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ON THE STOCHASTIC REGULARITY OF SEQUENCE TRANSFORMATIONS OPERATING IN A BANACH SPACE

I. Introduction. In numerical analysis, convergence acceleration methods have been studied for many years and applied to various situations (see [3]).

On the other hand, the jacknife, a well-known statistical procedure for bias reduction, has been studied in the recent years by several authors (for example, see [4]), who established a direct parallel between the jacknife statistic and the e_n-transformation which is a sequence transformation used in numerical analysis for accelerating the convergence of a sequence by extrapolation.

Thus the idea of applying sequence transformations studied in numerical analysis to sequences of random elements converging in a stochastic mode was born.

In this paper, we define a new notion of stochastic regularity and we develop linear transformations called summation processes applied to sequences of random elements in a Banach space. It is shown that the regular summation process defined in numerical analysis is not always regular for every mode of stochastic convergence.

II. Definitions and notations

II.1. *Definitions relating to numerical analysis.* Let E be a Banach space with norm $\| \|_E$, and let $\mathcal{S}(E)$ be the set of sequences whose terms are elements of E.

Let T be a transformation which transforms a sequence $(S_n) \in \mathcal{S}(E)$ into another sequence $(T_n) \in \mathcal{S}(E)$. We say that T *operates* in $\mathcal{S}(E)$. Let

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(S_n) converge to S. Then, if (T_n) also converges to S, we say that T is *regular for the sequence* (S_n).

If, for each converging sequence $(S_n) \in \mathcal{S}(E)$, (T_n) also converges to the same limit, we say that T is *regular for* $\mathcal{S}(E)$.

II.2. *Definitions relating to probability*

II.2.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let F be a separable Banach space with norm $\| \ \|_F$ and σ-field $\mathcal{B}$ of Borel sets. Let S be a measurable mapping from $(\Omega, \mathcal{A})$ into $(F, \mathcal{B})$; we call it an *F-valued random element.* Let (S_n) be a sequence of F-valued random elements (defined on the same field Ω); under the assumption of separability, it is known that $\|S_n - S\|_F$ is a random variable defined on Ω (see [1] and [5]).

II.2.2. Now, we recall the definitions of stochastic convergences.

(a) *Convergence in distribution*:

$$S_n \xrightarrow{\mathcal{D}} S \Leftrightarrow \text{ for all } B \in \mathcal{B} \text{ with } P(S \in \partial B) = 0,\ P(S_n \in B) \to P(S \in B).$$

(b) *Convergence in probability*:

$$S_n \xrightarrow{P} S \Leftrightarrow \forall \varepsilon > 0,\ P(\|S_n - S\|_F \geq \varepsilon) \to 0.$$

(c) *Almost sure convergence*:

$$S_n \xrightarrow{\text{a.s.}} S \Leftrightarrow P(\|S_n - S\|_F \to 0) = 1.$$

(d) *Almost complete convergence*:

$$S_n \xrightarrow{\text{a.c.}} S \Leftrightarrow \forall \varepsilon > 0, \quad \sum_{n \geq 1} P(\|S_n - S\|_F \geq \varepsilon) < \infty.$$

(e) *Convergence in the rth mean*:

$$S_n \xrightarrow{N_r} S \Leftrightarrow E(\|S_n - S\|_F)^r \to 0.$$

Let us recall the following implications:

$$\text{a.c.} \Rightarrow \text{a.s.} \Rightarrow P \Rightarrow \mathcal{D} \quad \text{and} \quad N_r \Rightarrow N_{r'} \Rightarrow P, \quad \text{with} \quad r' < r$$

(for the proof, see [2]).

II.2.3. Finally, we set:

(a) $\mathcal{L}_0(\Omega, \mathcal{A}, P, F)$ or $\mathcal{L}_0(P, F)$, the vector space of F-valued random elements, and $L_0(P, F) = \mathcal{L}_0(P, F)/\sim$, the quotient space by the equivalence relation "$S = T$ almost surely".

