APPLICATIONES MATHEMATICAE
23,1 (1995), pp. 13–23

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THE SOLUTION SET OF A DIFFERENTIAL INCLUSION ON A CLOSED SET OF A BANACH SPACE

Abstract. We consider differential inclusions with state constraints in a Banach space and study the properties of their solution sets. We prove a relaxation theorem and we apply it to prove the well-posedness of an optimal control problem.

1. Introduction. It is well known that the relaxation theorem is very useful in optimal control problems. For a differential inclusion with Lipschitz right hand side without state constraints, several papers [2, 5, 6, 9–11] yield results on the relaxation theorem and some other properties of the solution sets. In [7], the relaxation theorem for a semilinear evolution equation with state constraints was proved. In this paper, we consider the same problem for the differential inclusion system

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t,$$
$$x(0) = x_0 \quad \text{and} \quad x(t) \in K,\ 0 \le t \le T.$$

Here $K \subset X$ is a closet subset of a Banach space X, and $F : [0,T] \times K \to 2^X$ is a multifunction. Under weak conditions, we obtain results similar to [7]. We note that in our case, we require at each step a projection on the set K, since F is not defined outside K, and that this projection is not continuous. Moreover, in general there is no extension $\overline{F}$ of F to an open neighbourhood of K, so we cannot obtain our results from known results. Let us also mention that the viability problems for differential inclusions were studied in [1, 8] and well-posedness for differential inclusions on closed subsets of $\mathbb{R}^n$ was discussed in [4].

2. Preliminaries. Let $I = [0, T] \subset \mathbb{R}^1$ and μ be Lebesgue measure; let X be a Banach space and K be a closed subset of X. For $x \in K$, let

1991 *Mathematics Subject Classification*: 34A60, 34G10.
Key words and phrases: differential inclusion, relaxation theorem, well-posedness.

$d_K(x) = \inf\{\|x-y\| \mid y \in K\}$ be the distance from x to K. Also let $\pi_K(x) = \{y \in K \mid \|x-y\| = d_K(x)\}$ be the metric projection of x onto K and let

$$T_K(x) = \{v \in K \mid \liminf_{h\to 0}(1/h)d_K(x+hv) = 0\}$$

be the contingent cone to K at x. For $A, B \subset X$ denote by $d(A,B)$ the Hausdorff distance from A to B.

A multifunction $G : I \to 2^X$ is called *measurable* if there exists a sequence $\{g_n\}$ of measurable selections such that $G(t) \subset \mathrm{cl}\{g_n(t) \mid n \geq 0\}$. We observe that when X is separable and G has closed images this definition is the same as the usual one [3].

LEMMA 2.1 ([11]). *Assume that $F : [0,T] \times K \to 2^X$ is a multifunction with closed images such that*

(a) *for any $x \in K$, $F(\cdot, x)$ is measurable on I;*
(b) *for any $t \in I$, $F(t,\cdot)$ is continuous on K.*

Then for any measurable function $x(\cdot)$, $t \to F(t, x(t))$ is measurable on I.

LEMMA 2.2 ([11]). *Let $G : I \to 2^X$ be a measurable multifunction with closed images and $u(\cdot) : I \to X$ a measurable function. Then for any measurable function $r(t) > 0$, there exists a measurable selection g of G such that for almost all $t \in I$,*

$$\|g(t) - u(t)\| \leq d(u(t), G(t)) + r(t).$$

LEMMA 2.3 ([11]). *If $G : I \to 2^X$ is an integrable multifunction then, for any $x_0 \in X$,*

$$\overline{S_G(x_0)} = \overline{S_{\overline{\mathrm{co}}G}(x_0)},$$

where $S_G(x_0)$ denotes the solution set of the differential inclusion $\dot{x}(t) \in G(t)$ a.e. $t \in I$, $x(0) = x_0$.

