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MODULATING ELEMENT METHOD IN THE IDENTIFICATION OF A GENERALIZED DYNAMICAL SYSTEM

Abstract. In this paper the identification of generalized linear dynamical differential systems by the method of modulating elements is presented. The dynamical system is described in the Bittner operational calculus by an abstract linear differential equation with constant coefficients. The presented general method can be used in the identification of stationary continuous dynamical systems with compensating parameters and for certain nonstationary compensating or distributed parameter systems.

1. Introduction. The theoretical basis of operational calculus was markedly developed by the Polish mathematical school created by S. Bellert, R. Bittner and others (see [9]).

In his 1957 paper [1] and in its continuations S. Bellert formulated the general principles of operational calculus in linear spaces. Using operational calculus he also made efforts to create a uniform basis of dynamical systems theory [2, 3]. In these papers he noticed that using operational calculus "we avoid the necessity of creating separate theories for various system types".

R. Bittner has developed a similar concept of operational calculus since 1959 [4]. In the book [15] (which is a posthumous edition of Bellert's selected papers) J. Osiowski confirmed that Bittner brought the idea of the Bellert operational calculus to the shape of a compact and complete mathematical theory (see [5–7]).

Using the Bittner operational calculus, the authors [23, 24] presented certain generalizations of the identification method of a dynamical system by means of modulating functions (see [17, 11]).

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In [23], the identification of a system described by a differential equation of nth order required the knowledge of one modulating function and at least $n+1$ input signals, and the same number of output signals corresponding to them, while [24] required the knowledge of at least $n+1$ modulating functions, only one input signal and one output signal corresponding to it.

In this paper a further generalization of the modulating function method is given. Similar to the previous papers, there is also considered the problem of choosing the best model (equation) describing the dynamics of the studied system. For its identification we now have to know only one modulating function and only one pair of input and output signals corresponding to each other. The number of input and output signals required for the identification is important for economical reasons, when we take into account the costs of those signal measurements. It is also of particular importance in the cases when multiple measurements are troublesome or unfeasible.

Using the notion of the modulating element defined in the Bittner operational calculus we perform the identification of the dynamical system described by the following abstract linear differential equation with constant coefficients:

$$a_n S^n y + a_{n-1} S^{n-1} y + \ldots + a_1 S y + a_0 y = u. \tag{0}$$

Here S stands for an abstract derivative and u and y denote the input and output signals of the system, respectively.

The proposed identification is based on an optimization algorithm. The classical modulating function method, as given in [17], concerns the compensating constants system only. Moreover, the problem of optimization is not posed at all.

In our generalization of the modulating function method, using various representations of operational calculus we may identify various types of dynamical systems.

In this paper we discuss the identification of the stationary systems described by linear ordinary differential equations of second order (together with the interpolation of signals by means of splines) and of nonstationary first order systems described by linear ordinary differential equations and quasi-linear partial differential equations.

2. The operational calculus. In accordance with the notation used e.g. in [6], a *Bittner operational calculus* is a system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where L^0 and L^1 are linear spaces over the same field Γ of scalars, the linear operation $S : L^1 \to L^0$ (written $S \in L(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, the nonempty set Q is the set of indices q for the operations $T_q \in L(L^0, L^1)$ such that $ST_q w = w$, $w \in L^0$, called *integrals*, and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx, x \in L^1$, called *limit conditions*.

By induction we define a sequence of spaces $L^n, n \in \mathbb{N}$, such that

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}.$$

Then $\ldots \subset L^n \subset L^{n-1} \subset \ldots \subset L^1 \subset L^0$ and

$$S^n(L^{m+n}) = L^m,$$

where

$$L(L^n, L^0) \ni S^n := \underbrace{S \circ \ldots \circ S}_{n \text{ times}}, \qquad n \in \mathbb{N},\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The kernel of S, i.e. the set $\operatorname{Ker} S := \{c \in L^1 : Sc = 0\}$, is called the *space of constants* for the derivative S.

Let Q be a set which has more than one element. The mapping $I_{q_1}^{q_2} \in L(L^0, \operatorname{Ker} S)$ defined by

$$(1) \qquad I_{q_1}^{q_2} w := (T_{q_1} - T_{q_2})w = s_{q_2} T_{q_1} w, \qquad q_1, q_2 \in Q,\ w \in L^0,$$

is called the *operation of definite integration*.

Suppose that L^0 is an algebra and L^1 is its subalgebra. We say that the derivative S satisfies the *Leibniz condition* if

$$(2) \qquad S(x \cdot y) = Sx \cdot y + x \cdot Sy, \qquad x, y \in L^1.$$

We say that the limit condition s_q is *multiplicative* if

$$(3) \qquad s_q(x \cdot y) = s_q x \cdot s_q y, \qquad x, y \in L^1.$$

3. The system identification. Henceforth, we assume that

- $q_0, q_1, \ldots, q_m \in Q$, $m \geq n + 1$,
- L^0 is a real algebra, and L^1 is its subalgebra,
- the derivative S satisfies the Leibniz condition (2),
- the operations $s_{q_\nu}, q_\nu \in Q$, $\nu \in \overline{0, m} := \{0, 1, \ldots, m\}$, are multiplicative.

Let $R_{q_\mu}^{q_\nu} \in L(L^1, \operatorname{Ker} S)$ be defined by

$$R_{q_\mu}^{q_\nu} x := (s_{q_\nu} - s_{q_\mu})x = I_{q_\mu}^{q_\nu} Sx, \qquad q_\mu, q_\nu \in Q,\ x \in L^1.$$

By induction on $k \in \mathbb{N}$ we can prove [23, 21] the following formula of *integration by parts*:

$$(4) \qquad I_{q_\mu}^{q_\nu}(x \cdot S^k y) = \sum_{i=0}^{k-1} (-1)^i R_{q_\mu}^{q_\nu}(S^i x \cdot S^{k-i-1}) + (-1)^k I_{q_\mu}^{q_\nu}(S^k x \cdot y),$$

where $S^0 x := x$, $q_\mu, q_\nu \in Q$, $x, y \in L^k$, $k \in \mathbb{N}$.

Consider all the real systems whose dynamics, in suitable models of operational calculus, is described by an equation

$$a_n S^n y + a_{n-1} S^{n-1} y + \ldots + a_1 S y + a_0 y = u, \tag{5}$$

where $a_i \in \mathbb{R}^1$, $i \in \overline{0,n}$, $u \in L^0$, $y \in L^n$, $n \in \mathbb{N}$.

The model (5) will be called a *generalized linear dynamical differential stationary system with compensating constants*. The given element u and the unknown element y will be called the *input signal* (*control*) and *output signal* (*response*) of the system (5), respectively. The set Q will be called the set of *instants* (see [22–24]).

