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NON-PARALLEL PLANE RAYLEIGH BENARD CONVECTION IN CYLINDRICAL GEOMETRY

Abstract. This paper considers the effect of a perturbed wall in regard to the classical Benard convection problem in which the lower rigid surface is of the form $z = \varepsilon^2 g(s)$, $s = \varepsilon r$, in axisymmetric cylindrical polar coordinates (r, ϕ, z). The boundary conditions at $s = 0$ for the linear amplitude equation are found and it is shown that these conditions are different from those which apply to the nonlinear problem investigated by Brown and Stewartson [1], representing the distribution of convection cells near the center.

1. Introduction. The theoretical foundation of the onset of thermal instability in an infinite horizontal layer of fluid heated uniformly from below was laid by Rayleigh [7], who proved the validity of the principle of the exchange of stabilities. For the case of two free boundaries, several papers have recently appeared in which the effect of certain perturbations of the classical Benard problem is studied. For example, Daniels [3] has studied the effect of including distant conducting side-walls at $x = O(L)$ when the Rayleigh number, R, exceeds the classical critical R by $O(L^{-2})$, where L is large. Brown and Stewartson [1] have considered a similar problem but with distant cylindrical conducting boundary. And again Daniels [4] has studied the effect of centrifugal force in a shallow rotating cylinder or annulus. These studies all yield certain changes in the amplitude equation, which results from the balance of terms in the governing equation at $O(\varepsilon^3)$.

The changes from the classical problem are essentially extra terms in the amplitude equation or changes in its boundary conditions. It is hoped

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that experimental results will be more easily compared with these modified problems.

In this paper we are interested in the Benard convection problem with the lower surface being of the form $z = \varepsilon^2 g(s)$, in cylindrical geometry with axisymmetry. We refer to this new problem as the non-parallel plane problem in contrast to the parallel plane problem (Golbabai [6]), where the lower plane is given by $z = 0$. The upper boundary is $z = 1$. It is assumed that $g(s)$ is bounded so that for ε sufficiently small the surfaces do not intersect. We choose $g(0) = 0$ and g positive for $r = \infty$, the excess of the Rayleigh number above R_c (critical Rayleigh number) is assumed to be $O(\varepsilon^2)$, and the deviation of the lower surface from the plane case is $O(\varepsilon^2)$. We consider a fluid confined between two rigid boundaries $z = 1$ and $z = \varepsilon^2 g(s)$, where z, r are dimensionless cylindrical coordinates and ε is a small parameter. Gravity acts in the negative direction and the flow field extends to $r = 0$ and $r = \infty$. For definiteness the space cooordinates are made non-dimensional with respect to the fluid depth, d, the lower surface $z = \varepsilon^2 g(s)$ is kept at constant temperature θ_0^* and the upper surface $z = 1$ at constant temperature θ_1^*. The velocity component v is taken to be zero.

2. The governing equations of motion. The full set of equations in the Oberbeck–Boussinesq approximation for viscous, incompressible, axisymmetric flow can be expressed as follows:

$$u_r + \frac{u}{r} + w_z = 0, \tag{2.1}$$

$$u_t - \sigma\left(\nabla^2 u - \frac{u}{r^2}\right) + p_r = -(uu_r + wu_z), \tag{2.2}$$

$$w_t - \sigma(\nabla^2 w + \theta) + p_z = -(uw_r + ww_z), \tag{2.3}$$

$$\theta_t - \nabla^2\theta - Rw = -(u\theta_r + w\theta_z), \tag{2.4}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

where $u_r = \partial u/\partial r$, $u_z = \partial u/\partial z$ etc., and

$$R = g\alpha_0 d^3(\theta_0^* - \theta_1^*)/(k\nu) \quad \text{and} \quad \sigma = \nu/k$$

are the Rayleigh number and Prandtl number respectively.