3. Main results. Consider the differential inclusion

$$\text{(P)} \qquad \begin{aligned} &\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in I,\\ &x(0) = \xi \quad \text{and} \quad x(t) \in K,\ t \in I, \end{aligned}$$

where $F : [0,T] \times K \to 2^X$ is a multifunction with closed images and $K \subset X$ is a closed subset of X. We denote by $S_F(\xi)$ the solution set of (P) and by $S_{\overline{\mathrm{co}}F}(\xi)$ the solution set of the relaxation differential inclusion

$$(\overline{\mathrm{P}}) \qquad \begin{aligned} &\dot{x}(t) \in \overline{\mathrm{co}}F(t, x(t)) \quad \text{a.e. } t \in I,\\ &x(0) = \xi \quad \text{and} \quad x(t) \in K,\ t \in I. \end{aligned}$$

We assume that $F : [0,T] \times K \to 2^X$ satisfies the following hypotheses:

(H_1) $t \to F(t,x)$ is measurable for all $x \in K$;

(H_2) there exists $l(\cdot) \in L^1(I,\mathbb{R})$ such that for all $x, y \in X$,

$$d(F(t,x),F(t,y)) \le l(t)\|x-y\|;$$

(H_3) for all $(t,x) \in I \times K$, $F(t,x) \subset T_K(x)$; K is *proximal*, i.e., for any $x \in X$, $\pi_K(x) \neq \emptyset$;

(H_4) for any continuous function $x(\cdot) : I \to K$, $t \in F(t,x(t))$ is integrable.

THEOREM 3.1. *Let* $F : [0,T]\times K \to 2^X$ *be a multifunction with closed images satisfying* (H_1)–(H_4). *Let* $M = \exp(\int_0^T l(t)\,dt)$ *and let* $y(\cdot)$ *be an absolutely continuous function such that* $y(0)=\xi_0\in K$. *Let* $q(t)=\operatorname{ess\,sup}\{d(\dot y(t), F(t,z(t))) \mid z(t)$ *is a measurable selection of* $\pi_K(y(t))\}$ (*if* $y(t) \in K$ *for all* $t \in I$, *we let* $q(t) = d(\dot y(t), F(t,y(t)))$) *and let* $\int_0^T q(t)\,dt < \varepsilon$. *Then there exists* $\eta > 0$ *such that for all* $\xi \in (\xi_0 + \eta B) \cap K$, *there exists a solution* $x(\cdot)$ *of* (P) *such that*

$$\|x(\cdot) - y(\cdot)\|_{C(I,X)} \le 12M^5\varepsilon.$$

Proof. Let η be a positive number such that $\eta + \int_0^T q(t)\,dt < \varepsilon$; also let $m(t) = \int_0^t l(s)\,ds$. For any $\xi \in (\xi_0 + \eta B) \cap K$, we define $x_0(t,\xi) = \xi + \int_0^t \dot y(s)\,ds$. It is easy to see that $\|x_0(\cdot,\xi) - y(\cdot)\| \le \|\xi - \xi_0\| < \eta$. Let $z_0(t) \in \pi_K(x_0(t,\xi))$ be a measurable selection of $t \to \pi_K(x_0(t,\xi))$ and $z(t)$ be a measurable selection of $\pi_K(y(t))$. Then

$$\begin{aligned}
d(\dot x_0(t,\xi), F(t,z_0(t))) &= d(\dot y(t), F(t,z_0(t)))\\
&\le d(\dot y(t), F(t,z(t))) + l(t)\|z(t) - z_0(t)\|\\
&\le q(t) + l(t)\|z(t) - y(t)\| + l(t)\|y(t) - x_0(t,\xi))\|\\
&\quad + l(t)\|x_0(t,\xi)) - z_0(t)\|\\
&\le q(t) + l(t)\eta + l(t)d_K(x_0(t,\xi)) + l(t)d_K(y(t)).
\end{aligned}$$

By Proposition 1 in [2, p. 202], we have

$$\begin{aligned}
\frac{d}{dt}(d_K(y(t))) &\le d(\dot y(t), T_K(\pi_K(y(t)))) \le d(\dot y(t), T_K(z(t)))\\
&\le d(\dot y(t), F(t,z(t))) \le q(t)
\end{aligned}$$

and, since $d_K(y(0)) = 0$, we obtain

$$d_K(y(t)) \le \int_0^t q(s)\,ds.$$

Similarly, we get

$$d_K(x_0(t,\xi)) \le \int_0^t d(\dot{x}_0(s,\xi), F(s, z_0(s)))\,ds$$
$$\le \int_0^t (q(s) + l(s)\eta)\,ds + \int_0^t l(s) \int_0^s q(u)\,du\,ds$$
$$+ \int_0^t l(s) d_K(x_0(s,\xi))\,ds.$$

