APPLICATIONES MATHEMATICAE
23,1 (1995), pp. 37–49

S. TRYBUŁA (Wrocław)

A NOISY DUEL UNDER ARBITRARY MOTION. VII

1. Definitions and assumptions. In [17], [18] and in this paper an m versus n bullets noisy duel is considered in which duelists can move at will. It is assumed that Player I has greater maximal speed. The cases $m = 1, 2, 3$, $n = 1, 2, 3$ are solved. Let a be the point where Player I is at the beginning of the duel, $0 \le a < 1$ (Player II is at 1). In contrast to [11]–[16] where the duels are solved for small a, now we solve the duels for any $0 \le a < 1$.

In this paper we consider the cases $m = 1$, $n = 1$; $m = 1$, $n = 2$; $m = 2$, $n = 1$.

Let us define a game which will be called the *game* (m, n). Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \ge 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at the beginning of the duel the players are at distance 1 from each other and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys the opponent) if he fires when the distance between the players is $1 - s$. We assume that $P(s)$ is increasing and continuous in $[0, 1]$, has continuous second derivative in $(0, 1)$, and $P(s) = 0$ for $s \le 0$, $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The game is over if at least one player is destroyed or all bullets are shot. In the other case the duel lasts infinitely long and the payoff is zero.

The duel is noisy—each player hears every shot of his opponent.

As will be seen from the sequel, without loss of generality we can assume that Player II is motionless. Then $v_1 = 1$ and $v_2 = 0$. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at 1.

1991 *Mathematics Subject Classification*: Primary 90D26.
Key words and phrases: noisy duel, game of timing, zero-sum game.

We suppose that between successive shots of the same player there has to pass a time $\widehat{\varepsilon} > 0$. We also assume that the reader knows the papers [11]–[16] and remembers the definitions, notations and assumptions made there.

For general definitions and notations in the theory of games of timing see [3], [19]. For other results see [1], [2], [4]–[10], [20].

2. The duel (1, 1). Consider the case where Players I and II have one bullet each.

Let $K(\widehat{\xi}, \widehat{\eta})$ be the payoff function (the expected gain for Player I) for strategies $\widehat{\xi}$ and $\widehat{\eta}$ of Players I and II respectively and let a' mean (the strategy) that Player I (II) fires at distance $1 - a'$ if his opponent has not fired before. Any moment of time when Player I has been at a' will be denoted by $\langle\langle a' \rangle\rangle$. Since there can be many such moments, we denote by $\langle a' \rangle$ the earliest one.

Let $a_{mn} \in [0, 1]$. Denote by a^{ε}_{mn} a random moment of time with

$$\langle a_{mn} \rangle \le a^{\varepsilon}_{mn} \le \langle a_{mn} \rangle + \alpha(\varepsilon),$$

distributed according to an absolutely continuous probability distribution in the above interval, with $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Case 1: $Q(a) \ge Q(a_{11}) = 2 - \sqrt{2}$, $Q(s) = 1 - P(s)$. Consider the following strategies ξ and η of Players I and II.

Strategy of Player I. Reach the point a_{11}, and if Player II has not fired before, fire a shot at time a^{ε}_{11}.

Strategy of Player II. Fire at the earliest moment when Player I reaches the point a_{11} (i.e. at time $\langle a_{11} \rangle$). If he does not reach this point, do not fire.

The number a_{11} satisfies the condition

$$P(a_{11}) = \sqrt{2} - 1.$$

In [11] it is proved that ξ and η are ε-optimal strategies of Players I and II and the value of the game is

$$v_{11} = 1 - 2P(a_{11}) = 3 - 2\sqrt{2}.$$

3. Further definitions and assumptions. When Player I has fired all his bullets, his motion towards his opponent loses sense. Therefore we shall always assume that Player I escapes with maximal speed after firing all his bullets.

Suppose that Player I has fired all his bullets and he is escaping. In this case the best for Player II is to fire all his bullets immediately after the last shot of Player I. If, on the other hand, Player II has fired all his bullets and Player I survives and has bullets yet, the best for him is to reach the opponent and to achieve success surely.

Suppose now that the duel (m, n) begins when the distance between the players is $1 - a$. This duel will be denoted by $(m, n), \langle a \rangle$. To simplify considerations we count the time also from $t = a$. All other assumptions about the duel (m, n) made at the beginning of the paper also hold for the duel $(m, n), \langle a \rangle$.