We define the slow variable, s, by $s = \varepsilon r$, where ε is a small parameter. The boundary conditions are

$$u = w = \theta = 0 \quad \text{on } z = 1,\ z = \varepsilon^2 g(s). \tag{2.5}$$

3. Analysis of the base flow and steady state solution. For steady flow in the parallel plane problem, which is given by $\varepsilon = 0$, there is a solution of the form

$$u = 0, \quad w = 0, \quad \theta = 0, \quad p = \text{constant}.$$

For $\varepsilon \neq 0$ we denote the velocity component of the steady case flow by u_s, w_s, and the pressure and temperature by p_s and θ_s respectively. The boundary conditions for the base flow are

$$\begin{aligned} &u_s = w_s = \theta_s = 0 && \text{on } z = 1, \\ &u_s = w_s = 0, \quad \theta_s = R\varepsilon^2 g(s) && \text{on } z = \varepsilon^2 g(s). \end{aligned} \tag{3.1}$$

We also add the condition that

$$u_s \to 0 \quad \text{and} \quad w_s \to 0 \quad \text{as } s \to \infty. \tag{3.2}$$

For small ε we expand the perturbation u_s, w_s, θ_s and p_s in powers of ε and write:

$$\begin{aligned} u_s &= \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \\ w_s &= \varepsilon w_1 + \varepsilon^2 w_2 + \dots, \\ \theta_s &= \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \dots, \\ p_s &= \varepsilon p_1 + \varepsilon^2 p_2 + \dots \end{aligned} \tag{3.3}$$

The functions u_i, w_i, θ_i, p_i for $i = 1, 2, \dots$ are considered to depend on the two variables z and s.

Substituting the form of expansions (3.1)–(3.3) into (2.1)–(2.4), replacing $\partial/\partial r$ by $\varepsilon\partial/\partial s$, and equating powers of ε, we obtain a set of partial differential equations as follows:

From (2.1), we find that

$$\frac{\partial w_1}{\partial z} = 0, \quad u_i + s\left(\frac{\partial u_i}{\partial s} + \frac{\partial w_{i+1}}{\partial z}\right) = 0, \quad 1 = 1, 2, 3; \tag{3.4}$$

from (2.2) and (2.3),

$$\frac{\partial^2 u_1}{\partial z^2} = 0, \qquad \sigma\frac{\partial^2 u_{i+1}}{\partial z^2} - \frac{\partial p_i}{\partial s} = 0, \qquad i = 1, 2, \tag{3.5}$$

$$\frac{\partial p_1}{\partial z} - \sigma\theta_1 = 0, \quad \frac{\partial p_i}{\partial z} - \sigma\left(\theta_i + \frac{\partial^2 w_i}{\partial z^2}\right) = 0, \quad i = 2, 3; \tag{3.6}$$

finally, from (2.4),

$$\theta_1 = 0, \quad \frac{\partial^2 \theta_i}{\partial z^2} + R w_i = 0, \quad i = 2, 3, \tag{3.7}$$

$$\frac{1}{s}\frac{\partial \theta_2}{\partial s} + \frac{\partial^2 \theta_4}{\partial z^2} + R w_4 - w_2\frac{\partial \theta_2}{\partial z} = 0. \tag{3.8}$$

Now we define the boundary conditions on u_i, θ_i, w_i for $i = 1, 2, \dots$ From (3.1),

$$u_i = w_i = \theta_i = 0 \quad \text{on } z = 1, \ i = 1, 2, \dots \tag{3.9}$$

The boundary conditions on $z = \varepsilon^2 g$, for u_i, w_i, θ_i $(i = 1, 2, \ldots)$, are given by expansion of u_s, w_s, θ_s about $z = 0$, and the details are given for u_s only; we have

$$u_s(s, \varepsilon^2 g) = \varepsilon u_1(s, 0) + \varepsilon^2 u_2(s, 0) + \varepsilon^3 (g u_{1z} + u_3)_{z=0} + \varepsilon^4 (g u_{2z} + u_4)_{z=0} + \ldots \tag{3.10}$$

Therefore, since $u_s(s, \varepsilon^2 g) = 0$, we have

$$u_1 = u_2 = 0, \quad g u_{1z} + u_3 = g u_{2z} + u_4 = 0 \quad \text{on } z = 0,$$

where $u_{1z} = \partial u_i/\partial z$ $(i = 1, 2, 4)$. Similar arguments provide boundary conditions for w_i and θ_i on $z = 0$.