From Gronwall's inequality and by interchanging the order of integration, we obtain

$$d_K(x_0(t,\xi)) \le \int_0^t \exp(m(t) - m(s))(q(s) + l(s)\eta)\,ds$$
$$+ \int_0^t \exp(m(t) - m(s)) l(s) \int_0^s q(u)\,du\,ds$$
$$\le \int_0^t \exp(m(t) - m(s)) q(s)\,ds + (\exp(m(t)) - 1)\eta$$
$$+ \int_0^t (\exp(m(t) - m(s)) - 1) q(s)\,ds$$

and

$$d(\dot{x}_0(t,\xi), F(t, z_0(t))) \le q(t) + l(t)\exp(m(t))\eta$$
$$+ 2l(t)\exp(m(t)) \int_0^t \exp(-m(s)) q(s)\,ds.$$

Set $\delta_0(t) = \operatorname{ess\,sup}\{d(\dot{x}_0(t,\xi), F(t, z(t))) \mid z(t)$ is a measurable selection of $\pi_K(x_0(t,\xi))\}$. Then

$$\delta_0(t) \le q(t) + l(t)\exp(m(t))\eta + 2l(t)\exp(m(t)) \int_0^t \exp(-m(s)) q(s)\,ds,$$
$$d_K(x_0(t,\xi)) \le \int_0^t \delta_0(s)\,ds.$$

By Lemma 2.2, we can choose a measurable selection $v_1(t)$ of $F(t, z_0(t))$ such that

$$\|v_1(t) - \dot{x}_0(t,\xi)\| \le 2d(\dot{x}_0(t,\xi), F(t, z_0(t))) \le 2\delta_0(t).$$

Set $x_1(t) = \xi + \int_0^t v_1(s)\,ds$ and let $z_1(t)$ be a measurable selection of $\pi_K(x_1(t))$. Then

$$\|x_1(t) - x_0(t,\xi)\| \le \int_0^t \|v_1(s) - \dot{x}_0(s,\xi)\|\,ds \le 2\int_0^t \delta_0(s)\,ds,$$

since

$$\begin{aligned} d(\dot{x}_1(t), F(t, z_1(t))) &= d(v_1(t), F(t, z_1(t))) \le l(t)\|z_0(t) - z_1(t)\| \\ &\le l(t)\|z_0(t) - x_0(t,\xi)\| + l(t)\|x_0(t,\xi) - x_1(t)\| \\ &\quad + l(t)\|z_1(t) - x_1(t)\| \\ &\le 3l(t)\int_0^t \delta_0(s)\,ds + l(t)d_K(x_1(t)) \end{aligned}$$

and

$$\begin{aligned} d_K(x_1(t)) &\le \int_0^t d(\dot{x}_1(s), F(s, z_1(s)))\,ds \\ &\le 3\int_0^t \exp(m(t) - m(s))l(s)\int_0^s \delta_0(u)\,du\,ds \\ &\le 3\int_0^t (\exp(m(t) - m(s)) - 1)\delta_0(s)\,ds, \end{aligned}$$

so that

$$d(\dot{x}_1(t), F(t, z_1(t))) \le 3l(t)\exp(m(t))\int_0^t \exp(-m(s))\delta_0(s)\,ds.$$

Set $\delta_1(t) = \operatorname{ess\,sup}\{d(\dot{x}_1(t), F(t, z(t))) \mid z(t)$ is a measurable selection of $\pi_K(x_1(t))\}$. Then

$$\delta_1(t) \le 3l(t)\exp(m(t))\int_0^t \exp(-m(s))\delta_0(s)\,ds,$$

$$d_K(x_1(t)) \le \int_0^t \delta_1(s)\,ds.$$

We claim that we may define sequences $\{x_n\}$, $\{\delta_n\}$ of functions with the following properties:

(i) $\delta_n(t) = \operatorname{ess\,sup}\{d(\dot{x}_n(t), F(t, z(t))) \mid z(t)$ is a measurable selection of

$\pi_K(x_n(t))\}$ and

$$\delta_n(t) \le 3l(t)\exp(m(t)) \int_0^t \exp(-m(s))\delta_{n-1}(s)\,ds$$
$$\le 3^n l(t)\exp(m(t)) \int_0^t [(m(t)-m(s))^{n-1}/(n-1)!]$$
$$\times \exp(-m(s))\delta_0(s)\,ds,$$

(ii) $d_K(x_n(t)) \le \int_0^t \delta_n(s)\,ds$,
(iii) $\|\dot{x}_n(t) - \dot{x}_{n-1}(t)\| \le 2\delta_{n-1}(t)$.