We say that Player I *assures the value* u_1 *in limit* if for each $\varepsilon \geq 0$ and $\widehat{\varepsilon} > 0$ he has a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ such that

$$K(\xi_{\varepsilon\hat{\varepsilon}}, \widehat{\eta}) \geq u_1 - k_1(\varepsilon, \widehat{\varepsilon})$$

for any strategy $\widehat{\eta}$ of Player II, where $k_1(\varepsilon, \widehat{\varepsilon})$ tends to 0 as $\varepsilon \to 0$ and $\widehat{\varepsilon} \to 0$.

Similarly, Player II *assures the value* u_2 *in limit* if for each $\varepsilon \geq 0$ and $\widehat{\varepsilon} > 0$ he has a strategy $\eta_{\varepsilon\hat{\varepsilon}}$ such that

$$K(\widehat{\xi}, \eta_{\varepsilon\hat{\varepsilon}}) \leq u_2 + k_2(\varepsilon, \widehat{\varepsilon})$$

for any strategy $\widehat{\xi}$ of Player I, where $k_2(\varepsilon, \widehat{\varepsilon})$ tends to zero as $\varepsilon \to 0$ and $\widehat{\varepsilon} \to 0$.

Other notions defined below can be defined in a wider context. Since I want to be understood also by people not working in game theory, I define these notions in a simpler way but under the following additional assumption (satisfied in the paper):

(C) Players I and II assure the same value v^a_{mn} in limit.

The number v^a_{mn} will be called the *limit value* of the game.

Suppose that there is a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ of Player I in the duel $(m, n), \langle a \rangle$ assuring the value v^a_{mn} in limit. This strategy will be called *optimal* or *maximin in limit*.

Similarly we define a strategy of Player II which is optimal or minimax in limit.

If we have additionally

$$\lim_{\hat{\varepsilon} \to 0} k_1(\varepsilon, \widehat{\varepsilon}) \leq \varepsilon$$

then such a strategy is called ε*-optimal in limit*.

Consider a family $\mathcal{F}$ of strategies such that for each $\varepsilon \geq 0$ and $\widehat{\varepsilon} > 0$ there is a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ in $\mathcal{F}$ which is ε-optimal in limit. In the paper we only consider families $\mathcal{F}$ of strategies containing for each $\widehat{\varepsilon} > 0$ a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ such that $\varepsilon < \delta(\widehat{\varepsilon})$ and

$$\lim_{\hat{\varepsilon} \to 0} \delta(\widehat{\varepsilon}) = 0.$$

If Player I has such a family of strategies at his disposal, then he has a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ optimal in limit.

A similar fact is also true for Player II.

Now we determine strategies optimal in limit for Players I and II in the game $(1,1),\langle a\rangle$.

Case 2: $2-\sqrt{2}\ge Q(a)\ge 1/2$. Consider the following strategies ξ and η of Players I and II.

Strategy of Player I. If Player II has not fired before, fire at time a^ε and escape.

Strategy of Player II. Fire at time $\langle a\rangle$.

Now

$$v_{11}^a = 1-2P(a).$$

Suppose that Player II playing against ξ fires at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we obtain

$$K(\xi,\widehat{\eta}) = -P(a)+1-P(a) = 1-2P(a) = v_{11}^a.$$

Suppose that Player II has not fired before $\langle a\rangle+\alpha(\varepsilon)$. For such a strategy $\widehat{\eta}$ we obtain

$$K(\xi,\widehat{\eta}) \ge P(a)-(1-P(a))P(a)-k(\widehat{\varepsilon}) = P^2(a)-k(\widehat{\varepsilon}) \ge 1-2P(a)-k(\widehat{\varepsilon}).$$

On the other hand, suppose that Player I fires at $\langle a\rangle$. For such a strategy $\widehat{\xi}$,

$$K(\widehat{\xi},\eta) = 0 \le 1-2P(a)$$

if $P(a)\le 1/2$.

Suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$,

$$K(\widehat{\xi},\eta) \le 1-2P(a)+k(\widehat{\varepsilon}).$$

Thus in this case the strategies ξ and η are optimal in limit and $v_{11}^a = 1-2P(a)$.

Case 3: $Q(a)\le 1/2$. Consider ξ and η defined as follows:

Strategy of Player I. Fire at time $\langle a\rangle$ and escape.