(b) $\mathcal{L}_r(P, F)$, the vector subspace of $\mathcal{L}_0(P, F)$ defined by $S \in \mathcal{L}_r(P, F)$ iff $\int_\Omega \|S\|_F^r \, dP < \infty$, and $L_r(P, F) = \mathcal{L}_r(P, F)/\sim$. It is known that for $r > 0$, $L_r(P, F)$ is a Banach space with norm $\|S\|_{L_r} = (\int_\Omega \|S\|_F^r \, dP)^{1/r}$ associated with the convergence $S_n \xrightarrow{N_r} S$.

(c) $\mathcal{L}_\infty(P,F)$, the vector space of F-valued random elements S such that $\|S\|_F < \infty$ and $L_\infty(P,F) = \mathcal{L}_\infty(P,F)/\sim$. It is known that $L_\infty(P,F)$ is a Banach space with norm $\|S\|_{L_\infty} = \operatorname{ess\,sup} \|S\|_F$ (see [5]). (We write $S_n \overset{N_\infty}{\to} S \Leftrightarrow \|S_n - S\|_{L_\infty} \to 0$.)

III. Stochastic regularity of a sequence transformation

III.1. *General case.* Let $\mathcal{S}[\mathcal{L}_0(P,F)]$ be the set of sequences of F-valued random elements and let (S_n) be a sequence converging to $S \in \mathcal{L}_0(P,F)$ for one of the modes $\mathcal{M}$ defined in II.2.

Let T be a sequence transformation operating in F.

Taking $s_n = S_n(\omega)$ and $t_n = T_n(\omega)$, for $\omega \in \Omega$, if T_n is measurable for all n (which we suppose), we may consider that T operates in $\mathcal{L}_0(P,F)$ and transforms the sequence (S_n) into another sequence (T_n).

Definition III.1. We say that T is $\mathcal{M}$-*regular for* (S_n) if $S_n \overset{\mathcal{M}}{\to} S$ implies $T_n \overset{\mathcal{M}}{\to} S$.

Definition III.2. We say that T is $\mathcal{M}$-*regular for* $\mathcal{S}[\mathcal{L}_0(P,F)]$ if for every sequence (S_n) of F-valued elements such that $S_n \overset{\mathcal{M}}{\to} S$ we have $T_n \overset{\mathcal{M}}{\to} S$.

Remark. From the connections between the different modes of convergence, we immediately have the corresponding implications:

$$T \text{ a.c.-regular} \Rightarrow T \text{ a.s-regular} \Rightarrow T\ P\text{-regular} \Rightarrow T\ \mathcal{D}\text{-regular},$$
$$T\ N_r\text{-regular} \Rightarrow T\ N_{r'}\text{-regular with } r' < r.$$

This obviously holds for a sequence (S_n) well defined under the condition that (S_n) converges for the mode $\mathcal{M}$ concerned. Applying the definition of almost sure convergence, we obtain

Theorem III.1. *If T is regular for $\mathcal{S}(F)$, then T is a.s.-regular for $\mathcal{S}[\mathcal{L}_0(P,F)]$.*

III.2. *Finite summation process.* We call so the simplest linear transformation defined by $(a_0,\ldots,a_k) \in \mathbb{R}^k$ or $\mathbb{C}^k$, with k fixed in $\mathbb{N}$. (S_n) being a sequence in $\mathcal{S}(E)$, the sequence (T_n) is defined by $T_n = a_0S_n + \ldots + a_kS_{n+k}$.

Such a process is said to be *regular* if $a_0 + a_1 + \ldots + a_k = 1$. Under this assumption, clearly, for any Banach space E, the associated transformation operating in $\mathcal{S}(E)$ is regular for $\mathcal{S}(E)$.

Now, let $(S_n) \in \mathcal{S}[\mathcal{L}_0(P,F)]$. From Theorem III.1, we obviously have the following result concerning almost sure convergence:

THEOREM III.2. *Let* (S_n) *be a sequence of* F*-valued elements and* T *the preceding transformation operating in* $\mathcal{S}[\mathcal{L}_0(P,F)]$. *If the process is regular, then* T *is a.s.-regular for* $\mathcal{S}[\mathcal{L}_0(P,F)]$. ■

Now, taking $E = L_r(P,F)$ with norm $\|\ \|_{L_r}$, we obtain the result concerning convergence in the rth mean for all $r \in]0,\infty[$ and convergence in $L_\infty(P,F)$:

THEOREM III.3. *Under the assumptions of Theorem* III.2, T *is* N_r*-regular for* $\mathcal{S}[\mathcal{L}_r(P,F)]$ *for all* $r \in]0,\infty]$.