For $n = 1$ the above holds. Assume it holds up to i and let us show it holds for $i+1$. Let $z_i(t)$ be a measurable selection of $\pi_K(x_i(t))$ and let $v_{i+1}(t)$ be a measurable selection of $F(t, z_i(t))$ such that

$$\|v_{i+1}(t) - \dot{x}_i(t)\| \le 2d(\dot{x}_i(t), F(t, z_i(t))) \le 2\delta_i(t).$$

Set $x_{i+1}(t) = \xi + \int_0^t v_{i+1}(s)\,ds$. Then

$$\|x_{i+1}(t) - x_i(t)\| \le \int_0^t \|v_{i+1}(s) - \dot{x}_i(s)\|\,ds \le 2\int_0^t \delta_i(s)\,ds.$$

Let $z_{i+1}(t)$ be a measurable selection of $\pi_K(x_{i+1}(t))$. Then

$$d(\dot{x}_{i+1}(t), F(t, z_{i+1}(t))) \le l(t)\|z_i(t) - z_{i+1}(t)\|$$
$$\le l(t)\|z_i(t) - x_i(t)\| + l(t)\|x_i(t) - x_{i+1}(t)\|$$
$$+ l(t)d_K(x_{i+1}(t))$$
$$\le 3l(t)\int_0^t \delta_i(s)\,ds + l(t)d_K(x_{i+1}(t)),$$

since

$$d_K(x_{i+1}(t)) \le \int_0^t d(\dot{x}_{i+1}(s), F(s, z_{i+1}(s)))\,ds$$
$$\le 3\int_0^t \exp(m(t)-m(s))l(s)\int_0^s \delta_i(u)\,du\,ds$$
$$\le 3\int_0^t (\exp(m(t)-m(s)) - 1)\delta_i(s)\,ds.$$

Thus

$$d(\dot{x}_{i+1}(t), F(t, z_{i+1}(t))) \le 3l(t)\exp(m(t))\int_0^t \exp(-m(s))\delta_i(s)\,ds.$$

Therefore, set $\delta_{i+1}(t) = \operatorname{ess\,sup}\{d(\dot{x}_{i+1}(t), F(t, z(t))) \mid z(t)$ is a measurable selection of $\pi_K(x_{i+1}(t))\}$. We have

$$d_K(x_{i+1}(t)) \le \int_0^t \delta_{i+1}(s)\,ds,$$

$$\delta_{i+1}(t) \le 3l(t)\exp(m(t)) \int_0^t \exp(-m(s))\delta_i(s)\,ds.$$

Finally, it follows from (i) that

$$\begin{aligned}\delta_{i+1}(t) &\le 3l(t)\exp(m(t)) \int_0^t \exp(-m(s))3^i l(s)\exp(m(s)) \\ &\quad \times \int_0^s [(m(s)-m(u))^{i-1}/(i-1)!]\exp(-m(u))\delta_0(u)\,du\,ds \\ &\le 3^{i+1} l(t)\exp(m(t)) \int_0^t [(m(t)-m(s))^i/i!]\exp(-m(s))\delta_0(s)\,ds.\end{aligned}$$

Hence, the proof of our claim is complete.

Note that from (iii), we have

$$\begin{aligned}(*)\quad &\|x_{i+1}(\cdot) - x_n(\cdot)\| \\ &\le \int_0^T \|\dot{x}_{n+1}(t) - \dot{x}_n(t)\|\,dt \le 2\int_0^T \delta_n(t)\,dt \le 2\int_0^T 3^n l(t)\exp(m(t)) \\ &\quad \times \int_0^t [(m(t)-m(s))^{n-1}/(n-1)!]\exp(-m(s))\delta_0(s)\,ds\,dt \\ &\le 6[(3m(T))^{n-1}/(n-1)!]\int_0^T l(t)\exp(m(t))\int_0^t \exp(-m(s))\delta_0(s)\,ds\,dt \\ &\le 6[(3m(T))^{n-1}/(n-1)!]\int_0^T [\exp(m(T)-m(t))-1]\delta_0(t)\,dt.\end{aligned}$$

Thus, $\{x_n(\cdot)\}$ is a Cauchy sequence in $C(I,X)$ and so we may assume $x_n(\cdot)$ converges to $x(\cdot)$ in $C(I,X)$. Since $\int_0^T \delta_n(t)\,dt \to 0$ and $d_K(x_n(t))) \le \int_0^T \delta_n(s)\,ds$, we obtain $x(t) \in K$ for all $t \in I$.