Strategy of Player II. Fire at time $\langle a\rangle$.

Now

$$v_{11}^a = 0.$$

Suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we have

$$K(\xi,\widehat{\eta}) \ge P(a)-(1-P(a))P(a)-k(\widehat{\varepsilon}) = P^2(a)-k(\widehat{\varepsilon}) \ge v_{11}-k(\widehat{\varepsilon}).$$

Suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi},\eta) = 1-2P(a) \le v_{11}^a$$

if $P(a)\ge 1/2$.

Thus also in this case the strategies ξ and η are optimal in limit and the limit value of the game is $v_{11}^a = 0$.

4. The duels $(m,n), \langle 1, a \wedge c, a\rangle$ and $(m,n), \langle 2, a, a \wedge c\rangle$. Case $m = n = 1$. We have supposed that a time $\widehat{\varepsilon}$ has to elapse between successive shots of the same player. Let

$$(m,n), \langle 2, a, a \wedge c\rangle, \qquad 0 < c \le \widehat{\varepsilon},$$

be the duel in which Player I has m bullets, Player II has n bullets but if $c < \widehat{\varepsilon}$, then Player I can fire his bullets from time $\langle a\rangle$ on, and Player II from time $\langle a\rangle + c$ on. If $c = \widehat{\varepsilon}$ the rule is the same except that Player I is not allowed to fire at time $\langle a\rangle$.

Similarly we define the duel $(m,n), \langle 1, a \wedge c, a\rangle$.

For the properties of the duels $(m,n), \langle 1, a \wedge c, a\rangle$ and $(m,n), \langle 2, a, a \wedge c\rangle$ see [12], Section 5.

Now we determine strategies optimal in limit for the duel $(1,1), \langle 1, a \wedge c, a\rangle$.

Case 1: $Q(a) \ge 2 - \sqrt{2}$. The strategies ξ and η defined in the duel $(1,1), \langle a\rangle$ for $Q(a) \ge 2 - \sqrt{2}$ are now also optimal in limit.

Case 2: $Q(a) \le 2 - \sqrt{2}$. Let $\rangle t\langle$ be the point where Player I was at time t. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at time $\rangle\langle a\rangle + c\langle^{\varepsilon}$ and escape.

Strategy of Player II. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$.

Let $\overset{1}{v}{}_{mn}^{a}$ and $\overset{2}{v}{}_{mn}^{a}$ be the limit values of the games $(m,n), \langle 1, a \wedge c, a\rangle$ and $(m,n), (2, a, a \wedge c)$, respectively. We prove that in the considered case the strategies ξ and η are optimal in limit and

$$\overset{1}{v}{}_{11}^{a} = 1 - 2P(a).$$

For each strategy $\widehat{\xi}$ of Player I we obtain

$$K(\widehat{\xi}, \eta) \le -P(a) + 1 - P(a) + k(\widehat{\varepsilon}) = \overset{1}{v}{}_{11}^{a} + k(\widehat{\varepsilon}).$$

On the other hand, if Player II fires before $\langle a\rangle + c$ then

$$K(\xi, \widehat{\eta}) \ge 1 - 2P(a) - k(\widehat{\varepsilon}) = \overset{1}{v}{}_{11}^{a} - k(\widehat{\varepsilon}).$$

If Player II has not fired before $\rangle\langle a\rangle + c\langle + \alpha(\varepsilon)$ we obtain

$$K(\xi, \widehat{\eta}) \ge P(a) - (1 - P(a))P(a) - k(\widehat{\varepsilon}) = P^2(a) - k(\widehat{\varepsilon}) \ge 1 - 2P(a) - k(\widehat{\varepsilon})$$

if $a \ge a_{11}$.

Thus the strategies ξ and η are optimal in limit and the limit value of the game is $\overset{1}{v}{}_{11}^{a} = 1 - 2P(a)$.

The duel $(1,1), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \ge 2-\sqrt{2}$. The strategies ξ and η defined in the duel $(1,1), \langle a\rangle$ for $Q(a) \ge 2-\sqrt{2}$ are now also optimal in limit.

Case 2: $Q(a) \le 2-\sqrt{2}$. Define ξ and η as follows:

Strategy of Player I. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$, and escape.

Strategy of Player II. If Player I has not fired before, fire at $\langle a\rangle + c$.