The solutions of the equations (3.4)–(3.8) subject to the boundary conditions are the following:

$$u_1 = u_2 = 0, \quad w_1 = w_2 = w_3 = 0, \quad \theta_1 = \theta_3 = p_1 = 0, \tag{3.11}$$

$$\theta_2 = R(1 - z)g, \tag{3.12}$$

$$p_2 = \sigma R\left(z - \frac{z^2}{2}\right) + B(s), \tag{3.13}$$

$$u_3 = R\left(z^3 - \frac{z^4}{24} - \frac{z}{8}\right)\frac{dg}{ds} + \frac{1}{2}\sigma^{-1}(z^2 - z)\frac{dB}{ds}, \tag{3.14}$$

$$w_4 = R f_1(z)\left(\frac{d^2 g}{ds^2} + \frac{1}{s}\frac{dg}{ds}\right) + \sigma^{-1} f_2(z)\left(\frac{d^2 B}{ds^2} + \frac{1}{s}\frac{dB}{ds}\right), \tag{3.15}$$

where f_1 and f_2 are given by

$$f_1 = \frac{z^5}{120} + \frac{z^2}{16} - \frac{z^4}{24}, \qquad f_2 = \frac{z^2}{2} - \frac{z^3}{3}. \tag{3.16}$$

In obtaining these solutions, p_2 was first found in the form (3.13), where $B(s)$ is an unknown function at this stage.

In order that $u_3 \to 0$ as $s \to \infty$, we see from (3.14) that

$$dg/ds \to 0 \qquad \text{as } s \to \infty, \tag{3.17}$$

and

$$dB/ds \to 0 \qquad \text{as } s \to \infty. \tag{3.18}$$

From the condition $w_4 = 0$ on $z = 1$, and (3.17), we find that

$$\frac{dB}{ds} + (7\sigma R/20)\frac{dg}{ds} = 0, \tag{3.19}$$

and thus

$$B = (-\sigma 7R/20) g(s) + \text{const.} \tag{3.20}$$

Substituting (3.20) into (3.13)–(3.15) provides the explicit form of p_2, u_3, w_4. Summarizing, the expansions for the base flow, pressure and temperature can be written as follows:

$$u_s = \varepsilon^3 R\left(1 - \frac{z^4}{24} + \frac{z^3}{16} - 7\frac{z^2}{40} + \frac{z}{20}\right)\frac{dg}{ds} + \dots, \tag{3.21}$$

$$w_s = \frac{1}{120}\varepsilon^4(z^5 - 5z^4 + 7z^3 - 3z^3)\left(\frac{d^2g}{ds^2} + \frac{1}{s}\frac{dg}{ds}\right)R + \dots, \tag{3.22}$$

$$\theta_s = \varepsilon^2 R(1-z)g + \dots, \tag{3.23}$$

$$p_s = \varepsilon^2\sigma\left(z - \frac{z^2}{2} - \frac{7}{20}\right)g + \dots, \tag{3.24}$$

Note that as $s \to \infty$, we have zero fluid velocity and just a linear temperature variation.

4. The disturbance equations in matrix form. We continue with the case where the equation of the lower boundary is $z = \varepsilon^2 g(s)$, where $s = \varepsilon r$. In equations (2.1)–(2.4) we set

$$\begin{aligned} u &= u_s + \widehat{u}(r,z,t), \qquad \theta = \theta_s + \widehat{\theta}(r,z,t), \\ w &= w_s + \widehat{w}(r,z,t), \qquad p = p_s - \widehat{p}(r,z,t) \end{aligned} \tag{4.1}$$