To show that $x(\cdot)$ is a solution, we choose a sequence $\{z_n(t)\}$ of measurable selections of $\pi_K(x_n(t))$ and observe that

$$\begin{aligned}(**)\quad d(\dot{x}_n(t), F(t, x(t))) &\le d(\dot{x}_n(t), F(t, z_n(t))) \\ &\quad + l(t)\|z_n(t) - x_n(t)\| + l(t)\|x_n(t) - x(t)\|\end{aligned}$$

$$\leq \delta_n(t) + l(t)\Big(\int_0^t \delta_n(s)\,ds + \|x_n(t) - x(t)\|\Big).$$

Since $\{\dot{x}_n(\cdot)\}$ is a Cauchy sequence in $L^1(I,X)$, there exists a subsequence of $\{\dot{x}_n(t)\}$ which converges to $\dot{x}(t)$ a.e. $t \in I$. Passing to the limit in $(**)$, we find that $x(\cdot)$ is a solution.

From $(*)$, we have

$$\begin{aligned}
&\|x_n(\cdot) - y(\cdot)\| \\
&\quad \leq \|x_0(\cdot) - y(\cdot)\| + \|x_1(\cdot) - x_0(\cdot)\| + \ldots + \|x_n(\cdot) - x_{n-1}(\cdot)\| \\
&\quad \leq \eta + 2\int_0^T \delta_0(t)\,dt + 6\sum_{i=1}^{n-1}[(3m(T))^{i-1}/(i-1)!] \\
&\qquad \times \int_0^T [\exp(m(T) - m(t)) - 1]\delta_0(t)\,dt \\
&\quad \leq \eta + 2\int_0^T \delta_0(t)\,dt + 6\exp(3m(T))\int_0^T [\exp(m(T) - m(t)) - 1]\delta_0(t)\,dt \\
&\quad \leq \eta + 2\int_0^T \delta_0(t)\,dt + 6\exp(4m(T)) \\
&\qquad \times \int_0^T \exp(-m(t))\delta_0(t)\,dt - 6\exp(3m(T))\int_0^T \delta_0(t)\,dt \\
&\quad \leq 12M^5\varepsilon.
\end{aligned}$$

THEOREM 3.2. *Assume that* $F : [0,T] \times K \to 2^X$ *is a multifunction with closed images satisfying* (H_1)–(H_4). *Then for any* $x_0 \in K$,

$$\overline{S_F(x_0)} = \overline{S_{\overline{\mathrm{co}}F}(x_0)}.$$

Proof. It is enough to show that for any $x_0 \in K$, $S_{\overline{\mathrm{co}}F}(x_0) \subset \overline{S_F(x_0)}$. Let $y(\cdot) \in S_{\overline{\mathrm{co}}F}(x_0)$ and define $G(\cdot) = F(\cdot, y(\cdot))$. It is easy to see that $G : I \to 2^X$ satisfies the requirement of Lemma 2.3 and so

$$y(\cdot) \in S_{\overline{\mathrm{co}}G}(x_0) \subset \overline{S_G(x_0)}.$$

For any $\varepsilon > 0$, there exists $z \in S_G(x_0)$, i.e., $\dot{z}(t) \in F(t, y(t))$, $z(0) = x_0$, such that

$$\|z(\cdot) - y(\cdot)\|_G \leq \varepsilon/(12M^6).$$

For any $z_0(t) \in \pi_K(z(t))$ measurable, since

$$d(\dot{z}(t), F(t, z_0(t))) \leq l(t)\|z_0(t) - y(t)\| \leq l(t)\|z(t) - y(t)\| + l(t)d_K(z(t)),$$

we have

$$d_K(z(t)) \le \int_0^t d(\dot z(s), F(s, z_0(s)))\, ds,$$

so that

$$d(\dot z(t), F(t, z_0(t))) \le l(t)\|z(t) - y(t)\| + l(t) \int_0^t l(s) \exp(m(t) - m(s)) \|y(s) - z(s)\|\, ds,$$

and therefore

$$\int_0^T d(\dot z(t), F(t, z_0(t)))\, dt \le \|z(\cdot) - y(\cdot)\| \Big(\int_0^T l(t)\, dt + \int_0^T l(t) \int_0^t l(s) \exp(m(t) - m(s))\, ds\, dt \Big) \le \|z(\cdot) - y(\cdot)\| (\exp(m(T)) - 1) \le M \|z(\cdot) - y(\cdot)\|.$$