Concerning almost complete convergence, we have

THEOREM III.4. *Under the assumptions of Theorem* III.2, T *is a.c.-regular for* $\mathcal{S}[\mathcal{L}_0(P,F)]$.

Proof. Suppose that, for all $\varepsilon > 0$, $\sum_{n\in\mathbb{N}} P(\|S_n - S\|_F \ge \varepsilon) < \infty$.

Since $T_n - S = a_0(S_n - S) + \ldots + a_k(S_{n+k} - S)$, we have

$$\|T_n - S\|_F \le |a_0|\cdot\|S_n - S\|_F + \ldots + |a_k|\cdot\|S_{n+k} - S\|_F. \tag{1}$$

Now, $\|S_j - S\|_F < \varepsilon$ for all $j \in \{n,\ldots,n+k\}$ implies

$$\|T_n - S\|_F < \varepsilon(|a_0| + \ldots + |a_k|) = M\varepsilon$$

where M is a constant. Hence

$$P\Big(\bigcap_{j=n}^{n+k} \|S_j - S\|_F < \varepsilon\Big) \le P(\|T_n - S\|_F < M\varepsilon).$$

It follows that

$$P\Big(\bigcup_{j=n}^{n+k} \|S_j - S\|_F \ge \varepsilon\Big) \ge P(\|T_n - S\|_F \ge M\varepsilon)$$

and

$$P(\|T_n - S\|_F \ge M\varepsilon) \le \sum_{j=n}^{n+k} P(\|S_j - S\|_F \ge \varepsilon).$$

Writing this inequality for $n = 0,1,\ldots$ and summing we obtain

$$\sum_{n\in\mathbb{N}} P(\|T_n - S\|_F \ge M\varepsilon) \le (k+1)\sum_{n\in\mathbb{N}} P(\|S_n - S\|_F \ge \varepsilon).$$

Taking $\varepsilon = \varepsilon'/M$, for any $\varepsilon' > 0$ we obtain $\sum_{n\in\mathbb{N}} P(\|T_n - S\|_F \ge \varepsilon') < \infty$. ■

Concerning convergence in probability, we have

THEOREM III.5. *Under the assumptions of Theorem* III.2, T *is* P*-regular for* $\mathcal{S}[\mathcal{L}_0(P,F)]$.

Proof. Suppose that, for all $\varepsilon > 0$, $P(\|S_n - S\|_F \ge \varepsilon) \to 0$. We may write (1). Then, for each $\varepsilon > 0$, $\|T_n - S\|_F \ge \varepsilon$ implies $|a_0|\cdot\|S_n - S\|_F + \ldots + |a_k|\cdot\|S_{n+k} - S\|_F \ge \varepsilon$. Hence

$$(2)\quad P\{\|T_n - S\|_F \ge \varepsilon\} \le P\{|a_0|\cdot\|S_n - S\|_F + \ldots + |a_k|\cdot\|S_{n+k} - S\|_F \ge \varepsilon\}.$$

But we know that the convergence in probability of random variables is compatible with the vector space structure of $\mathbb{R}$. Then $\|S_n - S\|_F \xrightarrow{P} 0$ implies $|a_0|\cdot\|S_n - S\|_F + \ldots + |a_k|\cdot\|S_{n+k} - S\|_F \xrightarrow{P} 0$ and the result comes from (2). ∎

R e m a r k. Concerning convergence in distribution, the following example proves that we do not obtain a similar result.

Take $k = 1$, $a_0 = a_1 = 1/2$ and $F = \mathbb{R}$. The sequence (S_n) is defined by

$$S_{2n} = S_0, \quad S_{2n+1} = -S_0 \quad \text{for } n \in \mathbb{N}.$$

Suppose now that S_0 has a symmetric distribution different from the Dirac measure δ_0 (that is, $P\{S_0 \ne 0\} > 0$). It follows that S_n has the same distribution as S_0, hence $S_n \xrightarrow{\mathcal{D}} S_0$. But $T_n = 0$, for all n, and thus $T_n \overset{\mathcal{D}}{\nrightarrow} S_0$.