Assume that the pair $(u, y) \in L^0 \times L^n$ satisfies (5) with the given coefficients $a_0, a_1, \ldots, a_n$. Then for every $f \in L^n$ we have

$$a_n f S^n + a_{n-1} f S^{n-1} + \ldots + a_1 f S y + a_0 f y = f u. \tag{6}$$

Acting by $I_{q_{\nu-1}}^{q_\nu}$ on both sides and then using (4) we obtain

$$\sum_{i=1}^{n} a_i \Big[\sum_{j=0}^{i-1} (-1)^j R_{q_{\nu-1}}^{q_\nu} (S^j f \cdot S^{i-j-1} y) \Big] + \sum_{i=0}^{n} (-1)^i a_i I_{q_{\nu-1}}^{q_\nu} (S^i f \cdot y) = I_{q_{\nu-1}}^{q_\nu} (f u), \tag{7}$$

where $\nu \in \overline{1,m}$.

Assume that $f \in L^n$ satisfies

$$f \notin \operatorname{Ker} S^n, \quad s_{q_\nu} S^i f = 0, \quad \nu \in \overline{0,m},\ i \in \overline{0,n-1}. \tag{8}$$

Then $f \in L^n$ will be called a *modulating element* of (5) corresponding to $q_0, q_1, \ldots, q_m \in Q$.

With the above assumptions, we obtain from (7),

$$\sum_{i=0}^{n} (-1)^i a_i I_{q_{\nu-1}}^{q_\nu} (S^i f \cdot y) = I_{q_{\nu-1}}^{q_\nu} (f u), \qquad \nu \in \overline{1,m}.$$

The system may be written in the form

$$\sum_{i=0}^{n} a_i \overline{v}_i = \overline{w},$$

where

$$\overline{v}_i := \begin{bmatrix} (-1)^i I_{q_0}^{q_1} (S^i f \cdot y) \\ \vdots \\ (-1)^i I_{q_{m-1}}^{q_m} (S^i f \cdot y) \end{bmatrix}, \quad \overline{w} := \begin{bmatrix} I_{q_0}^{q_1} (f u) \\ \vdots \\ I_{q_{m-1}}^{q_m} (f u) \end{bmatrix}, \quad i \in \overline{0,n}. \tag{9}$$

From (1) it follows that

$$\overline{v}_i, \overline{w} \in (\operatorname{Ker} S)_m := \bigoplus_{\nu=1}^{m} \operatorname{Ker} S, \qquad i \in \overline{0,n},$$

where $\oplus$ is direct sum.

In this paper, by *identification of a dynamical system* (5) we shall understand the problem of choosing the coefficients of (5) with given $u^* \in L^0, y^* \in L^n$ so that for some modulating element $f \in L^n$ the functional

$$J_f(a_0, a_1, \ldots, a_n) := \Big\| \sum_{i=0}^{n} a_i \overline{v}_i^* - \overline{w}^* \Big\| \tag{10}$$

(called the *identification performance index*) attains its minimum, where $\|\cdot\|$ is the norm induced by the scalar product $(\cdot \mid \cdot)$ in a fixed Hilbert space H and $\overline{v}_i^*, \overline{w}^* \in (\operatorname{Ker} S)_m$, $i \in \overline{0,n}$, are vectors of the form (9) determined for the signals u^*, y^*.

Assume that $B := \{\overline{v}_0^*, \overline{v}_1^*, \ldots, \overline{v}_n^*\}$ is a set of linearly independent vectors in H. Then

$$\operatorname{Lin} B := \Big\{ \overline{w} = \sum_{i=0}^{n} a_i \overline{v}_i^* : a_i \in \mathbb{R}^1, i \in \overline{0,n} \Big\}$$

is a closed subspace of H.

Now, we determine a vector

$$\overline{w}^0 = a_0^0 \overline{v}_0^* + a_1^0 \overline{v}_1^* + \ldots + a_n^0 \overline{v}_n^* \in \operatorname{Lin} B$$

which is the nearest (with respect to the norm $\|\cdot\|$) to the given vector $\overline{w}^*$. This means that we shall find real numbers $(a_0^0, a_1^0, \ldots, a_n^0)$ such that

$$\|\overline{w}^0 - \overline{w}^*\| = J_f(a_0^0, a_1^0, \ldots, a_n^0) = \min\{J_f(a_0, a_1, \ldots, a_n) : a_i \in \mathbb{R}^1, i \in \overline{0,n}\}.$$

From the orthogonal projection theorem (Th. 2 of [12]) we infer the existence and uniqueness of $\overline{w}^0$ and the orthogonality of $\overline{w}^0 - \overline{w}^*$ to every $\overline{v}_j^* \in B, j \in \overline{0,n}$. Therefore

$$(\overline{w}^0 - \overline{w}^* \mid \overline{v}_j^*) = \Big(\sum_{i=0}^{n} a_i^0 \overline{v}_i^* - \overline{w}^* \,\Big|\, \overline{v}_j^* \Big) = 0, \qquad j \in \overline{0,n}.$$

Hence

$$\sum_{i=0}^{n} a_i^0 b_{ij} = c_j, \qquad i, j \in \overline{0,n}, \tag{11}$$

where

$$b_{ij} := (\overline{v}_i^* \mid \overline{v}_j^*), \quad c_j := (\overline{w}^* | \overline{v}_j^*), \quad i, j \in \overline{0,n}. \tag{12}$$

The linear system (11) has exactly one solution, because the matrix $[(\overline{v}_i^* \mid \overline{v}_j^*)]_{(n+1)\times(n+1)}$ is non-singular (this is equivalent to the linear independence of $\overline{v}_0^*, \overline{v}_1^*, \ldots, \overline{v}_n^*$).

As in [22], one can prove that by increasing the order of equation (5) the exactness of identification will not deteriorate, more precisely, the identification performance index given by (10) will not increase.

Fix $G := \{g_{-1}, g_0, \ldots, g_r\} \subset L^n$. Assume that $u^*, y^* \in \operatorname{Lin} G$. Moreover, let V_i^ν, W^ν denote the νth coordinates of $\overline{v}_i^*$ and $\overline{w}^*$, respectively. Then, by (9), we obtain

$$V_i^\nu = (-1)^i \sum_{j=-1}^{r} \beta_j d_{i,\nu,j}, \qquad W^\nu = \sum_{j=-1}^{r} \alpha_j d_{0,\nu,j}, \tag{13}$$

where

$$u^* = \sum_{j=-1}^{r} \alpha_j g_j, \qquad y^* = \sum_{j=-1}^{r} \beta_j g_j$$

and

$$d_{i,\nu,j} := I_{q_{\nu-1}}^{q_\nu}(S^i f \cdot g_j) \in \operatorname{Ker} S, \qquad i \in \overline{0,n},\ \nu \in \overline{1,m},\ j \in \overline{-1,r}. \tag{14}$$

The above formulas are useful for the identification of dynamical systems differing by the signals $u^*, y^* \in \operatorname{Lin} G$ only, since the constants $d_{i,\nu,j}$ are independent of the signals.

