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S. LEWANOWICZ (Wrocław)

A FAST ALGORITHM FOR THE CONSTRUCTION OF RECURRENCE RELATIONS FOR MODIFIED MOMENTS

Abstract. A new approach is presented for constructing recurrence relations for the modified moments of a function with respect to the Gegenbauer polynomials.

1. Introduction. Let w be a weight function on the interval $(-1,1)$. We call the integrals

$$m_k[w] \equiv m_k^\lambda[w] := \int_{-1}^{1} w(x) C_k^\lambda(x)\,dx \qquad (\lambda > -1/2) \tag{1.1}$$

modified moments of w with respect to the Gegenbauer polynomials

$$C_k^\lambda(x) := \frac{(-1)^n (2\lambda)_k}{2^k k! (\lambda+\frac{1}{2})_k (1-x^2)^{\lambda-1/2}} \boldsymbol{D}^k\{(1-x^2)^{k+\lambda-1/2}\},$$

where $\boldsymbol{D} = d/dx$ (see, e.g., [2, Vol. 1, §10.9], or [9, Vol. 1, §8.3]), or the *Gegenbauer moments*, for short. Here $(a)_m$ is the Pochhammer symbol given by

$$(a)_0 := 1, \quad (a)_m := a(a+1)\ldots(a+m-1) \quad (m = 1,2,\ldots).$$

Modified moments, provided they are accurately computable, are used in the generation of nonstandard orthogonal polynomials (see [3, 4, 5] and the references given therein) which have applications in many areas (e.g. numerical quadrature, summation of series, approximation). The *Chebyshev moments*,

$$\tau_k[w] := \int_{-1}^{1} w(x) T_k(x)\,dx = \begin{cases} m_0^0[w] & (k=0), \\ \frac{1}{2} k m_k^0[w] & (k \ge 1) \end{cases} \tag{1.2}$$

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are needed in the numerical evaluation of certain difficult integrals by the so-called modified Clenshaw–Curtis method (see, e.g., [10, 11, 12]). Sometimes, by a stroke of luck, modified moments are explicitly known. More frequently, however, they are computed from a recurrence relation of the form

$$Lm_k[w] = \varrho(k), \tag{1.3}$$

judiciously employed. Here L is a difference operator,

$$L := \sum_{j=u}^{u+r} \lambda_j(k) E^j,$$

where $\lambda_u, \lambda_{u+1}, \ldots, \lambda_{u+r}$ ($\lambda_u \not\equiv 0$, $\lambda_{u+r} \not\equiv 0$) are known (rational) functions in k, $u \in \mathbb{Z}$, $r \in \mathbb{Z}_+$ is referred to as the *order* of L, and E^j ($j \in \mathbb{Z}$) is the jth shift operator, acting on the variable k:

$$E^j \mu(k) = \mu(k+j)$$

for any function $\mu : \mathbb{Z} \to \mathbb{Z}$. (We write I for E^0.)

In the cited references, such recurrence relations are constructed by *ad hoc* methods. A systematic way of constructing a recurrence (1.3) under the assumption that the function w satisfies a linear differential equation

$$\sum_{i=0}^{n} p_{ni}(x) \boldsymbol{D}^i w = q, \tag{1.4}$$

where p_{ni} are polynomials, and q is a known function, was described in [6]. The first algorithm given therein (which, however, is the most complex) leads to a recurrence of the lowest possible order. This algorithm seems to be of great theoretical value. For instance, it helped us to obtain some partial results on certain hypergeometric sums which were later generalized to the form given in [7]. Unfortunately, the degree of complexity of the algorithm grows quickly with n, so that the calculations may be very tedious.

The aim of the present paper is to propose a new algorithm, which in the author's belief is equivalent to the best algorithm of [6]. As a final result we obtain a pair P, L of difference operators and a function $\psi(k)$ such that the recurrence (1.3) holds with $\varrho(k) = Pm_k[q] - \psi(k)$. The order of the recurrence equals

$$\operatorname{ord}(P) + 2 \max_{0 \le i \le n,\, p_{ni} \not\equiv 0} (\deg p_{ni} - i),$$

where $\operatorname{ord}(P)$ is the order of the operator P.

In applications, the order n of the differential equation (1.4) is usually not greater than 3. We give the closed-form expressions for P, L and $\varrho(k)$ for the cases $n = 1$ and $n = 2$, and show how the case $n = 3$ may be treated with a little effort, using the results given for $n \le 2$.

Let us remark that the algorithm can be easily extended to the case of arbitrary $n > 3$.

It should be noticed that P, L and $\varrho(k)$ are given in terms of certain basic difference operators (see Section 2). A scalar form of the recurrence relation (1.3) may be obtained using a language for symbolic manipulation, as for instance Maple [1].

In Section 2, we give some important properties of the Gegenbauer moments (1.1). Section 3 contains the main result of the paper—formulae for P, L and $\varrho(k)$. In Section 4 we give an illustrative example.