(where the minus sign with perturbed pressure is merely for convenience) to obtain the equations for small disturbances $\widehat{u}, \widehat{w}, \widehat{\theta}, \widehat{p}$; these are assumed to be sufficiently small for non-linear products of these terms to be neglected in the governing equations. The functions u_s, w_s, θ_s, p_s are the steady solutions of (2.1)–(2.4) which are given in (3.21)–(3.24). Upon substitution of (4.1) into (2.1)–(2.4) we then obtain the linear system

$$\frac{\partial\widehat{u}}{\partial r} + \frac{\widehat{u}}{r} + \frac{\partial\widehat{w}}{\partial z} = 0, \tag{4.2}$$

$$\frac{\partial\widehat{u}}{\partial t} + u_s\frac{\partial\widehat{u}}{\partial r} + \widehat{u}\frac{\partial u_s}{\partial r} + \widehat{w}\frac{\partial u_s}{\partial z} + w_s\frac{\partial\widehat{u}}{\partial z} = \widehat{p}_r + \sigma\left(\nabla^2\widehat{u} - \frac{\widehat{u}}{r^2}\right), \tag{4.3}$$

$$\frac{\partial\widehat{w}}{\partial t} + u_s\frac{\partial\widehat{w}}{\partial r} + \widehat{u}\frac{\partial w_s}{\partial r} + \widehat{w}\frac{\partial w_s}{\partial z} + w_s\frac{\partial\widehat{w}}{\partial z} = \widehat{p}_z + \sigma(\nabla^2\widehat{w} + \widehat{\theta}), \tag{4.4}$$

$$\frac{\partial\widehat{\theta}}{\partial t} + u_s\frac{\partial\widehat{\theta}}{\partial r} + \widehat{u}\frac{\partial\theta_s}{\partial r} + \widehat{w}\frac{\partial\theta_s}{\partial z} + w_s\frac{\partial\widehat{\theta}}{\partial z} = R\widehat{w} + \nabla^2\widehat{\theta} \tag{4.5}$$

and

$$\widehat{u} = \widehat{\theta} = \widehat{w} = 0 \quad \text{on } z = 1,\ z = \varepsilon^2 g(s). \tag{4.6}$$

We now introduce the notation

$$\widehat{u} = \partial u/\partial z, \qquad \widetilde{\theta} = \partial\widetilde{\theta}/\partial z, \tag{4.7}$$

and in equation (4.3) we write

$$\nabla^2\widehat{u} = \frac{\partial^2\widehat{u}}{\partial r^2} + \frac{1}{r}\frac{\partial\widehat{u}}{\partial r} + \frac{\partial\widetilde{u}}{\partial z}. \tag{4.8}$$

Then (4.3) expresses $\partial\widetilde{u}/\partial z$ in terms of $\widehat{u}, \widetilde{u}, \widehat{w}, \widehat{p}$ and their derivatives with respect to r and t. We next note that the derivative of (4.2) with respect to z may be written as

$$\frac{\partial^2 \widehat{w}}{\partial z^2} = -\left(\frac{\widetilde{u}}{r} + \frac{\partial \widehat{u}}{\partial r}\right). \tag{4.9}$$

In our analysis we shall ignore powers of ε^n for $n \geq 3$. With this assumption and (3.21)–(3.24) we observe that

$$u_s = 0, \quad w_s = 0, \quad \theta_s = \varepsilon^2 Rg(1 - z). \tag{4.10}$$

Now we introduce the extended flow vector

$$\underline{U} = \left[\widetilde{u}, \frac{\partial \widehat{p}}{\partial r}, \frac{\partial \widehat{\theta}}{\partial r}, \widehat{u}, \frac{\partial \widehat{w}}{\partial r}, \frac{\partial \widehat{\theta}}{\partial r}\right]^{\mathrm{tr}}, \tag{4.11}$$

where tr denotes the transpose and $\widetilde{u}, \widetilde{\theta}$ are given in (4.7). Substituting (4.10) in (4.3)–(4.5), we find that