We now prove that

$$\overset{2}{v}{}^a_{11} = P^2(a).$$

For each strategy $\widehat{\eta}$ of Player I we obtain

$$K(\xi, \widehat{\eta}) \ge P(a) - (1-P(a))P(a) - k(\widehat{\xi}) = P^2(a) - k(\widehat{\varepsilon}).$$

On the other hand, if Player I fires at $\langle a\rangle + c$ (call this strategy $\widehat{\xi}$) then

$$K(\widehat{\xi}, \eta) \le k(\widehat{\varepsilon}) \le P^2(a) + k(\widehat{\varepsilon}).$$

Finally, if Player I has not fired before or at $\langle a\rangle + c$ we obtain

$$K(\widehat{\xi}, \eta) \le 1 - 2P(a) + k(\widehat{\varepsilon}) \le P^2(a) + k(\widehat{\varepsilon})$$

if $a \ge a_{11}$.

Thus the strategies ξ and η are optimal in limit and the limit value of the game is $\overset{2}{v}{}^a_{11} = P^2(a)$.

5. Results for the duel (1,1). Let $Q(a) = 1 - P(a)$. We have

$$\overset{1}{v}{}^a_{11} = \begin{cases} v_{11} = 3 - 2\sqrt{2} \cong 0.171573 & \\ \qquad & \text{if } Q(a) \ge Q(a_{11}) = 2-\sqrt{2} \cong 0.585787, \\ 1 - 2P(a) & \text{if } Q(a) \le Q(a_{11}); \end{cases}$$

$$v^a_{11} = \begin{cases} 3 - 2\sqrt{2} & \text{if } Q(a) \ge Q(a_{11}), \\ 1 - 2P(a) & \text{if } Q(a_{11}) \ge Q(a) \ge 1/2, \\ 0 & \text{if } Q(a) \le 1/2; \end{cases}$$

$$\overset{2}{v}{}^a_{11} = \begin{cases} 3 - 2\sqrt{2} & \text{if } Q(a) \ge Q(a_{11}), \\ P^2(a) & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

6. The duel $(m,1), \langle a\rangle$**,** $m \ge 2$**.** Consider the case where Player I has m bullets, $m \ge 2$, and Player II has one bullet. In this case we define strategies ξ and η of these players as follows.

Strategy of Player I. Reach the point a_{m1} and if Player II has not fired before, fire a shot at $\langle a_{m1}\rangle$ and play ε-optimally the resulting duel.

STRATEGY OF PLAYER II. If Player I reaches the point a_{m1} and has not fired, fire a shot at a^{ε}_{m1}. If he has not reached this point, do not fire.

The number a_{m1} satisfies the condition

$$P(a_{m1}) = \frac{P(a_{11})}{1 + (m-1)P(a_{11})}.$$

"Play ε-optimally" means: apply an ε-optimal strategy.

In [12] it is proved that if $m \geq 2$ and $a \leq a_{m1}$ then the above strategy ξ is ε-maximin and the strategy η is ε-minimax (for properly chosen $\alpha(\varepsilon)$). The value of the game $(m,1), \langle a \rangle = (m,1)$ is given by the formula

$$v^a_{m1} = v_{m1} = \frac{1 + (m-3)P(a_{11})}{1 + (m-1)P(a_{11})}.$$

7. The duel (1, 2)

The duel $(1,2), \langle a \rangle$. Cases 1 and 2 are solved in [12].

Case 3: $Q(a) \leq Q(a_{12}) \cong 0.730812$. Define ξ and η as follows:

STRATEGY OF PLAYER I. Fire at $\langle a \rangle$ and escape.

STRATEGY OF PLAYER II. Fire at $\langle a \rangle$ and play optimally the resulting duel.

We have

$$v^a_{12} = -Q^2(a)P(a).$$

"Play optimally" means: apply a strategy optimal in limit.

