1-q)(1-p)})-\exp(z_q)} & \text{otherwise, } \forall p\in(0,1),\end{cases}$$

where z_q is defined as in Example 4.

6. If

$$F(x,y)=\begin{cases}0 & \text{if } (x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+,\\ 0.6 & \text{if } 0<x\le 1,\ 0<y\le 1,\\ 0.7 & \text{if } 0<x\le 1,\ y>1,\\ 0.7 & \text{if } x>1,\ 0<y\le 1,\\ 1 & \text{otherwise,}\end{cases}$$

then

$$x_p=y_p=\begin{cases}0 & \text{if } 0<p<0.7,\\ 1 & \text{otherwise,}\end{cases}$$

$$F_p^{(1)}(x)=\begin{cases}0 & \text{if } x\le 0,\\ \max\left(\dfrac{4.9-6p}{7(1-p)},\dfrac{1}{3}\right) & \text{if } 0<x\le 1,\\ 1 & \text{otherwise,}\end{cases}$$

$$F_p^{(2)}(x)=\begin{cases}0 & \text{if } x\le 0,\\ \min\left(\dfrac{6}{7},\dfrac{p+1.1}{3p}\right) & \text{if } 0<x\le 1,\\ 1 & \text{otherwise,}\end{cases}$$

$$\widehat{x}_{q|p}^{(1)}=\begin{cases}1 & \text{if } \max\left(\dfrac{1}{3},\dfrac{4.9-6p}{7(1-p)}\right)\le q<1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\widehat{x}_{q|p}^{(2)}=\begin{cases}1 & \text{if } \min\left(\dfrac{6}{7},\dfrac{p+1.1}{3p}\right)\le q<1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\widehat{\mu}_{X,Y}^{(1)}(q,p)=\begin{cases}1 & \text{if } \max\left(\dfrac{1}{3},\dfrac{4.9-6p}{7(1-p)}\right)\le q<0.7,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\widehat{\mu}_{X,Y}^{(2)}(q,p)=\begin{cases}1 & \text{if } \min\left(\dfrac{6}{7},\dfrac{p+1.1}{3p}\right)\le q<1,\\ 0 & \text{otherwise.}\end{cases}$$

One can see that X and Y are positively quadrant dependent.

7. If

$$F(x,y)=\begin{cases}0 & \text{if } (x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+,\\ \frac{1}{2}x & \text{if } 0\le x\le 1,\ 0<y\le 1,\\ \frac{1}{2}x & \text{if } 0\le x\le \frac{1}{2},\ y>1,\\ \frac{1}{2} & \text{if } x>1,\ 0<y\le 1,\\ x-\frac{1}{4} & \text{if } \frac{1}{2}<x\le 1,\ y>1,\\ \frac{1}{2}x+\frac{1}{4} & \text{if } 1<x\le\frac{3}{2},\ y>1,\\ 1 & \text{otherwise,}\end{cases}$$

then

$$x_q = \begin{cases} 2q & \text{if } 0 < q \le \frac{1}{4}, \\ q + \frac{1}{4} & \text{if } \frac{1}{4} < q \le \frac{3}{4}, \\ 2q - \frac{1}{2} & \text{if } \frac{3}{4} < q < 1, \end{cases} \qquad y_p = \begin{cases} 0 & \text{if } 0 < p < \frac{1}{2}, \\ 1 & \frac{1}{2} \le p < 1, \end{cases}$$

$$\overline{\overline{x}}_{q|p} = \begin{cases} q & \text{if } 0 < p < \frac{1}{2}, \\ q + \frac{1}{2} & \text{if } \frac{1}{2} \le p < 1, \end{cases}$$

$$x^{(1)}_{q|p} = \begin{cases} q + \frac{1}{2} & \text{if } 0 < p < \frac{1}{2}, \\ \infty & \text{if } \frac{1}{2} \le p < 1 \text{ (by convention)}, \end{cases}$$

$$x^{(2)}_{q|p} = \begin{cases} \infty & \text{if } 0 < p < \frac{1}{2} \text{ (by convention)}, \\ q & \text{if } \frac{1}{2} \le p < 1. \end{cases}$$

Hence,

$$\overline{\overline{\mu}}^{(1)}_{X,Y}\left(\frac{1}{2}, p\right) = \begin{cases} \frac{1}{2(1-p)} & \text{if } 0 < p < \frac{1}{2}, \\ \frac{1}{4p-1} & \text{if } \frac{1}{2} \le p < 1, \end{cases}$$

$$\overline{\overline{\mu}}^{(2)}_{X,Y}\left(\frac{1}{2}, p\right) = \begin{cases} \frac{1}{3-4p} & \text{if } 0 < p < \frac{1}{2}, \\ \frac{1}{2p} & \text{if } \frac{1}{2} \le p < 1. \end{cases}$$