4. Examples. A. Let $L^n := C^n(\mathbb{R}^1, \mathbb{R}^1)$, $n \in \mathbb{N}_0$, and

$$S := \frac{d}{dt}, \quad T_q := \int_q^t, \quad s_q :=|_{t=q}, \quad q \in Q := \mathbb{R}^1.$$

With the natural definition of multiplication the spaces $L^n, n \in \mathbb{N}_0$, are algebras such that $L^n \subset L^{n-1}, n \in \mathbb{N}$, whereas the derivative S satisfies the Leibniz condition and the operations $s_q, q \in Q$, are multiplicative.

As $\operatorname{Ker} S$ is the space of constant functions on $\mathbb{R}^1$, isomorphic to $\mathbb{R}^1$, for the Hilbert space H we take the real space l_m^2 with the inner product

$$(\overline{a} \mid \overline{b}) = \sum_{\nu=1}^{m} a_\nu b_\nu, \qquad \overline{a}, \overline{b} \in l_m^2, \tag{15}$$

and the norm

$$\|\overline{a}\| = \Big(\sum_{\nu=1}^{m} a_\nu^2\Big)^{1/2}, \qquad \overline{a} \in l_m^2.$$

Then (5) reads

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_0 y(t) = u(t), \tag{16}$$

where $u = u(t)$ is the input signal and $y = y(t)$ is the output signal of the system to be identified.

The identification algorithm of the system (16) comprises:

1) the algorithm of approximation of the input signal and the output signal,

2) the algorithm of choosing the coefficients of the differential equation,

3) the algorithm of verification of the model.

Let us discuss the identification algorithm for the second order equation

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = u(t). \tag{17}$$

1) From the values $\widetilde{u}_i = \widetilde{u}(t_i)$ of the input signal and the values $\widetilde{y}_i = \widetilde{y}(t_i)$ of the output signal, obtained from measurements on the real system at times $t_i = t_0 + ih, h = (t_{4k} - t_0)/(4k)$, $i \in \overline{0, 4k+1}$, we determine the functions $u^* = u^*(t)$ and $y^* = y^*(t)$ which are used in the identification of the coefficients of the system (17). We take interpolating cubic splines for approximation of the input and output signals.

Assume that $u^*, y^* \in \operatorname{Lin} G$, where $G = \{g_{-1}, g_0, \ldots, g_{4k+1}\}$, $k \geq 3$, and the g_j are cubic basic splines given by (see [10, 19])

$$g_j = \Phi_j(t) = \frac{1}{h^3}\begin{cases} (t - t_{j-2})^3 & \text{for } t \in [t_{j-2}, t_{j-1}], \\ h^3 + 3h^2(t - t_{j-1}) + 3h(t - t_{j-1})^2 - 3(t - t_{j-1})^3 & \\ & \text{for } t \in [t_{j-1}, t_j], \\ h^3 + 3h^2(t_{j+1} - t) + 3h(t_{j+1} - t)^2 - 3(t_{j+1} - t)^3, & \\ & \text{for } t \in [t_j, t_{j+1}], \\ (t_{j+2} - t)^3 & \text{for } t \in [t_{j+1}, t_{j+2}], \\ 0 & \text{for other } t \in \mathbb{R}^1, \end{cases} \tag{18}$$

$j \in \overline{-1, 4k+1}$ (Fig. 1). Obviously, G is a subset of $L^2 = C^2(\mathbb{R}^1, \mathbb{R}^1)$.

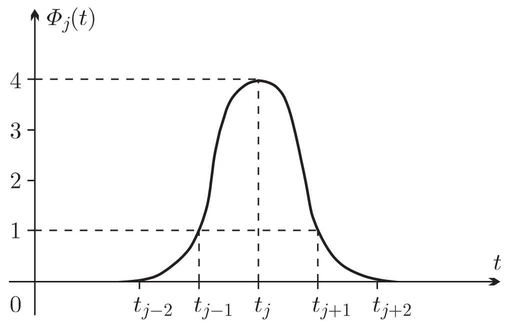

Fig. 1. The basic function $\Phi_j(t)$

The functions interpolating the input and output signals are determined by the coefficients α_j, β_j, $j \in \overline{-1, 4k+1}$, respectively, i.e.

$$u^* = \sum_{j=-1}^{4k+1} \alpha_j \Phi_j(t), \qquad y^* = \sum_{j=-1}^{4k+1} \beta_j \Phi_j(t) \quad [10]. \tag{19}$$

The coefficients of (19) are obtained from measurement data of $\widetilde{u}(t_i)$, $\widetilde{y}(t_i)$ and from the boundary conditions for the derivatives

$$\begin{aligned} \dot{u}^*(t_0) &= \dot{u}_0^*, & \dot{u}^*(t_{4k}) &= \dot{u}_{4k}^*, \\ \dot{y}^*(t_0) &= \dot{y}_0^*, & \dot{y}^*(t_{4k}) &= \dot{y}_{4k}^*, \end{aligned}$$

which we can approximate by difference quotients obtaining

$$\dot{u}_0^* \approx \frac{\widetilde{u}(t_1) - \widetilde{u}(t_0)}{h}, \qquad \dot{u}_{4k}^* \approx \frac{\widetilde{u}(t_{4k+1}) - \widetilde{u}(t_{4k})}{h}, \tag{20}$$

$$\dot{y}_0^* \approx \frac{\widetilde{y}(t_1) - \widetilde{y}(t_0)}{h}, \qquad \dot{y}_{4k}^* \approx \frac{\widetilde{y}(t_{4k+1}) - \widetilde{y}(t_{4k})}{h}. \tag{21}$$

Substituting the interpolating points $(t_i, \widetilde{u}_i), (t_i, \widetilde{y}_i), i \in \overline{0, 4k}$, into (19) and $(t_i, \dot{u}_i^*)$, $(t_i, \dot{y}_i^*)$, $i = 0, 4k$, into the derivatives of the functions (19) we obtain two systems of $4k+3$ equations with $4k+3$ unknowns (cf. [10]):

$$\begin{cases} -\alpha_{-1} + \alpha_1 = \frac{1}{3} h \dot{u}_0^*, \\ \alpha_{-1} + 4\alpha_0 + \alpha_1 = \widetilde{u}_0, \\ \dots\dots\dots\dots\dots\dots \\ \alpha_{4k-1} + 4\alpha_{4k} + \alpha_{4k+1} = \widetilde{u}_{4k}, \\ -\alpha_{4k-1} + \alpha_{4k+1} = \frac{1}{3} h \dot{u}_{4k}^*, \end{cases} \qquad \begin{cases} -\beta_{-1} + \beta_1 = \frac{1}{3} h \dot{y}_0^*, \\ \beta_{-1} + 4\beta_0 + \beta_1 = \widetilde{y}_0, \\ \dots\dots\dots\dots\dots\dots \\ \beta_{4k-1} + 4\beta_{4k} + \beta_{4k+1} = \widetilde{y}_{4k}, \\ -\beta_{4k-1} + \beta_{4k+1} = \frac{1}{3} h \dot{y}_{4k}^*. \end{cases} \tag{22}$$

After elimination of α_{-1}, α_{4k+1} and β_{-1}, β_{4k+1}, we obtain systems with tridiagonal coefficient matrices with dominating main diagonal. An algorithm (see Fig. 3, Interpol algorithm) of solving that type of systems of linear equations is presented in [10, 14].