Suppose Player II does not fire at $\langle a \rangle$. By assumption, he fires immediately after the shot of Player I. Then we have

$$K(\xi, \widehat{\eta}) \geq P(a) - Q(a)(1 - Q^2(a)) - k(\widehat{\varepsilon}) \geq -Q^2(a)P(a) - k(\widehat{\varepsilon}).$$

On the other hand, suppose Player I does not fire at $\langle a \rangle$. We have

$$\begin{aligned} K(\widehat{\xi}, \eta) &\leq -P(a) + Q(a)\, \overset{2}{v}{}^a_{11} + k(\widehat{\varepsilon}) \\ &= -1 + (1 + v_{11})Q(a) + k(\widehat{\varepsilon}) \leq -Q^2(a)P(a) + k(\widehat{\varepsilon}) \end{aligned}$$

if $0.585787 \cong Q(a_{11}) \leq Q(a) \leq Q(a_{12}) \cong 0.730842$ and

$$\begin{aligned} K(\widehat{\xi}, \eta) &\leq -P(a) + Q(a)\, \overset{2}{v}{}^a_{11} + k(\widehat{\varepsilon}) \\ &= -1 + 2Q(a) - 2Q^2(a) + Q^3(a) + k(\widehat{\varepsilon}) \\ &\leq -Q^2(a) + Q^3(a) + k(\widehat{\varepsilon}) \end{aligned}$$

if $Q(a) \leq Q(a_{11})$.

Thus if $Q(a) \leq Q(a_{12})$ then the strategies ξ and η are optimal in limit and the limit value of the game is $v^a_{12} = -Q^2(a) + Q^3(a)$.

The duel $(1,2), \langle 1, a \wedge c, a \rangle$. Cases 1 and 2 are solved in [12].

Case 3: $Q(a) \le Q(a_{11}) = 0.585787$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at time $\langle a\rangle + c$ and escape. If he fired, play optimally the resulting duel.

Strategy of Player II. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$, and play optimally the resulting duel.

Now

$$\overset{1}{v}{}^{a}_{12} = -P(a) + Q(a)\,\overset{2}{v}{}^{a}_{11} = -1 + 2Q(a) - 2Q^2(a) + Q^3(a) \tag{1}$$

if $Q(a) \le Q(a_{11})$.

It is easy to see that Player II always assures this value.

On the other hand, suppose that Player II also fires at $\langle a\rangle + c$. For such a strategy $\widehat{\eta}$ we obtain

$$\begin{aligned} K(\xi,\widehat{\eta}) &\ge -Q^2(a)P(a) - k(\widehat{\varepsilon}) \\ &\ge -1 + 2Q(a) - 2Q^2(a) + Q^3(a) - k(\widehat{\varepsilon}) = \overset{1}{v}{}^{a}_{12} - k(\widehat{\varepsilon}) \end{aligned}$$

if $Q(a) \le Q(a_{11})$.

Finally, suppose that Player II does not fire before or at $\langle a\rangle + c$. We obtain

$$\begin{aligned} K(\xi,\widehat{\eta}) &\ge P(a) - Q(a)(1 - Q^2(a)) - k(\widehat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^3(a) - k(\widehat{\varepsilon}) \\ &\ge -1 + 2Q(a) - 2Q^2(a) + Q^3(a) - k(\widehat{\varepsilon}) \end{aligned}$$

if $Q(a) \le Q(a_{11})$.

Thus ξ and η are optimal in limit and the limit value of the game is given by (1).

The duel $(1,2), \langle 2, a, a \wedge c\rangle$. Also now, Cases 1 and 2 are solved in [12].

Case 3: $Q(a) \le Q(\check{a}_{12}) \cong 0.780539$. Define ξ and η as follows:

Strategy of Player I. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$, and escape.

Strategy of Player II. If Player I has not fired before, fire at time $\langle a\rangle + c$ and play optimally the resulting duel.

Now we have

$$\overset{2}{v}{}^{a}_{12} = P(a) - Q(a)(1 - Q^2(a)) = 1 - 2Q(a) + Q^3(a).$$

For any strategy $\widehat{\eta}$ of Player I,

$$K(\xi,\widehat{\eta}) \ge P(a) - Q(a)(1 - Q^2(a)) - k(\widehat{\varepsilon}) = \overset{2}{v}{}^{a}_{12} + k(\widehat{\varepsilon}).$$

On the other hand, if Player I also fires at $\langle a\rangle + c$ then

$$\begin{aligned} K(\widehat{\xi},\eta) &\le -Q^2(a)P(a) + k(\widehat{\varepsilon}) \\ &\le 1 - 2Q(a) + Q^3(a) + k(\widehat{\varepsilon}) = \overset{2}{v}{}^{a}_{12} + k(\widehat{\varepsilon}). \end{aligned}$$