$$\begin{aligned}
&\frac{\partial \widehat{u}}{\partial z} = \sigma^{-1}\frac{\partial \widehat{p}}{\partial r} - L\widehat{u} + \sigma^{-1}\frac{\partial \widehat{u}}{\partial t}, \\
&\frac{\partial^2 \widehat{p}}{\partial z \partial r} = \sigma\left(L\left(\widehat{u} + \frac{\partial \widehat{w}}{\partial r}\right) + \frac{\partial \widehat{\theta}}{\partial r} + \frac{\partial^2 \widehat{w}}{\partial r \partial t}\right), \\
&\frac{\partial^2 \widehat{\theta}}{\partial z \partial r} = -R\frac{\partial \widehat{w}}{\partial r} - L\left(\frac{\partial \widehat{\theta}}{\partial r}\right) + \frac{\partial^2 \theta}{\partial r \partial t} + \frac{\partial \widehat{w}}{\partial r}\frac{\partial \theta_s}{\partial z}, \\
&\frac{\partial \widehat{u}}{\partial z} = \widetilde{u}, \quad \frac{\partial^2 \widehat{w}}{\partial z \partial r} = -L\widehat{u}, \quad \frac{\partial^2 \widehat{\theta}}{\partial r \partial z} = \frac{\partial \widetilde{\theta}}{\partial r},
\end{aligned} \tag{4.12}$$

where L is a linear operator: $L = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) - 1/r^2$. This formulation enables us to write the equation (4.12) in matrix form:

$$\frac{\partial \underline{U}}{\partial z} = A\underline{U} + B\frac{\partial \underline{U}}{\partial t}, \tag{4.13}$$

where A and B are 6×6 matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & -L & 0 & 0 \\ L & 0 & 0 & 0 & L & 1 \\ 0 & 0 & 0 & 0 & X & -L \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad X = (1 + \varepsilon^2 g)R,$$

$$B = \begin{pmatrix} \underline{0} & \begin{matrix} \sigma^{-1} & 0 & 0 \\ 0 & \sigma^{-1} & 0 \\ 0 & 0 & 1 \end{matrix} \\ \underline{0} & \underline{0} \end{pmatrix},$$

where $\underline{0}$ is the 3×3 zero matrix. We can write $A = A_1 - LA_2 + R(1+\varepsilon^2 g)A_3$, where A_1, A_2, A_3 are constant 6×6 matrices. From (4.6),

$$\widehat{u} = \frac{\partial \widehat{w}}{\partial r} = \frac{\partial \widehat{\theta}}{\partial r} = 0 \quad \text{on } z = 1. \tag{4.14}$$

On $z = \varepsilon^2 g$, we have, for example, $\partial\widehat{w}/\partial r = 0$, so that

$$\frac{\partial \widehat{w}}{\partial r} = \frac{\partial \widehat{w}(r,0)}{\partial r} + \varepsilon^2 g \frac{\partial^2 w(r,0)}{\partial z \partial r} + \dots \tag{4.15}$$

and so within the order of approximation considered here, $\partial\widehat{w}/\partial r = 0$ on $z = 0$. The complete set of boundary conditions can be conveniently labeled as follows:

γ: the last three components of $\underline{U}$ are zero on $z = 0$ and $z = 1$,

or alternatively:

$$\gamma : C\underline{U} = 0 \text{ on } z = 0,\ z = 1, \tag{4.16}$$

where $C = \left(\begin{smallmatrix} \underline{0} & \underline{0} \\ \underline{0} & I_3 \end{smallmatrix}\right)$, and I_3 is the unit 3×3 matrix.