Solving the systems (22) we obtain the interpolated (in $[t_0, t_{4k}]$) input signal u^* and output signal y^* in the form (19) of cubic splines.

2) From (8) it follows that every function $f = f(t) \in C^2(\mathbb{R}^1, \mathbb{R}^1) \setminus \operatorname{Ker} d^2/dt^2$ satisfying

$$f^{(i)}(q_\nu) = 0, \qquad \nu \in \overline{0, m},\ m \geq 3,\ i = 0, 1,$$

may be a modulating element of the system (17) corresponding to $q_\nu \in \mathbb{R}^1$.

In particular, the function f defined by

$$f(t) = \begin{cases} \Phi_2(t) & \text{for } t \in [t_0, t_4], \\ \Phi_6(t) & \text{for } t \in [t_4, t_8], \\ \dots & \\ \Phi_{4k-2}(t) & \text{for } t \in [t_{4k-4}, t_{4k}], \end{cases}$$

is a modulating function of the system (17) corresponding to $q_\nu = t_{4\nu}$, $\nu \in \overline{0,k}$ (Fig. 2).

That function can also be represented in the form

$$f(t) = \Phi_{4j-2}(t) \quad \text{for } t \in [t_{4j-4}, t_{4j}],\ j \in \overline{1,k}. \tag{23}$$

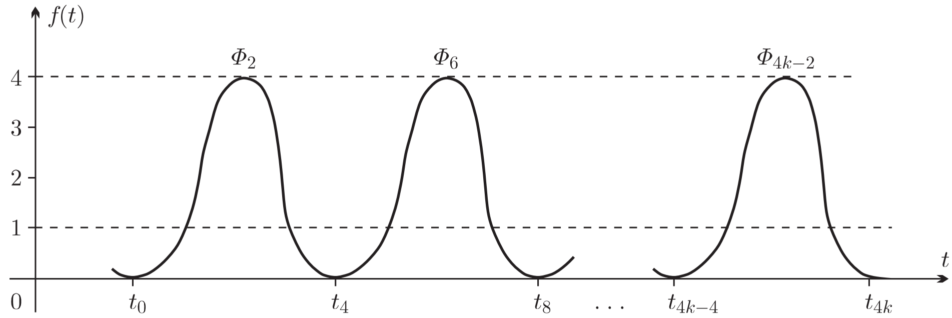

Fig. 2. The modulating function of the system (17)

With $q_\nu = t_{4\nu}$ we have $I_{q_{\nu-1}}^{q_\nu} = \int_{t_{4\nu-4}}^{t_{4\nu}}$. By (18) and (23), the formula (14) takes the form

$$d_{i,\nu,j} = \int\limits_{t_{4\nu-4}}^{t_{4\nu}} \Phi_{4\nu-2}^{(i)}(t)\Phi_j(t)\,dt, \qquad i = 0,1,2,\ \nu \in \overline{1,k},\ j \in \overline{-1,4k+1}. \tag{24}$$

The coefficients (24) are determined in [14] (cf. [19]). Their values for $i = 0,1,2$ are

$$d_{0,\nu,j} = h \begin{cases} \frac{1}{140} & \text{for } j = 4\nu-5,\ j = 4\nu+1, \\ \frac{6}{7} & \text{for } j = 4\nu-4,\ j = 4\nu, \\ \frac{1191}{140} & \text{for } j = 4\nu-3,\ j = 4\nu-1, \\ \frac{604}{35} & \text{for } j = 4\nu-2, \\ 0 & \text{for } |4\nu-2-j| > 3, \end{cases}$$

$$d_{1,\nu,j} = \begin{cases} \frac{1}{20} & \text{for } j = 4\nu-5, \\ \frac{14}{5} & \text{for } j = 4\nu-4, \\ \frac{49}{4} & \text{for } j = 4\nu-3, \\ 0 & \text{for } j = 4\nu-2, \\ -\frac{49}{4} & \text{for } j = 4\nu-1, \\ -\frac{14}{5} & \text{for } j = 4\nu, \\ -\frac{1}{20} & \text{for } j = 4\nu+1, \\ 0 & \text{for } |4\nu-2-j| > 3, \end{cases}$$

$$d_{2,\nu,j} = \frac{1}{h}\begin{cases} \frac{3}{10} & \text{for } j = 4\nu-5,\ j = 4\nu+1, \\ \frac{36}{5} & \text{for } j = 4\nu-4,\ j = 4\nu, \\ \frac{9}{2} & \text{for } j = 4\nu-3,\ j = 4\nu-1, \\ -24 & \text{for } j = 4\nu-2, \\ 0 & \text{for } |4\nu-2-j| > 3, \end{cases}$$

for $\nu \in \overline{1,k}$, $j \in \overline{-1,4k+1}$. Therefore, by (13), we obtain

$$\begin{aligned} V_0^\nu &= \big[\tfrac{1}{140}(\beta_{4\nu-5}+\beta_{4\nu+1}) + \tfrac{6}{7}(\beta_{4\nu-4}+\beta_{4\nu}) \\ &\quad + \tfrac{1191}{140}(\beta_{4\nu-3}+\beta_{4\nu-1}) + \tfrac{604}{35}\beta_{4\nu-2}\big]h, \end{aligned} \tag{25}$$

$$V_1^\nu = \tfrac{1}{20}(\beta_{4\nu+1}-\beta_{4\nu-5}) + \tfrac{14}{5}(\beta_{4\nu}-\beta_{4\nu-4}) + \tfrac{49}{4}(\beta_{4\nu-1}-\beta_{4\nu-3}), \tag{26}$$

$$\begin{aligned} V_2^\nu &= \big[\tfrac{3}{10}(\beta_{4\nu-5}+\beta_{4\nu+1}) + \tfrac{36}{5}(\beta_{4\nu-4}+\beta_{4\nu}) \\ &\quad + \tfrac{9}{2}(\beta_{4\nu-3}+\beta_{4\nu-1}) - 24\beta_{4\nu-2}\big]/h, \end{aligned} \tag{27}$$

$$\begin{aligned} W^\nu &= \big[\tfrac{1}{140}(\alpha_{4\nu-5}+\alpha_{4\nu+1}) + \tfrac{6}{7}(\alpha_{4\nu-4}+\alpha_{4\nu}) \\ &\quad + \tfrac{1191}{140}(\alpha_{4\nu-3}+\alpha_{4\nu-1}) + \tfrac{604}{35}\alpha_{4\nu-2}\big]h, \end{aligned} \tag{28}$$

for $\nu \in \overline{1,k}$.

The above formulas contain no definite integrals. This is important for numerical calculations.