Finally, if Player I does not fire before or at $\langle a\rangle + c$ then

$$K(\widehat{\xi},\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{11} + k(\widehat{\varepsilon}) = -1 + (1+v_{11})Q(a) + k(\widehat{\varepsilon})$$
$$\le 1 - 2Q(a) + Q^3(a) + k(\widehat{\varepsilon})$$

if $Q(a) \ge Q(a_{11})$ and

$$S(Q) = Q^3(a) - (3+v_{11})Q(a) + 2 \ge 0. \tag{2}$$

We have

$$S'(Q) < 0, \quad S(Q(a_{11})) = 0.343145, \quad S(Q(\breve{a}_{12})) = S(0.780539) = 0.$$

Thus inequality (2) holds if

$$Q(a_{11}) \le Q(a) \le Q(\breve{a}_{12}) = 0.780539.$$

When $Q(a) \le Q(a_{12})$ we obtain

$$K(\widehat{\xi},\eta) \le P(a) + Q(a)\overset{2}{v}{}^{a}_{11} + k(\widehat{\varepsilon})$$
$$= -1 + 2Q(a) - 2Q^2(a) + Q^3(a) + k(\widehat{\varepsilon})$$
$$\le 1 - 2Q(a) + Q^3(a) + k(\widehat{\varepsilon}).$$

Thus if $Q(a) \le Q(\breve{a}_{12}) \cong 0.780539$ then the strategies ξ and η are optimal in limit and $\overset{2}{v}{}^{a}_{12} = 1 - 2Q(a) + Q^3(a)$ is the limit value of the game.

8. Results for the duel (1,2). We have

$$\overset{1}{v}{}^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \ge Q(a_{12}) \cong 0.853553, \\ -1 + (1+v_{11})Q(a) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787 \text{ (see [12])}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^3(a) & \text{if } Q(a) \le Q(a_{11}); \end{cases}$$

$$v^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1+v_{11})Q(a) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\widehat{a}_{12}) \cong 0.730812 \text{ (see [12])}, \\ -Q^2(a) + Q^3(a) & \text{if } Q(a) \le Q(\widehat{a}_{12}); \end{cases}$$

$$\overset{2}{v}{}^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \ge Q(a_{12}), \\ -1 + (1+v_{11})Q(a) & \\ & \text{if } Q(a_{12}) \ge Q(a) \ge Q(\breve{a}_{12}) \cong 0.780539 \text{ (see [12])}, \\ 1 - 2Q(a) + Q^3(a) & \text{if } Q(a) \le Q(\breve{a}_{12}). \end{cases}$$

9. The duel (2,1)

The duel $(2,1), \langle a\rangle$

Case 1: $Q(a)\ge Q(a_{21})\cong 0.707107$. The optimal (in limit) strategies ξ and η are given in Section 6. For these strategies,

$$(3)\quad Q(a_{21})=\frac{\sqrt{2}}{2}\cong 0.707107,\qquad v_{12}^{a}=\frac{1-P(a_{11})}{1+P(a_{11})}=\sqrt{2}-1\cong 0.414214.$$

Case 2: $0.668179\cong Q(\widehat{a}_{21})\le Q(a)\le Q(a_{21})$. Define strategies ξ and η of Players I and II as follows:

Strategy of Player I. Fire at $\langle a\rangle$ and play optimally afterwards.

Strategy of Player II. If Player I has not fired before, fire at a^{ε}.

Now

$$v_{21}^{a}=P(a)+Q(a)\overset{1}{v}{}_{11}^{a}.$$

Suppose that Player II fires at $\langle a\rangle$. We obtain

$$K(\xi,\widehat{\eta})\ge Q^{2}(a)-k(\widehat{\varepsilon})\ge P(a)+Q(a)\overset{1}{v}{}_{11}^{a}-k(\widehat{\varepsilon})=1-(1-v_{11})Q(a)-k(\widehat{\varepsilon})$$

if $Q(a)\ge Q(a_{11})$. Then

$$Q^{2}(a)+(1-v_{11})Q(a)-1\ge 0$$

if $Q(a)\ge Q(a_{11})$. This inequality is satisfied if

$$Q(a)\ge Q(\widehat{a}_{21})\cong 0.668179.$$

On the other hand, suppose that Player I does not fire before $\langle a\rangle+\alpha(\varepsilon)$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi},\eta')\le 1-2P(a)+k(\widehat{\varepsilon})\le 1-(1-v_{11})Q(a)+k(\widehat{\varepsilon})$$

if

$$Q(a)\le\frac{2}{3-v_{11}}=\frac{\sqrt{2}}{2}\cong 0.707107.$$

Thus if

$$0.668179\cong Q(a_{21})\le Q(a)\le Q(a_{21})\cong 0.707107$$

then the strategies ξ and η are optimal in limit and the limit value of the game is

$$v_{21}^{a}=1-(1-v_{11})Q(a).$$

Case 3: $Q(a)\le Q(\widehat{a}_{21})\cong 0.668179$. Define ξ and η as follows:

Strategy of Player I. Fire at $\langle a\rangle$ and play optimally afterwards.