5. Expansion procedure and derivation of the amplitude equation. If $g(s) = 0$ (or $\varepsilon = 0$) then we have the standard linear parallel plane problem, so that we may expect that the critical disturbance of the plane problem will have a corresponding perturbed solution in the non-parallel case. We assume $g(s)$ remains $O(1)$ for $r = O(\varepsilon^{-1})$, and look for a steady solution of (4.13) in this region of the form

$$\underline{U} = e^{i\alpha r}\underline{V}(z,s) + \text{c.c.}, \tag{5.1}$$

with $R = R_c + \varepsilon^2\beta$, where ε is fixed by the size of the depression in the lower surface ($z = \varepsilon^2 g$), and β is an arbitrary parameter which represents an $O(\varepsilon^2)$ variation in R about R_c, and the symbol c.c. denotes the complex conjugate. Now we expand the complex function $\underline{V}$ in powers of ε:

$$\underline{V} = \varepsilon \underline{F}_1 + \varepsilon^2 \underline{F}_2 + \varepsilon^3 \underline{F}_3 + \dots, \tag{5.2}$$

where $\underline{F}_i$ is a function of s and z. On substituting (5.1)–(5.2) into (4.13) and equating powers of ε^n, we obtain a set of equations as follows:

$$L_0\underline{F}_1 = 0, \qquad \gamma, \tag{5.3}$$

$$L_0\underline{F}_2 = -i\alpha_c A_2 L_1 \underline{F}_1, \qquad \gamma, \tag{5.4}$$

$$L_0\underline{F}_3 = (R_c g + \beta)A_3\underline{F}_1 - i\alpha A_2 L_1 \underline{F}_2 - A_2 L_2 \underline{F}_1 \tag{5.5}$$

with

$$C(\underline{F}_3 - g\underline{F}_1) = 0 \quad \text{on } z = 0, \quad C\underline{F}_3 = 0 \quad \text{on } z = 1,$$

where

$$
\begin{aligned}
L_0 &= \frac{\partial}{\partial z} - (A_1 + \alpha^2 A_2 + R_c A_3), \\
L_1 &= \frac{\partial}{\partial s} + \frac{1}{2s}, \\
L_2 &= \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} - \frac{1}{s^2},
\end{aligned} \tag{5.6}
$$

A_1, A_2, A_3 are constant 6×6 matrices, C and γ are given in (4.16) and α, R_c are the critical wave and Rayleigh number respectively.

A general solution of (5.3) can be written as

$$\underline{F}_1 = \overline{A}(s)\underline{f}_1(z), \tag{5.7}$$

where

$$L_0 \underline{f}_1(z) = 0, \qquad \gamma,$$

and $\underline{f}_1(z)$ is the critical eigenfunction of the standard parallel plane problem (S. Chandrasekhar [2]). The solution given in (5.7) contains an amplitude function, $\overline{A}(s)$, which is determined by the solvability condition obtained at the higher order $O(\varepsilon^2)$. Upon using (5.7), equation (5.4) becomes

$$L_0 \underline{F}_2 = -i\alpha A_2 L_1 \overline{A}(s)\underline{f}_1, \qquad \gamma, \tag{5.8}$$

and has a solution for $\underline{F}_2$ if the adjoint condition

$$i\alpha L_i \overline{A}(s) \int_0^1 \underline{f}(z)(A_2 \underline{f}_1)\, dz = 0$$

is satisfied; the adjoint condition here is similar to that of Eagles [5], where $\underline{f}(z)$ is the adjoint function.

From the boundary conditions on $\underline{F}_2$, (5.4), and from the right-hand side of (5.8), $\underline{F}_2$ may be expressed as

$$\underline{F}_2 = -i\left(\frac{d\overline{A}}{ds} + \frac{\overline{A}}{2s}\right)\underline{f}_2 + \overline{A}_1(s)\underline{f}_1, \tag{5.9}$$

where $\overline{A}_1(s)$ is an unknown function at this stage, and

$$L_0 \underline{f}_2 = 2\alpha A_2 \underline{f}_1, \qquad \gamma.$$

A solution for $\underline{F}_3$ exists if the adjoint condition is satisfied, which results in the amplitude equation

$$\alpha\left(\frac{d^2\overline{A}}{ds^2} + \frac{1}{s}\frac{d\overline{A}}{ds} - \frac{\overline{A}}{4s^2}\right) + \overline{A}\{(b - cRc)g + b\beta\} = 0. \tag{5.10}$$