Using (25)–(28) we can determine the coefficients of equations (11). Namely, from (15) and (12) it follows that

$$b_{ij} = \sum_{\nu=1}^{k} V_i^\nu V_j^\nu, \quad c_j = \sum_{\nu=1}^{k} W^\nu V_j^\nu, \quad i,j = 0,1,2. \tag{29}$$

Solving (11) we obtain a model of the dynamical system (17):

$$a_2^0\ddot{y}(t) + a_1^0\dot{y}(t) + a_0^0 y(t) = u(t), \tag{30}$$

where a_0^0, a_1^0, a_2^0 are the optimal coefficients of (30) in $[t_0, t_{4k}]$.

3) The value of the functional (10) at the optimal point (a_0^0, a_1^0, a_2^0) serves to assess the identification method used. In the considered model of operational calculus we have

$$J_f(a_0^0, a_1^0, a_2^0) = \sqrt{\sum_{\nu=1}^{k}\Big(\sum_{i=0}^{2} a_i^0 V_i^\nu - W^\nu\Big)^2}, \tag{31}$$

where f is the modulating function (23).

Another way of assessing the identification method is the computation of the absolute errors

$$\Delta_f y(t_i) = |\widetilde{y}(t_i) - y(t_i)|, \qquad i \in \overline{0,4k}, \tag{32}$$

between the measured values $\widetilde{y}_i = \widetilde{y}(t_i)$ and the output signal $y(t_i)$ obtained from (30) as a response to $u(t_i) = u^*(t_i)$.

We can accept that the system is "well" identified if

$$\max\{\Delta_f y(t_i) : i \in \overline{0, 4k}\} \leq \delta,$$

where δ is the absolute error of measurement of the output signal.

In order to determine the absolute errors (32) we first have to solve the initial value problem

$$a_2^0 \ddot{y}(t) + a_1^0 \dot{y}(t) + a_0^0 y(t) = u^*(t), \qquad y(t_0) = \widetilde{y}_0,\ \dot{y}(t_0) = \dot{y}_0^*. \tag{33}$$

Figure 3 presents the whole scheme of the algorithm ([1]). To solve (11) the Gauss elimination method was applied.

Table 1 contains the results of the identification of the equation (17) for the observation interval $[0, 2.4]$ in the case of sinusoidal input signal. In particular, the table contains optimal values of the coefficients of (17) and the identification performance index. Moreover, the results of measurements $\widetilde{y}(t_i)$ of the output signal, the values $y(t_i)$ obtained from the model and the absolute errors $\Delta y(t_i)$ at $t_i = 0.2i, i \in \overline{0, 12}$, are listed.

TABLE 1

Identification of an ordinary differential equation of the second order

$a_2\ddot{y} + a_1\dot{y} + a_0 y = u$		$a_0^0 = 1.95100373$	
$t_0 = 0,\ t_{4k} = 2.4$		$a_1^0 = 2.89063531$	
$k = 3$		$a_2^0 = 1.04296329$	
$\widetilde{u}(t_i) = \sin t_i$		$J = 1.04774\text{E}{-09}$	
t	$\widetilde{y}$	y	Δy
0.0	0	0	0
0.2	0.001	1.87108E−03	8.71077E−04
0.4	0.008	8.89653E−03	8.96531E−04
0.6	0.023	2.40286E−02	1.02863E−03
0.8	0.047	0.048037949	1.03795E−03
1.0	0.079	0.080195102	1.19510E−03
1.2	0.117	0.118774478	1.77448E−03
1.4	0.159	0.161434070	2.43407E−03
1.6	0.201	0.205508083	4.50808E−03
1.8	0.243	0.248236869	5.23687E−03
2.0	0.279	0.286950117	7.95012E−03
2.2	0.310	0.319213313	0.009213313
2.4	0.335	0.342943448	7.94345E−03
2.6	0.343	0.356497392	1.34974E−02

([1]) For all numerical examples there exist programs in Basic 1.1 for Amstrad CPC 6128.

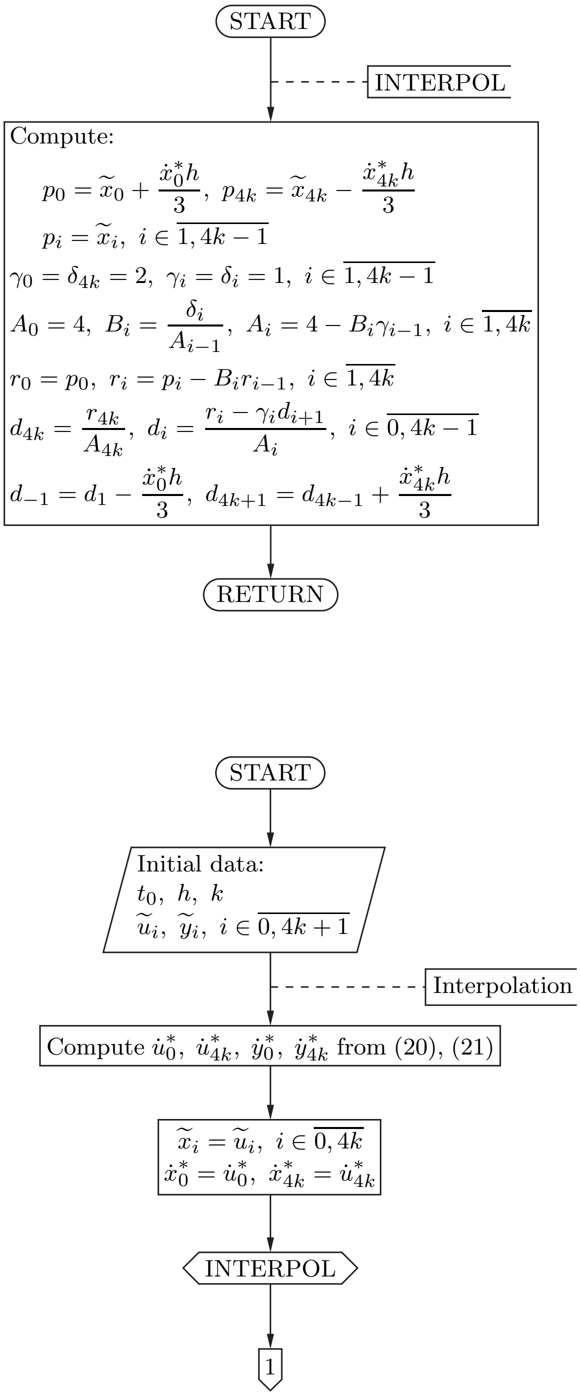

Fig. 3. The identification algorithm for the equation $a_2\ddot{y} + a_1\dot{y} + a_0 y = u$