Set $q(t) = \operatorname{ess\,sup}\{d(\dot z(t), F(t, z_0(t))) \mid z_0(t)$ is a measurable selection of $\pi_K(z(t))\}$. Then

$$\int_0^T q(t)\, dt \le M \|z(\cdot) - y(\cdot)\| \le \varepsilon/(12M^5).$$

By Theorem 3.1, there exists $x(\cdot) \in S_F(x_0)$ such that $\|x(\cdot) - z(\cdot)\| < \varepsilon$. Thus

$$d(y(\cdot), S_F(x_0)) \le \|x(\cdot) - y(\cdot)\| \le \|x(\cdot) - z(\cdot)\| + \|z(\cdot) - y(\cdot)\| \le (1 + 1/(12M^6))\varepsilon.$$

Since ε is arbitrary, $y \in \overline{S_F(x_0)}$.

4. An application. Let X be a Banach space and Y be a separable Banach space. Also let K, K_ε $(0 < \varepsilon \le 1)$ be closed subsets of X and let $U(\cdot) : I \to 2^X$ be a measurable multifunction with nonempty closed values.

Consider a function $f : I \times X \times Y \times [0,1] \to X$. We will assume the following hypotheses:

(1) For all $(x, u, \varepsilon) \in X \times Y \times [0,1]$, $t \to f(t, x, u, \varepsilon)$ is measurable, and for every $t \in I$, $(x, u, \varepsilon) \to f(t, x, u, \varepsilon)$ is continuous.

(2) There exists $l(\cdot) \in L^1(I, \mathbb{R}^+)$ such that for almost every $t \in I$ and for all $u \in U(t)$ and $0 \le \varepsilon \le 1$,

$$\|f(t, x', u, \varepsilon) - f(t, x'', u, \varepsilon)\| \le l(t)\|x' - x''\|.$$

(3) For almost every $t \in I$ and for all $x \in X$ and $0 \le \varepsilon \le 1$ the set $F(t,x,\varepsilon) = f(t,x,U(t),\varepsilon)$ is closed and contained in $l(t)B$.

(4) $F(t,x) = F(t,x,0) \subset T_K(x)$ for $(t,x) \in I \times K$.

(5) $\bigcup_{0<\varepsilon<1} K_\varepsilon$ is compact, K is proximal and $\limsup_{\varepsilon\to 0} K_\varepsilon \subset K$, where the lim sup is defined in the Kuratowski sense, i.e.,

$$\limsup_{\varepsilon\to 0} K_\varepsilon = \{x \in X \mid \liminf_{\varepsilon\to 0} d(x, K_\varepsilon) = 0\}.$$

(6) Let $g : X \to \mathbb{R}$ be continuous. Consider the optimal control problem

$$(\mathrm{P}_\varepsilon) \qquad J(u,\varepsilon) = g(x(T)) \to \inf$$

subject to

$$\dot{x}(t) = f(t,x,u,\varepsilon), \quad x(0) = x_0, \tag{4.1}$$

$$x(t) \in K_\varepsilon, \tag{4.2}$$

where $u \in U_{\mathrm{ad}} = \{u(\cdot) : I \to Y \mid u(t) \in U(t) \text{ is measurable}\}$.

We denote the value of (P_ε) by V_ε and the value of the original problem (P_0) $(\varepsilon = 0)$ by V; we say that (P_ε) is *well-posed* if $V_\varepsilon \to V$ as $\varepsilon \to 0$.

To prove well-posedness, we need the following hypothesis:

(7) There exists a minimizing sequence $\{u_n\}$ for (P_0) such that if $x_n(\cdot,\varepsilon)$ and $x_n(\cdot)$ are solutions of (4.1), (4.2) and of the original equation $(\varepsilon = 0)$ respectively with $u_n(\cdot)$, then $x_n(T,\varepsilon) \to x_n(T)$ as $\varepsilon \to 0$.