Strategy of Player II. Fire at $\langle a\rangle$.

Now

$$v_{21}^{a}=Q^{2}(a).$$

Suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\widehat{\eta}$ we obtain

$$K(\xi,\widehat{\eta}) \geq P(a)+Q(a)\,\overset{1}{v}{}^{a}_{11}-k(\widehat{\varepsilon}) = 1-2Q(a)+2Q^2(a)-k(\widehat{\varepsilon}) \geq Q^2(a)-k(\widehat{\varepsilon})$$

if $Q(a) \leq Q(a_{11}) \cong 0.585787$, and

$$K(\xi,\widehat{\eta}) \geq P(a)+Q(a)v_{11}-k(\widehat{\varepsilon}) \geq Q^2(a)-k(\widehat{\varepsilon}),$$

i.e.

$$Q^2(a)-(1-v_{11})Q(a)-1 \leq 0$$

if $Q(a) \geq Q(a_{11})$.

Solving the above inequality under the condition $Q(a) \geq Q(a_{11})$ we obtain

$$Q(a_{11}) \leq Q(a) \leq Q(\widehat{a}_{21}) \cong 0.668179.$$

On the other hand, suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\widehat{\xi}$ we obtain

$$K(\widehat{\xi},\eta) \leq 1-2P(a)+k(\widehat{\varepsilon}) = 2Q(a)-1+k(\widehat{\varepsilon}) \leq Q^2(a)+k(\widehat{\varepsilon}),$$

which is always satisfied.

The duel $(2,1), \langle 1, a\wedge c, a\rangle$

Case 1: $Q(a) \geq Q(a_{21}) = \sqrt{2}/2 \cong 0.707107$. This case is solved in [12].

Case 2: $Q(a) \leq Q(a_{21})$. Define ξ and η as follows:

Strategy of Player I. If Player II has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II. Fire at t, $\langle a\rangle < t < \langle a\rangle + c$.

Now

$$\overset{1}{v}{}^{a}_{12} = 1-2P(a).$$

For each strategy $\widehat{\xi}$ of Player I we have

$$K(\widehat{\xi},\eta) \leq 1-2P(a)+k(\widehat{\varepsilon}).$$

On the other hand, suppose that Player II fires at $\langle a\rangle + c$. Then

$$K(\xi,\widehat{\eta}) \geq Q^2(a)-k(\widehat{\varepsilon}) \geq 1-2P(a)-k(\widehat{\varepsilon}).$$

Finally, suppose that Player II fires after $\langle a\rangle + c$ or does not fire at all. In this case

$$K(\xi,\widehat{\eta}) \geq P(a)+Q(a)v_{11}-k(\widehat{\varepsilon}) \geq 1-2P(a)+k(\widehat{\varepsilon})$$

if $Q(a) \geq Q(a_{11})$, which gives

$$Q(a) \leq \frac{2}{3-v_{11}} = \frac{\sqrt{2}}{2} \cong 0.707107.$$

Moreover,

$$\begin{aligned} K(\xi,\widehat{\eta}) &\ge P(a) + Q(a)\,\overset{1}{v}{}^{a}_{11} - k(\widehat{\varepsilon}) \\ &= 1 - Q(a) + Q(a)(-1 + 2Q(a)) - k(\widehat{\varepsilon}) \\ &\ge -1 + 2Q(a) - k(\widehat{\varepsilon}) \end{aligned}$$

if $Q(a) \le Q(a_{11})$.

Thus if $Q(a) \le Q(a_{21}) \cong 0.707107$ then the strategies ξ and η are optimal in limit and the limit value of the game is $\overset{1}{v}{}^{a}_{21} = 1 - 2P(a)$.