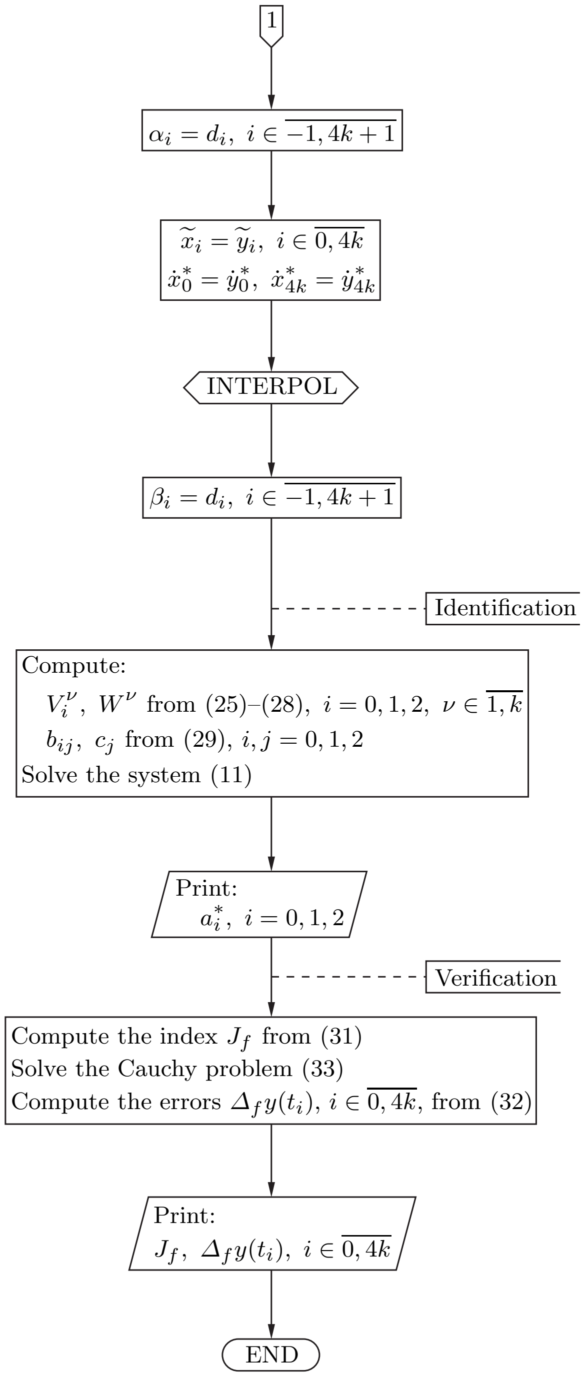
1
$\alpha_i = d_i,\ i \in \overline{-1, 4k+1}$
$\widetilde{x}_i = \widetilde{y}_i,\ i \in \overline{0, 4k}$
$\dot{x}_0^* = \dot{y}_0^*,\ \dot{x}_{4k}^* = \dot{y}_{4k}^*$
INTERPOL
$\beta_i = d_i,\ i \in \overline{-1, 4k+1}$
Identification
Compute:
$V_i^\nu,\ W^\nu$ from (25)–(28), $i = 0, 1, 2,\ \nu \in \overline{1, k}$
$b_{ij},\ c_j$ from (29), $i, j = 0, 1, 2$
Solve the system (11)
Print:
$a_i^*,\ i = 0, 1, 2$
Verification
Compute the index J_f from (31)
Solve the Cauchy problem (33)
Compute the errors $\Delta_f y(t_i)$, $i \in \overline{0, 4k}$, from (32)
Print:
$J_f,\ \Delta_f y(t_i),\ i \in \overline{0, 4k}$
END

Fig. 3 (cont.)

B. Let $CO(L^0, L^1, \widehat{S}, \widehat{T}_q, \widehat{s}_q, q, Q)$ be an operational calculus satisfying the same assumptions as previously. Moreover, suppose L^0 is an algebra with unit.

Consider the equation

$$a_1 p_1 \widehat{S} y + a_0 p_0 y = p, \tag{34}$$

where $p_0, p_1 \in \mathrm{Inv}(L^0)$, $p \in L^0$, $y \in L^1$, $a_0, a_1 \in \mathbb{R}^1$ and $\mathrm{Inv}(L^0)$ denotes the set of invertible elements in L^0.

To determine a_0 and a_1 we can apply our identification method for the new operational calculus with

$$Sx := A\widehat{S}x, \quad T_q w := \widehat{T}_q(A^{-1}w), \quad s_q x := \widehat{s}_q x, \tag{35}$$

where $A := p_0^{-1} p_1 \in L^0$, $x \in L^1$, $w \in L^0$, $q \in Q$. In this case (34) is a particular form of (5), i.e.

$$a_1 Sy + a_0 y = u,$$

where $u := p_0^{-1} p \in L^0$. Moreover, S satisfies the Leibniz condition and the limit conditions s_q, $q \in Q$, are multiplicative [13].

The application of the new operational calculus to (34) makes it possible to use the modulating element method for identification of certain types of nonstationary compensating or distributed parameter systems.

B.1. For the operational calculus in which

$$L^n := C^n([t_0, t_m], \mathbb{R}^1), \qquad n = 0, 1,$$

and

$$\widehat{S} := \frac{d}{dt}, \quad \widehat{T}_q := \int_q^t, \quad \widehat{s}_q := |_{t=q}, \qquad q \in Q := [t_0, t_m],$$

we can consider the structure of algebra and the Hilbert space as in Example A. Moreover, (34) takes the form

$$a_1 p_1(t) \dot{y}(t) + a_0 p_0(t) y(t) = p(t), \tag{36}$$

where $p_0(t), p_1(t) \neq 0$ for every $t \in [t_0, t_m]$. The equation (36) describes the dynamics of a nonstationary compensating constants system.

Every function $0 \not\equiv f(t) \in C^1([t_0, t_m], \mathbb{R}^1)$ satisfying

$$f(t_\nu) = 0, \qquad t_\nu \in Q, \ \nu \in \overline{0, m}, \ m \geq 2,$$

is a modulating element corresponding to $q_\nu = t_\nu \in Q$.

Now, let $p^* \in C^0([t_0, t_m], \mathbb{R}^1)$, $y^* \in C^1([t_0, t_m], \mathbb{R}^1)$ denote functions approximating the input and output signals of the system (36) in $[t_0, t_m]$ (on the basis of measurements of a real system). Then $u^* = p^*(t)/p_0(t)$.

Using the operational calculus with S, T_q and s_q defined by (35), where $A = A(t) = p_1(t)/p_0(t)$, we can determine the νth coordinates V_i^ν, W^ν of

$\overline{v}_i^*, \overline{w}^*$. Namely, from (9) it follows that

$$V_0^\nu = \int_{t_{\nu-1}}^{t_\nu} \frac{f(t)p_0(t)y^*(t)}{p_1(t)}\,dt, \qquad V_1^\nu = -\int_{t_{\nu-1}}^{t_\nu} f'(t)y^*(t)\,dt,$$

$$W^\nu = \int_{t_{\nu-1}}^{t_\nu} \frac{f(t)p^*(t)}{p_1(t)}\,dt, \qquad \nu \in \overline{1,m}.$$

Hence, from (15) and (12) we obtain the formulas for the coefficients of (11):

$$b_{ij} = \sum_{\nu=1}^{m} V_i^\nu V_j^\nu, \quad c_j = \sum_{\nu=1}^{m} W^\nu V_j^\nu, \quad i,j = 0,1.$$

Solving (11) we obtain the optimal coefficients a_0^0, a_1^0 of (36). The value of the identification performance index (10) is computed from the formula

$$J_f(a_0^0, a_1^0) = \sqrt{\sum_{\nu=1}^{m} (a_0^0 V_0^\nu + a_1^0 V_1^\nu - W^\nu)^2},$$

where f is a fixed modulating function of the system (36) corresponding to $t_\nu \in [t_0, t_m], \nu \in \overline{0,m}$.