THEOREM 4.1. *If hypotheses* (1)–(7) *hold, then the problem* (P_ε) *is well-posed.*

Proof. By (7), there exist a minimizing sequence $\{u_n(\cdot)\}$ for (P_0) and solutions $x_n(\cdot)$ of (4.1) and (4.2) (for $\varepsilon = 0$) with respect to $u_n(\cdot)$ such that $g(x_n(T,\varepsilon)) \to g(x_n(T))$ as $\varepsilon \to 0$. Also note that $V_\varepsilon \le g(x_n(T,\varepsilon))$. So we get

$$\limsup_{\varepsilon\to 0} V_\varepsilon \le V. \tag{4.3}$$

On the other hand, let $\varepsilon_n \to 0$ $(\varepsilon_n < 1)$. Choose admissible state-control pairs (x_n, u_n) for (4.1) and (4.2) such that

$$J(u_n, \varepsilon_n) \le V(\varepsilon_n) + 1/n. \tag{4.4}$$

We note that $x_n(t) \in K_{\varepsilon_n} \subset \bigcup_{0<\varepsilon<1} K_\varepsilon$ and $\|\dot{x}_n(t)\| \le l(t)$. From the Ascoli–Arzelà theorem, taking a subsequence and keeping the same notations we may assume that $x_n(\cdot) \to x(\cdot)$ in $C(0,T;X)$ and $\dot{x}_n \xrightarrow{w} \dot{x}(\cdot)$ in $L^1(I,X)$.

It is easy to show that (see [6])

$$\dot{x}(t) \in \overline{\mathrm{co}} \limsup_{n\to\infty} F(t, x_n(t), \varepsilon_n) \subset \overline{\mathrm{co}} F(t, x(t))$$

($t \to F(t,x,\varepsilon)$ is measurable; $x \to F(t,x,\varepsilon)$ is $l(t)$-Lipschitz and $\varepsilon \to F(t,x,\varepsilon)$ is continuous).

By hypothesis (5), we get

$$x(t) \in \limsup_{\varepsilon \to 0} K_\varepsilon \subset K.$$

From the definition of F and hypotheses (1)–(5), we know that F and K satisfy the hypotheses (H_1)–(H_4). By Theorem 3.2, there exists a sequence $\{x_m(\cdot)\}$ of solutions of the differential inclusions

$$\dot{x}_m(t) \in F(t, x_m(t)), \quad x_m(0) = x_0 \quad \text{and} \quad x_m(t) \in K \tag{4.5}$$

such that $x_m(\cdot) \to x(\cdot)$ in $C(0, T; X)$. From [3, p. 214], there exists a sequence $\{u_m(t)\} \in U(t)$ of measurable functions such that

$$\dot{x}_m(t) = f(t, x_m(t), u_m(t), 0), \quad x_m(0) = x_0 \quad \text{and} \quad x_m(t) \in K. \tag{4.6}$$

Hence, we get $g(x(T)) = \lim_{m\to\infty} g(x_m(T)) \geq V$. Note that by passing to the limit in (4.4), we obtain

$$V \leq g(x(T)) = \lim_{n\to\infty} g(x_n(T, \varepsilon_n)) = \lim_{n\to\infty} J(u_n, \varepsilon_n) \leq \lim_{n\to\infty} V(\varepsilon_n). \tag{4.7}$$

From (4.4)–(4.7), we deduce $V(\varepsilon) \to V$ as $\varepsilon \to 0$.

References

[1] J.-P. Aubin, *Viability Theory*, Birkhäuser, Boston, 1991.

[2] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.

[3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[4] A. Cellina and V. Staicu, *Well posedness for differential inclusions on closed sets*, J. Differential Equations 92 (1991), 2–13.

[5] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[6] H. Frankowska, *A priori estimates for operational differential inclusions*, J. Differential Equations 84 (1990), 100–128.

[7] N. S. Papageorgiou, *Relaxability and well-posedness for infinite dimensional optimal control problems*, Indian J. Pure Appl. Math. 21 (1990), 513–526.

[8] S. Shi, *Viability theorems for a class of differential-operator inclusions*, J. Differential Equations 79 (1989), 232–257.

[9] A. A. Tolstonogov, *The solution set of a differential inclusion in a Banach space. II*, Sibirsk. Mat. Zh. 25 (4) (1984), 159–173 (in Russian).

[10] A. A. Tolstonogov and P. I. Chugunov, *The solution set of a differential inclusion in a Banach space. I*, ibid. 24 (6) (1983), 144–159 (in Russian).

[11] Q. J. Zhu, *On the solution set of differential inclusions in Banach space*, J. Differential Equations 93 (1991), 213–237.

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Received on 18.10.1993