5. Applications. We only point out one application of our concept. Note that the property (7) of Theorem 1 furnishes a qualitative and at the same time quantitative characteristic of dependence for many classes of positive (negative) quadrant dependent random variables. It is known that the sequences $\{M_n = \max(X_1, \ldots, X_n) : n \ge 1\}$, $\{m_n = \min(X_1, \ldots, X_n) : n \ge 1\}$, $\{S_n = \sum_{j=1}^n X_j : n \ge 1\}$ and $\{S_n^* = \max(S_1, \ldots, S_n) : n \ge 1\}$ are sequences of associative and hence positively (negatively) quadrant dependent random variables when $\{X_n : n \ge 1\}$ is a sequence of independent random variables (cf. [1], [2], [8]). We shall see that in particular cases we can have quantitative characteristics of dependence for the above classes of random variables.

Example 1. Let U and V be independent random variables with continuous and strictly monotone distribution functions.

(i) Write $X = U$, $Y = \max(U, V)$, and put $F_X = F$. Then

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} 1 \quad \text{if } q \ge (F(y_p) - p)/(1-p), \\ \dfrac{x_{q(1+p(1-F(y_p))/(F(y_p)-p))} - x_q}{x_{q+(1-q)p} - x_q} \quad \text{otherwise}; \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \frac{x_q - x_{qF(y_p)}}{x_q - x_{qp}}, \qquad (q,p) \in (0,1) \times (0,1),$$

and if $F_U = F_V$ we have

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} 1 & \text{if } q \ge \sqrt{p}/(1+\sqrt{p}), \\ \dfrac{x_{q(1+\sqrt{p})} - x_q}{x_{q+(1-q)p} - x_q} & \text{otherwise;} \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \frac{x_q - x_{q\sqrt{p}}}{x_q - x_{qp}}, \qquad (q,p) \in (0,1)\times(0,1).$$

In the special case with $F(x) = 1/2 + (1/\pi)\arctan(x/t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}\backslash\{0\}$ (a Cauchy distribution function), we see that $x_q = t \cdot \tan[(q-1/2)\pi]$, and

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} 1 \quad \text{if } q \ge \sqrt{p}/(1+\sqrt{p}), \\ \dfrac{\sin q\sqrt{p}\pi \cos(1/2-(1-p)(1-q))\pi}{\sin[(1-q)p\pi]\cos[q(1+\sqrt{p})-1/2]\pi} & \text{otherwise;} \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \frac{\sin q(1-\sqrt{p})\pi \cos(qp-1/2)\pi}{\sin q(1-p)\pi \cos(q\sqrt{p}-1/2)\pi}, \qquad (q,p) \in (0,1)\times(0,1).$$

(ii) Define $X = U, Y = \min(U,V)$, and set $F_X = F$. Then

$$\mu^{(1)}_{X,Y}(q,p) = \frac{x_{q+(1-q)F(y_p)} - x_q}{x_{q+(1-q)p} - x_q}, \qquad (q,p) \in (0,1)\times(0,1);$$

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} 1 \quad \text{if } q \le F(y_p)/p, \\ \dfrac{x_q - x_{q-(1-p)(F(y_p)/p)(1-q)/(1-F(y_p)/p)}}{x_q - x_{qp}} & \text{otherwise,} \end{cases}$$

and when $F_U = F_V$ we have

$$\mu^{(1)}_{X,Y}(q,p) = \frac{x_{q+(1-q)p/(1+\sqrt{1-p})} - x_q}{x_{q+(1-q)p} - x_q}, \qquad (q,p) \in (0,1)\times(0,1);$$

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} 1 & \text{if } q \le 1/(1+\sqrt{1-p}), \\ \dfrac{x_q - x_{q-(1-q)\sqrt{1-p}}}{x_q - x_{qp}} & \text{otherwise.} \end{cases}$$

In the special case with $F(x) = 1 - 1/x$, $x \ge 1$ (a Pareto distribution), we see that $x_q = 1/(1-q)$, and

$$\mu^{(1)}_{X,Y}(q,p) = (1-p)^{3/2}/(p(1+\sqrt{1-p})), \qquad (q,p) \in (0,1)\times(0,1);$$

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} 1 & \text{if } q \le 1/(1+\sqrt{1-p}), \\ (1-qp)\sqrt{1-p}/(q(1+\sqrt{1-p})) & \text{if } q > 1/(1+\sqrt{1-p}). \end{cases}$$

Letting $p = 3/4$ we have

$$\mu^{(1)}_{X,Y}(q,3/4) = 1/9, \qquad q \in (0,1);$$

$$\mu^{(2)}_{X,Y}(q,3/4) = \begin{cases} 1 & \text{if } q \le 2/3, \\ (4-3q)/(12q) & \text{if } q > 2/3. \end{cases}$$