In order to determine the absolute errors

$$\Delta_f y(t_i) = |y(t_i) - y^*(t_i)|$$

we must first solve the initial value problem

$$a_1^0 p_1(t)\dot{y}(t) + a_0^0 p_0(t) y(t) = p^*(t), \qquad y(t_0) = y^*(t_0) = y_0^*.$$

A computer program for the identification algorithm has been elaborated. In this program the Simpson method of computing definite integrals and the Cramer method of solving linear algebraic equations are used. That program was used for the numerical example shown in Table 2. The behaviour of the optimal coefficients of the equation with two modulating functions corresponding to $t_\nu, \nu \in \overline{0,3}$, was studied. Table 2, apart from the modulating functions and the values of $t_\nu, \nu \in \overline{0,3}$, contains the general form of the differential equation describing the dynamical system, the forms of p^* and y^* of the approximated input and output signals, the number k of parts into which the integration interval $[t_0, t_m]$ was divided for the Simpson method, the values of the optimal coefficients a_0^0, a_1^0, of the identification performance index J_f, of the function y^* approximating the output signal and of the function y obtained from the model at $t_i = 0.6i, i \in \overline{0,5}$, and of the absolute errors $\Delta_f y$ at these points.

B.2. In the operational calculus with the derivative

$$\widehat{S}x(z,t) := \frac{\partial x(z,t)}{\partial z} + \frac{\partial x(z,t)}{\partial t},$$

TABLE 2

Identification of an ordinary differential equation of the first order

$t_0 = 0,\ t_1 = 1$		$a_1(0.5t^2 + t + 1)\dot{y} + a_0(t + 1)y = p$			
$t_2 = 2,\ t_3 = 3$ $k = 64$		$p^* = t + 1$ $y^* = -0.001t^5 + 0.006t^4 + 0.105t^3 + 0.375t^2 + 0.5t$			
f		$\sin[(t - t_0)(t - t_1)(t - t_2)(t - t_3)]$		$(t - t_0)(t - t_1)(t - t_2)(t - t_3)$	
a_0^0		−3.01639875		−3.0163477	
a_1^0		2.00463942		2.00460815	
J_f		2.80909E−09		3.29272E−10	
t	y^*	y	$\Delta_f y$	y	$\Delta_f y$
0.0	0	0	0	0	0
0.6	0.45837984	0.457921251	4.58589E−04	0.45792809	4.51750E−04
1.2	1.33139328	1.331037450	3.55835E−04	1.33105641	3.36871E−04
1.8	2.77144992	2.770779430	6.70488E−04	2.77081710	6.32819E−04
2.4	4.93095936	4.931174050	2.14692E−04	4.93123817	2.78803E−04
3.0	7.95300000	7.967295470	0.014295470	7.96739479	1.43948E−02

the integrals

$$\widehat{T}_q w(z,t) := \int_q^t w(z - t + \tau, \tau)\, d\tau$$

and the limit conditions

$$\widehat{s}_q x(z,t) := x(z - t + q, q),$$

where $q \in Q := [t_0, t_m], x = x(z,t) \in L^1 := C^2(\mathbb{R}^1 \times [t_0, t_m], \mathbb{R}^1), w = w(z,t) \in L^0 := C^1(\mathbb{R}^1 \times [t_0, t_m], \mathbb{R}^1)$ the equation (34) takes the form of a quasi-linear partial differential equation

$$a_1 p_1(z,t)\left(\frac{\partial y(z,t)}{\partial z} + \frac{\partial y(z,t)}{\partial t}\right) + a_0 p_0(z,t) y(z,t) = p(z,t), \tag{37}$$

where $p_0(z,t), p_1(z,t) \neq 0$ for every $(z,t) \in \mathbb{R}^1 \times [t_0, t_m]$.

The equation (37) describes the dynamics of a nonstationary distributed parameters system.

It is easy to verify that with the usual multiplication of functions the derivative $\widehat{S}$ satisfies the Leibniz condition, and the limit conditions $\widehat{s}_q$, $q \in Q$, are multiplicative. Moreover,

$$\operatorname{Ker} S = \{c(z,t) \in L^1 : c(z,t) = \varphi(z - t),\ \varphi \in C^2(\mathbb{R}^1, \mathbb{R}^1)\}.$$

Notice that every function $c(z,t) \in \operatorname{Ker} \widehat{S}$, restricted to the rectangle $P = [z_0, z_m] \times [t_0, t_m]$, may be treated as an element of $H_1 = L^2(P, \mathbb{R}^1)$. In this connection, when identifying the system (37) in the product P, we may take for H the direct sum

$$H = \bigoplus_{\nu=1}^{m} H_1,$$

because $\overline{v}_i^*, \overline{w}^* \in (\operatorname{Ker} S)_m$.

In this space the inner product is given by

$$(\overline{\alpha} \mid \overline{\beta}) = \sum_{\nu=1}^{m} \int_{t_0}^{t_m} \int_{z_0}^{z_m} \alpha_\nu(z,t)\beta_\nu(z,t)\,dz\,dt, \qquad \overline{\alpha}, \overline{\beta} \in H. \tag{38}$$

The induced norm is then

$$\|\overline{\alpha}\| = \sqrt{\sum_{\nu=1}^{m} \int_{t_0}^{t_m} \int_{z_0}^{z_m} [\alpha_\nu(z,t)]^2\,dz\,dt}, \qquad \overline{\alpha} \in H.$$

From the form of limit conditions in this operational calculus it follows that every function

$$0 \not\equiv f(t) \in C^2([t_0, t_m], \mathbb{R}^1), \qquad f(t_\nu) = 0,\ \nu \in \overline{0, m},\ m \ge 2,$$

is a modulating element corresponding to $q_\nu = t_\nu \in Q$.

Let $p^* = p^*(z,t), y^* = y^*(z,t)$ denote the approximated input and output signals of the system (37) in the rectangle P, on the basis of which we are to identify its coefficients a_0 and a_1. Then $u^* = p^*(z,t)/p_0(z,t)$.