The duel $(2,1), \langle 2, a, a \wedge c\rangle$

Case 1: $Q(a) \ge Q(a_{21}) \cong \sqrt{2}/2$. This case is solved in [12]. See also (2) and Section 6. In the case $Q(a) \ge \sqrt{2}/2$,

$$\overset{1}{v}{}^{a}_{21} = \overset{2}{v}{}^{a}_{21} = v^{a}_{21}.$$

Case 2: $Q(a) \le Q(a_{21}) \cong 0.707107$.

Strategy of Player I. Fire at time t, $\langle a\rangle < t < \langle a\rangle + c$, and play optimally the resulting duel.

Strategy of Player II. If Player I has not fired before, fire at time $\rangle\langle a\rangle + c\langle^{\varepsilon}$. If he fired, play optimally the resulting duel.

We have

$$v^{a}_{21} = P(a) + Q(a)\,\overset{1}{v}{}^{a}_{11} \quad \text{if } Q(a) \le Q(a_{21}) \cong 0.707107.$$

The proof is omitted.

10. Results for the duel (2, 1). We have

$$\overset{1}{v}{}^{a}_{21} = \begin{cases} v_{21} = \sqrt{2} - 1 \cong 0.414214 & \\ \qquad\qquad \text{if } Q(a) \ge Q(a_{21}) = \sqrt{2}/2 \cong 0.707107, & \\ 1 - 2P(a) \quad \text{if } Q(a) \le Q(a_{21}); & \end{cases}$$

$$v^{a}_{21} = \begin{cases} \sqrt{2} - 1 \text{ if } Q(a) \ge Q(a_{21}), & \\ 1 - (1 - v_{11})Q(a) & \\ \qquad \text{if } Q(a_{21}) \ge Q(a) \ge Q(\widehat{a}_{21}) \cong 0.668179, & \\ Q^2(a) \quad \text{if } Q(a) \le Q(\widehat{a}_{21}); & \end{cases}$$

$$\overset{2}{v}{}^{a}_{21} = \begin{cases} \sqrt{2} - 1 & \text{if } Q(a) \ge Q(a_{21}), \\ 1 - (1 - v_{11})Q(a) & \text{if } Q(a_{21}) \ge Q(a) \ge Q(a_{11}) \cong 0.585787, \\ 2Q^2(a) - 2Q(a) + 1 & \text{if } Q(a) \le Q(a_{11}). \end{cases}$$

For other noisy duels see [2], [6], [11]–[18].

References

[1] A. Cegielski, *Game of timing with uncertain number of shots*, Math. Japon. 31 (1986), 503–532.

[2] M. Fox and G. Kimeldorf, *Noisy duels*, SIAM J. Appl. Math. 17 (1969), 353–361.

[3] S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. 2, Addison-Wesley, Reading, Mass., 1959.

[4] G. Kimeldorf, *Duels: an overview*, in: *Mathematics of Conflict*, North-Holland, 1983, 55–71.

[5] K. Orłowski and T. Radzik, *Discrete silent duels with complete counteraction*, Optimization 16 (1985), 419–429.

[6] T. Radzik, *General noisy duels*, Math. Japon. 36 (1991), 827–857.

[7] R. Restrepo, *Tactical problems involving several actions*, in: *Contributions to the Theory of Games*, Vol. III, Ann. of Math. Stud. 39, Princeton Univ. Press, 1957, 313–335.

[8] A. Styszyński, *An n-silent-vs.-noisy duel with arbitrary accuracy functions*, Zastos. Mat. 14 (1974), 205–225.

[9] Y. Teraoka, *A single bullet duel with uncertain information available to the duelists*, Bull. Math. Statist. 18 (1979), 69–83.

[10] S. Trybuła, *Solution of a silent duel with arbitrary motion and arbitrary accuracy function*, Optimization 27 (1993), 151–172.

[11]–[16] —, *A noisy duel under arbitrary moving. I–VI*, Zastos. Mat. 20 (1990), 491–495, 497–516, 517–530; Zastos. Mat. 21 (1991), 43–61, 63–81, 83–98.

[17]–[18] —, *A noisy duel under arbitray motion. VIII–IX*, Applicationes Math., to appear.

[19] N. N. Vorob'ev, *Foundations of the Theory of Games. Uncoalition Games*, Nauka, Moscow, 1984 (in Russian).

[20] E. B. Yanovskaya, *Duel-type games with continuous firing*, Engrg. Cybernetics 1969 (1), 15–18.

STANISŁAW TRYBUŁA
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

Received on 30.12.1993