Here

$$a = -\alpha \int_0^1 \underline{f}(z)(A_2\underline{f}_2)\,dz, \quad b = \int_0^1 \underline{f}(z)(A_3\underline{f}_1)\,dz,$$

$$c = \sum_{k=4}^{6} f_k \frac{df_{1k}}{dz} \qquad \text{on } z = 0,$$

where $f_k(z)$ is the kth component of the adjoint function $\underline{f}(z)$, and $f_{1k}(z)$ is the kth component of the eigenfunction $\underline{f}_1(z)$ and A_2, A_3 are given in (5.6). Now let $A_0(s) = \overline{A}s^{1/2}$. From (5.10), we see that

$$\frac{d^2 A_0}{ds^2} + (\delta_1 + \delta_2 g)A_0 = 0, \tag{5.11}$$

where $\delta_1 = b\beta/a$, $\delta_2 = (b - cR_c)/a$.

The equation (5.11) is similar to that obtained by Eagles [5] in two-dimensional case, where the boundary conditions are defined at $s = \pm\infty$. In our case, we have no information about the boundary condition at the center, $s = 0$, and we shall investigate this by using a matching procedure.

6. The inner solution and investigating the amplitude equation. In the neighbourhood of $r = 0$ the function $g(s)$ tends to zero and we look for a linearized solution of (2.1)–(2.4) in which $R = R_c$ and the components of the disturbance are given in terms of Bessel functions. An equivalent solution has been found in the stress-free case by Brown and Stewartson [1]. One solution is

$$\theta = h(z)J_0(\alpha r), \quad u = f(z)J_0'(\alpha r), \quad w = g(z)J_0(\alpha r)$$

and a second solution can be found by writing

$$\begin{aligned} u &= \overline{f}(z)J_0'(\alpha r) + rf(z)J_0''(\alpha r), \\ w &= \overline{g}(z)J_0(\alpha r) + rg(z)J_0'(\alpha r), \\ \theta &= \overline{h}(z)J_0(\alpha r) + rh(z)J_0'(\alpha r), \end{aligned} \tag{6.1}$$

where $f, g, h, \overline{f}, \overline{g}, \overline{h}$ can be found numerically.

The general solution for θ in the inner zone may now be written as

$$\theta_I = \lambda J_0(\alpha r)h + \mu\{J_0(\alpha r)\overline{h} + J_0'(\alpha r)h\}. \tag{6.2}$$

Here θ_I denotes the inner solution and we use θ_0 to denote the solution already found. In order to match (6.2) with the outer solution (5.1), we need to know the behavior of the amplitude function $A_0(s)$ as $s \to 0$, which is found to be

$$A_0 \sim \overline{a} + \overline{b}s + \ldots \qquad (s \to 0), \tag{6.3}$$

where $\overline{a}$, $\overline{b}$ are arbitrary constants and $s = \varepsilon r$.

From (5.1), (5.7), (5.11), in the outer region where $g(s) \neq 0$, θ_0 is given by

$$\theta_0 \sim \frac{e^{i\alpha r}}{s^{1/2}}\left\{hA_0 - i\varepsilon\frac{\partial A_0}{\partial s}\overline{h} + \dots\right\} + \text{c.c.} \tag{6.4}$$

Now the asymptotic expansion of (6.2) for large r is

$$\theta_i \sim \left(\frac{2}{\pi\alpha r}\right)^{1/2}\left\{\left(\lambda - \frac{3\mu}{8\alpha}\right)h\cos\widehat{r} + \mu rh\sin\widehat{r}\right\}, \tag{6.5}$$

where $\widehat{r} = \alpha r - \pi/4$. Substituting (6.3) into (6.4) we obtain

$$\theta_0 \sim \frac{e^{i\alpha r}}{s^{1/2}}\{\overline{a}h + \varepsilon\overline{b}rh - i\overline{b}\varepsilon\overline{h}\} + \text{c.c.}, \tag{6.6}$$

and comparing (6.5) with (6.6), we see that a match of the terms in h and rh is secured if, respectively,

$$\begin{aligned} \overline{a} &= \left(\frac{\varepsilon}{2\pi\alpha}\right)^{1/2}\left(\lambda - \frac{3\mu}{8\alpha}\right)e^{-i\pi/4}, \\ \varepsilon^{1/2}\overline{b} - &\left(\frac{\pi\alpha}{2}\right)^{-1/2} e^{-i\pi/4} = 0. \end{aligned} \tag{6.7}$$