From the definition of integrals and limit conditions in the operational calculus (35), where $A = A(z,t) = p_1(z,t)/p_0(z,t)$, it follows that the νth coordinates of $\overline{v}_i^*$, $\overline{w}^*$ are

$$\begin{aligned}
V_0^\nu(z,t) &= \int_{t_{\nu-1}}^{t_\nu} \frac{f(\tau)p_0(z-t+\tau,\tau)y^*(z-t+\tau,\tau)}{p_1(z-t+\tau,\tau)}\,d\tau,\\
V_1^\nu(z,t) &= -\int_{t_{\nu-1}}^{t_\nu} f'(\tau)y^*(z-t+\tau,\tau)\,d\tau,\\
W^\nu(z,t) &= \int_{t_{\nu-1}}^{t_\nu} \frac{f(\tau)p^*(z-t+\tau,\tau)}{p_1(z-t+\tau,\tau)}\,d\tau, \qquad \nu \in \overline{1, m}.
\end{aligned} \tag{39}$$

Form (38) and (12), we obtain formulas for the coefficients of (11):

$$\begin{aligned}
b_{ij} &= \sum_{\nu=1}^{m} \int_{t_0}^{t_m} \int_{z_0}^{z_m} V_i^\nu(z,t)V_j^\nu(z,t)\,dz\,dt,\\
c_j &= \sum_{\nu=1}^{m} \int_{t_0}^{t_m} \int_{z_0}^{z_m} W^\nu(z,t)V_j^\nu(z,t)\,dz\,dt, \qquad i,j = 0,1.
\end{aligned} \tag{40}$$

The identification performance index (10) is calculated from

$$(41)\qquad J_f(a_0^0,a_1^0)=\|a_0^0\overline{v}_0^*+a_1^0\overline{v}_1^*-\overline{w}^*\|$$
$$=\Big(\sum_{\nu=1}^{m}\Big[\sum_{i,j=0}^{1}a_i^0a_j^0(V_i^\nu\mid V_j^\nu)-2\sum_{i=0}^{1}a_i^0(W^\nu\mid V_i^\nu)+(W^\nu\mid W^\nu)\Big]\Big)^{1/2},$$

where f is a fixed modulating function of (37) corresponding to $t_\nu\in[t_0,t_m]$, $\nu\in\overline{0,m}$, the inner product is

$$(\alpha\mid\beta)=\int_{t_0}^{t_m}\int_{t_0}^{t_m}\alpha(z,t)\beta(z,t)\,dz\,dt$$

and V_i^ν, W^ν are defined by (39).

TABLE 3

Identification of a partial differential equation of the first order

$t_0=0,\ t_1=0.5,\ t_2=1$	$a_1(z^2t^2+1)(\frac{\partial y}{\partial z}+\frac{\partial y}{\partial t})+a_0(2t^2+3)y=p(z,t)$		
$k=6$	$p^*=4.1t^3+2.9z^2t^2+8.2zt^2+11.8z+6.2t+3$ $y^*=2z+t$		
f	a_0^0	a_1^0	J_f
$\sin(t-t_0)(t-t_1)(t-t_2)$	2.01556215	1.048092380	3.25104E−03
$(t-t_0)(t-t_1)(t-t_2)$	2.01555038	1.048084830	3.25094E−03
$\sin(t-t_0)\sin(t-t_1)\sin(t-t_2)$	2.01007362	1.048158110	2.50256E−03
$[\sin(t-t_0)(t-t_1)(t-t_2)]^2$	1.95011143	0.969249844	1.70858E−04
$[(t-t_0)(t-t_1)(t-t_2)]^2$	1.95013695	0.969294625	1.70816E−04
$[\sin(t-t_0)\sin(t-t_1)\sin(t-t_2)]^2$	1.95841726	0.975006209	9.84764E−05

In the numerical example shown in Table 3, $z_0=t_0, z_m=t_m$. In this case the values of the definite integrals in (40), (41) given in the form

$$K_\nu=\int_{t_0}^{t_m}\int_{t_0}^{t_m}\Big[\int_{t_{\nu-1}}^{t_\nu}\gamma(z,t,\tau)\,d\tau\cdot\int_{t_{\nu-1}}^{t_\nu}\delta(z,t,\tau)\,d\tau\Big]\,dz\,dt$$

are approximated by the following formula:

$$K_\nu\approx\sum_{i,j=1}^{k}\Big[\sum_{l=1}^{k}\gamma(\xi_i,\xi_j,\eta_l)\cdot\sum_{l=1}^{k}\delta(\xi_i,\xi_j,\eta_l)\Big]h^2h_\nu^2,$$

where

$$h=\frac{t_m-t_0}{k},\quad \xi_i=\frac{t_m^{i-1}+t_m^i}{2},\quad t_m^j=t_0+jh,\quad t_m^0=t_0,\quad t_m^k=t_m,$$

$$h_\nu = \frac{t_\nu - t_{\nu-1}}{k}, \qquad \eta_i = \frac{t_\nu^{i-1} + t_\nu^i}{2},$$
$$t_\nu^j = t_{\nu-1} + jh_\nu, \quad t_\nu^0 = t_{\nu-1}, \quad t_\nu^k = t_\nu, \quad i \in \overline{1,k},\ j \in \overline{0,k}.$$

This example may be generalized to the case of distributed parameter systems described by a partial differential equation

$$a_1 p_1(t_1,\ldots,t_r)\left(b_1\frac{\partial y}{\partial t_1} + \ldots + b_r\frac{\partial y}{\partial t_r}\right) + a_0 p_0(t_1,\ldots,t_r)y = p(t_1,\ldots,t_r)$$

using the operational calculus with the derivative $\widehat{S} = \sum_{i=1}^r b_i \partial/\partial t_i$ (cf. [13]).

5. Conclusions. In this paper an application of the Bittner operational calculus to identification of generalized dynamical differential systems was presented. The identification method is so general that it may be used for any real system whose dynamics, in a proper model of operational calculus, is described by the equation (0). So, we have obtained the modulating element method of identification for many systems described by means of various types of ordinary and partial differential equations. From Example A it follows that every stationary compensating constants system in the classical sense is a stationary compensating constants system in the operational sense. From Examples B.1 and B.2 it follows that some nonstationary compensating or distributed parameter systems in the classical sense are stationary compensating constant systems in the operational sense.

The presented solution of the identification problem is determined by the choice of the identifying pair (u^*, y^*) and the modulating element f. The resulting coefficients

$$a_i^0 = a_i^0(u^*, y^*, f), \qquad i \in \overline{0,n},$$

of model (0) are the best with respect to the identification performance index J_f (cf. [8]). In this connection, we can set the problem of choosing the best modulating element:

Let $f, F \in L^n$ be modulating elements of the system (0) (corresponding to $q_0, q_1, \ldots, q_m \in Q$) with the same identifying pair (u^*, y^*). Which of the numbers

$$J_f(a_0^0, a_1^0, \ldots, a_n^0), \qquad J_F(A_0^0, A_1^0, \ldots, A_n^0)$$

is smaller? For some remarks relating to this question, see [23].

In particular, for the operational calculus with $S = d/dt$ we have obtained a modification of the continuous systems identification method by means of the modulating function. The problem of choosing the best model and the problem of signal interpolation by means of splines are not considered in the classical modulating function method (see [17, 11]). Here the splines have also been used to design the modulating function. This is important in numerical computations, as it reduces the time and errors of calculations.

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