(iii) Assume that (cf. [7])

$$F_U(u) = \begin{cases} e^u & \text{if } u < 0, \\ 1 & \text{if } u \ge 0, \end{cases} \qquad F_V(v) = \begin{cases} 0 & \text{if } v < 0, \\ 1 - e^{-v} & \text{if } v \ge 0, \end{cases}$$

and put $X = U$, $Y = U + V$. Then

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} \dfrac{\ln(4p(1-p)/q)}{2\ln(1+(1-q)p/q)} & \text{for } q < p/(1-p),\ p \in (0,1/2), \\ 1 & \text{for } q \ge p(1-p),\ p \in (0,1/2), \\ \dfrac{\ln(1/q)}{2\ln(1+(1-q)p/q)} & \text{for } q \in (0,1),\ p \in [1/2,1); \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} \dfrac{\ln(1+\sqrt{1-q})/(2p)}{\ln(1/p)} & \text{for } q \in (0,1),\ p \in (0,1/2), \\ \dfrac{\ln(1+\sqrt{1-4qp(1-p)})/(2p)}{\ln(1/p)} & \text{for } q \in (0,1),\ p \in [1/2,1). \end{cases}$$

(iv) Assume that U and V are independent random variables uniformly distributed on $(0,1)$, and put $X = U$, $Y = U + V$. Then

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} 1 \quad \text{for } q > (\sqrt{2p}-p)/(1-p),\ p \in (0,1/2), \\ (\sqrt{(1-\sqrt{2p})^2 + 2q(1-p)} - (1-\sqrt{2p}) - q)/((1-q)p) \\ \qquad\qquad \text{for } q \le (\sqrt{2p}-p)/(1-p),\ p \in [1/2,1), \\ (1-\sqrt{2(1-p)}/(1+\sqrt{q}))/p \quad \text{for } q \in (0,1),\ p \in [1/2,1); \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \begin{cases} (1-\sqrt{2p}/(1+\sqrt{1-q}))/(1-p) \quad \text{for } q \in (0,1),\ p \in (0,1/2), \\ \Big[\sqrt{(1-\sqrt{2(1-p)})^2 + 2(1-q)p} - (1-\sqrt{2(1-p)}) - (1-q)\Big] \\ \qquad\qquad \times (q(1-p))^{-1} \\ \qquad\qquad \text{for } q \ge (1-\sqrt{2(1-p)})/p,\ p \in [1/2,1), \\ 1 \qquad\qquad \text{for } q \le (1-\sqrt{2(1-p)})/p,\ p \in [1/2,1). \end{cases}$$

Moreover, we note that in this case

$$\mu^{(1)}_{X,Y}(p,q) = \mu^{(2)}_{X,Y}(1-p,1-q).$$

R e m a r k. Note that the correlation coefficient $\varrho_{X,Y} = 1/\sqrt{2}$ equals one of the values of the monotone dependence function. Observe that a "complete dependence" ($\mu^{(1)}_{X,Y}(q,p) = \mu^{(2)}_{X,Y}(1-q,1-p) = 1 = \sqrt{2}\varrho_{X,Y}$) occurs only when $q > (\sqrt{2p}-p)/(1-p)$ if $p \in (0,1/2)$.

EXAMPLE 2. Let $\mathbb{X} = (X_1,\ldots,X_n)$ be a random sample from a population having continuous and strictly increasing distribution functions. Put $X_{(1)} = \min(X_1,\ldots,X_n)$, $X_{(n)} = \max(X_1,\ldots,X_n)$. The following quantities show us that $X_{(1)}$ and $X_{(n)}$ are quadrant positively dependent random variables. For instance, in the case when $X := \min(X_1, X_2)$ and $Y := \max(X_1, X_2)$ the degree of that dependence is characterized by

$$\mu^{(1)}_{X,Y}(q,p) = \begin{cases} \dfrac{x_{q(1+\sqrt{p})/2} - x_{q/(1+\sqrt{1-q})}}{x_{(q+(1-q)p)/(1+\sqrt{(1-p)(1-q)})} - x_{q/(1+\sqrt{1-q})}} & \text{if } q < \sqrt{p}/(1+\sqrt{p}), \\ \dfrac{x_{(q+(1-q)p)/(1+\sqrt{(1-p)(1-q)})} - x_{q/(1+\sqrt{1-q})}}{x_{q/(1+\sqrt{1-q})} - x_{qp/(1+\sqrt{1-qp})}} & \text{if } q \ge \sqrt{p}/(1+\sqrt{p}); \end{cases}$$

$$\mu^{(2)}_{X,Y}(q,p) = \frac{x_{q/(1+\sqrt{1-q})} - x_{q\sqrt{p}/(1+\sqrt{1-q})}}{x_{q/(1+\sqrt{1-q})} - x_{qp/(1+\sqrt{1-qp})}}, \qquad (q,p) \in (0,1) \times (0,1).$$

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