Therefore, from matching conditions (6.7) we can list the boundary conditions for the amplitude equation as

$$A_0(0) = b_1 e^{-i\pi/4}, \qquad A_0'(0) = b_2 e^{-i\pi/4}, \tag{6.8}$$

where b_1 and b_2 are real constants. Note that these conditions are quite different from those which apply to the non-linear problem studied by Brown and Stewartson [1].

Now suppose that $g(s) \neq 0$ and define $g_1(s) = -\delta_2 g(s)$, so that the amplitude equation is given by $A_0'' + (\delta_1 - g_1(s))A_0 = 0$, where $g_1(s) \geq 0$. In view of the conditions (6.8), we set

$$A_0 = e^{-i\pi/4}(\overline{A}_1(s) + i\overline{A}_2(s)), \tag{6.9}$$

where $\overline{A}_1(s)$ and $\overline{A}_2(s)$ are assumed to be real functions. Then

$$\begin{aligned} \overline{A}_1(0) &= b_1, & \overline{A}_2(0) &= 0, \\ \overline{A}_1'(0) &= 0, & \overline{A}_2'(0) &= b. \end{aligned} \tag{6.10}$$

Let $\widetilde{A} = \overline{A}_1 + i\overline{A}_2$ and $\widetilde{A} = \overline{R}(s)e^{i\psi(s)}$. Then $\overline{R}$ and ψ satisfy the equations

$$\overline{R}'' + (\delta_1 - g_1)\overline{R} - (\psi')^2\overline{R} = 0, \qquad 2\psi'\overline{R}' + \psi''\overline{R} = 0, \tag{6.11}$$

with boundary conditions

$$\overline{R}(0) = b_1, \qquad \psi(0) = 0,$$
$$\overline{R}'(0) = 0, \qquad \psi'(0) = b_2/b_1. \tag{6.12}$$

From (6.11), $\psi' = c/\overline{R}^2$, where c is an arbitrary constant, and the boundary conditions at $s = 0$ imply that $c = b_1 b_2$. Hence

$$\psi = b_1 b_2 \int ds/\overline{R}^2. \tag{6.13}$$

Now if we impose the condition that $\overline{R} \to 0$ $(s \to \infty)$, then we choose $c = 0$, so that

$$\overline{R}'' + (\delta_1 - g_1)\,\overline{R} = 0, \quad \overline{R}'(0) = 0, \quad \overline{R} \to 0 \quad (s \to \infty). \tag{6.14}$$

This system is the same as the linear form considered by Eagles [5], where the function $g_1(s)$ is taken to be $g_1(s) = (\tanh(s/\sqrt{2})^2$ and for $\delta_1 = 1/2$ there is a solution $\overline{R} = b_1 \operatorname{sech}(s/\sqrt{2})$ with $\overline{R} \to 0$ as $s \to \infty$, representing the distribution of convection cells concentrated near the center.

Discussion. It is well known that the base flow in the parallel plane problem ($g(s) = 0$, or $\varepsilon = 0$) is unstable for $R > R_c$, and that for $R > R_c$ a pattern of convection cells or rolls is set up. In the non-parallel plane case we see that the local Rayleigh number is larger near $s = 0$ than at $s = \infty$, so that the convection cells occur in the center more readily than away from the center and the positive value of $g(s)$ for $s > 0$ causes an effective increase in the critical Rayleigh number over that for the plane case, where $g(s) = 0$. Finally, it should be pointed out that the boundary conditions at $s = 0$ for the linear amplitude equation differ from those which apply to the non-linear problem investigated by Brown and Stewartson [1].

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