III.3. *Summation process.* A *summation process* is the linear transformation defined by an infinite triangular matrix $A = (a_k^j)_{k\in\mathbb{N},\,0\le j\le k}$ where the a_k^j's are constants of $\mathbb{C}$ or $\mathbb{R}$. It transforms a sequence $(s_n) \in \mathcal{S}(F)$ into the sequence $(t_k^{(n)})$ defined by $t_k^{(n)} = a_k^0 s_n + \ldots + a_k^k s_{n+k}$, and a sequence $(S_n) \in \mathcal{S}[\mathcal{L}_0(p, F)]$ into the sequence $(T_k^{(n)})$ with $T_k^{(n)}(\omega) = t_k^{(n)}$ for $\omega \in \Omega$.

Such a transformation is said to be *regular* (or A is regular) if it satisfies the assumptions

(i) $\sum_{j=0}^k |a_k^j| \le M$ for all $k \in \mathbb{N}$,

(ii) $\lim_{k\to\infty} a_k^j = 0$ for all $j \in \mathbb{N}$,

(iii) $\lim_{k\to\infty} \sum_{j=0}^k a_k^j = 1$.

It is said to be *total* (or A is total) if (iii) becomes

(iii′) $\sum_{j=0}^k a_k^j = 1$ for all $k \in \mathbb{N}$.

Clearly, in the case of a total process, for each fixed k in $\mathbb{N}$ the transformation $T_{(k)}$ which transforms the sequence (S_n) into the sequence $(T_k^{(n)})_{n\in\mathbb{N}}$ is a regular finite summation process as studied in II.

In the following, we consider the transformation $T^{(n)}$ which transforms (S_n) into the sequence $(T_k^{(n)})_{k\in\mathbb{N}}$ with $T_k^{(n)} = a_k^0 S_n + \ldots + a_k^k S_{n+k}$, for n fixed. First, let us recall a well known theorem:

TOEPLITZ THEOREM. *Let E be a Banach space, (S_n) a sequence in $\mathcal{S}(E)$ and (T_k) the sequence in $\mathcal{S}(E)$ transformed by a summation process $T_k = a_k^0 S_0 + \ldots + a_k^k S_k$. Then a necessary and sufficient condition that, for all converging sequences (S_n), the sequence (T_k) converges to the same limit, is that the process is regular.*

For the proof see [7]. ■

Now, suppose the sequence (S_n) does converge to S for a mode $\mathcal{M}$ of stochastic convergence.

Concerning almost sure convergence, the following result comes from the Toeplitz theorem with $E = F$. For n fixed, it is obvious that the properties of convergence are the same for $T^{(n)}$ and $T^{(0)}$.

THEOREM III.6. *Let* (S_n) *be a sequence of* F*-valued elements and* $T^{(n)}$ (n *fixed*) *the transformation operating in* $\mathcal{S}[\mathcal{L}_0(P,F)]$ *associated with a regular summation process. Then* $T^{(n)}$ *is a.s.-regular for* $\mathcal{S}[\mathcal{L}_0(P,F)]$. ■

Now, taking $E = L_r(P,F)$ with norm $\| \|_{L_r}$ in the Toeplitz theorem yields a result on convergence in the rth mean, for all $r \in]0,\infty[$, and convergence in $L_\infty(P,F)$.

THEOREM III.7. *Under the assumptions of Theorem* III.6, $T^{(n)}$ *is* N_r*-regular for* $\mathcal{S}[\mathcal{L}_r(P,F)]$.

Remark 1. Concerning convergence in distribution, the following example proves that we do not obtain $\mathcal{D}$-regularity.

Take $a_k^j = 1/(k+1)$ $(j = 0,1,\ldots,k)$ and $F = \mathbb{R}$. The sequence (S_n) is defined by

$$S_{2n} = S_0, \quad S_{2n+1} = -S_0 \quad \text{for } n \in \mathbb{N},$$

and suppose that S_0 has a symmetric distribution different from δ_0. Thus $S_n \xrightarrow{\mathcal{D}} S_0$. But the sequence $(T_k^{(n)})$ does not converge in distribution to S_0 since $T_{2i}^{(n)} = S_0/(2i+1)$ for all $i \in \mathbb{N}$ and $T_{2i+1}^{(n)} = 0$ for all $i \in \mathbb{N}$; hence $T_k^{(n)} \xrightarrow{\text{a.s.}} 0$ as $k \to \infty$. ■

Remark 2. Concerning convergence in probability, the following example proves that we do not obtain P-regularity.

Take $k \in \mathbb{N}^*$, $a_j^k = 1/k$ for $j = 1,\ldots,k$ and $F = \mathbb{R}$. Let $(S_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables with distribution functions

$$F_n(x) = \begin{cases} 1 - 1/(x+n) & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

Then

$$\forall \varepsilon > 0,\ P(|S_n| \ge \varepsilon) = \frac{1}{\varepsilon + n}, \quad \text{i.e.} \quad S_n \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Let

$$T_k^{(1)} = a_k^1 S_1 + \ldots + a_k^k S_k = \frac{S_1 + \ldots + S_k}{k},$$

Now we prove that $T_k^{(1)} \overset{P}{\nrightarrow} 0$. Let $M_k = \sup(S_1 \ldots S_k)$. Since $S_i \geq 0$, we have

$$\frac{M_k}{k} \geq \varepsilon \Rightarrow \frac{S_1 + \ldots + S_k}{k} \geq \varepsilon,$$

which implies

$$P(M_k/k \geq \varepsilon) \leq P(T_k^{(1)} \geq \varepsilon). \tag{1}$$

On the other hand,

$$\begin{aligned} P(M_k < x) &= P(S_1 < x) \ldots P(S_k < x) \\ &= \left(1 - \frac{1}{1+x}\right) \ldots \left(1 - \frac{1}{k+x}\right) < \left(1 - \frac{1}{k+x}\right)^k. \end{aligned} \tag{2}$$

It follows that

$$P(M_k/k < \varepsilon) = P(M_k < k\varepsilon) < \left(1 - \frac{1}{k\varepsilon + k}\right)^k.$$

From (1) we conclude that

$$P(T_k^{(1)} \geq \varepsilon) \geq 1 - P(M_k/k < \varepsilon) > 1 - \left(1 - \frac{1}{k\varepsilon + k}\right)^k.$$

Finally, $\lim_{k\to\infty} P(T_k^{(1)} \geq \varepsilon) \geq 1 - e^{-1/(1+\varepsilon)} \neq 0$ and $T_k^{(1)} \overset{P}{\nrightarrow} 0$.

R e m a r k 3. Finally, concerning almost complete convergence, we give an example proving that we do not obtain a.c.-regularity.

As in the preceding example, take

$$T_k^{(1)} = \frac{S_1 + \ldots + S_k}{k},$$

where (S_n) is a sequence of independent random variables with distribution functions

$$F_n(x) = \begin{cases} 1 - 1/(x + n^2) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then

$$\forall \varepsilon > 0, \quad \sum_{n \in \mathbb{N}^*} P(|S_n| \geq \varepsilon) = \sum_{n \in \mathbb{N}^*} \frac{1}{\varepsilon + n^2} < \infty.$$

Hence $S_n \overset{\text{a.c.}}{\to} 0$. Let us prove that $T_k^{(1)} \overset{\text{a.c.}}{\nrightarrow} 0$. Defining M_k as in the preceding example, we have (1) and (2). It follows that

$$P(M_k < x) < \left(1 - \frac{1}{k^2 + x}\right)^k$$

and

$$P(M_k/k < \varepsilon) = P(M_k < k\varepsilon) < \left(1 - \frac{1}{k^2 + k\varepsilon}\right)^k.$$

Hence

$$P(T_k^{(1)} \geq \varepsilon) \geq 1 - P(M_k/k < \varepsilon) > 1 - \left(1 - \frac{1}{k^2 + k\varepsilon}\right)^k.$$

But

$$1 - \left(1 - \frac{1}{k^2 + k\varepsilon}\right)^k = 1 - e^{kL_n(1 - \frac{1}{k^2+k\varepsilon})} \underset{k\to\infty}{\sim} \frac{k}{k^2 + k\varepsilon} \sim \frac{1}{k},$$

which implies $\sum_{k=1}^{\infty} P(T_k^{(1)} \geq \varepsilon) = \infty$ and $T_k^{(1)} \overset{\text{a.c.}}{\nrightarrow} 0$.

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