2. Basic identities

LEMMA 2.1 [6]. *The Gegenbauer moments* (1.1) *satisfy the identities*

$$m_k[xw(x)] = Xm_k[w], \tag{2.1}$$

$$Dm_k[\boldsymbol{D}w] = m_k[w] + D\varphi_k[w], \tag{2.2}$$

where X and D are the second-order difference operators

$$X := \frac{k+2\lambda-1}{2k+2\lambda}E^{-1} + \frac{k+1}{2k+2\lambda}E, \tag{2.3}$$

$$D := \frac{1}{2k+2\lambda}\{E^{-1} - E\}, \tag{2.4}$$

and

$$\varphi_k[w] \equiv \varphi_k^\lambda[w] := [w(x)C_k^\lambda(x)]_{x=-1}^{x=1}. \tag{2.5}$$

It is easy to generalize equation (2.1) to the form

$$m_k[pw] = p(X)m_k[w] \qquad (p \text{ a polynomial}). \tag{2.6}$$

In Lemmas 2.2–2.4 we give identities which may be considered as generalizations of (2.2). We shall need some notation.

For $i = 0, 1, \ldots$ and $\sigma = \pm 1$ define the difference operator

$$A_i^{(\sigma)} := I - \sigma\alpha_i(k)E, \tag{2.7}$$

where

$$\alpha_i(k) := \frac{(2k+2\lambda+1)_2}{(2k+2\lambda+i+1)_2}.$$

Notice that

$$A_0^{(\sigma)} = I - \sigma E, \qquad A_1^{(\sigma)} = I - \sigma\frac{2k+2\lambda+1}{2k+2\lambda+3}E.$$

Further, let

$$S_{ij}^{(\sigma)} := A_i^{(\sigma)}A_{i-1}^{(\sigma)}\cdots A_j^{(\sigma)}, \tag{2.8}$$

$$P_j^{(\sigma)} := S_{j-1,0}^{(\sigma)} \qquad (j = 0, 1, \ldots). \tag{2.9}$$

We adopt the convention that $S_{ij}^{(\sigma)} = I$ for $i < j$.

Also, we will use the notation

$$\kappa(k) := (k+1)(k+2\lambda-1), \tag{2.10}$$

$$\mu_i(k) := 2^{-\lfloor (i+1)/2 \rfloor}(2k+2\lambda+1)_i \quad (i = 0, 1, \ldots). \tag{2.11}$$

Finally, let us introduce the differential operators

$$\boldsymbol{U} := (x^2-1)\boldsymbol{D} + (3-2\lambda)x\boldsymbol{I}, \qquad \boldsymbol{G} := \boldsymbol{U}\boldsymbol{D}, \tag{2.12}$$

$$\boldsymbol{V}_\sigma := (x+\sigma)\boldsymbol{D} + (3/2-\lambda)\boldsymbol{I}, \qquad \boldsymbol{H}_\sigma := \boldsymbol{V}_\sigma \boldsymbol{D} \quad (\sigma = \pm 1). \tag{2.13}$$

Here $\boldsymbol{I}$ is the identity operator.

Now we are able to prove the following.

LEMMA 2.2. *Let $\boldsymbol{Q}_1$ be any of the following first-order differential operators*:

$$\boldsymbol{Q}_1 := \begin{cases} \boldsymbol{U} & \text{(case 1A)}, \\ \boldsymbol{V}_\sigma \quad (\sigma = \pm 1) & \text{(case 1B)}, \\ \boldsymbol{D} & \text{(case 1C)}. \end{cases} \tag{2.14}$$

Then the identity

$$Q_1 m_k[\boldsymbol{Q}_1 w] = M_1 m_k[w] + \tau_k^{(1)}[w] \tag{$*$}$$

holds with

$$Q_1 := \begin{cases} I & \text{(case 1A)}, \\ P_1^{(\sigma)} & \text{(case 1B)}, \\ D & \text{(case 1C)}, \end{cases} \qquad M_1 := \begin{cases} \kappa(k)D & \text{(case 1A)}, \\ \mu_1(k)P_1^{(-\sigma)} & \text{(case 1B)}, \\ I & \text{(case 1C)}, \end{cases}$$

$$\tau_k^{(1)}[w] := \begin{cases} 0 & \text{(case 1A)}, \\ P_1^{(\sigma)}\varphi_k[(x+\sigma)w] & \text{(case 1B)}, \\ D\varphi_k[w] & \text{(case 1C)}. \end{cases}$$

Proof. Case 1A. As $m_k[(x^2-1)\boldsymbol{D}f] = Hm_k[f]$, where $H := (k+2\lambda-2)_2 E^{-1} - (k+1)_2 E$ (cf. [6, Eq. (20)]), we have

$$m_k[\boldsymbol{Q}_1 w] = m_k[\boldsymbol{U}w] = \{H + (3-2\lambda)X\}m_k[w] = (k+1)(k+2\lambda-1)Dm_k[w].$$

Thus identity $(*)$ holds with Q_1, M_1 and $\tau_k^{(1)}[w]$ given in the lemma.

Case 1B. It can be checked that the operator $Q := (k+2\lambda-1)I + \sigma(k+2)E$ satisfies the following equations:

$$P_1^{(\sigma)}(X+\sigma I) = QD,$$
$$Q + (\tfrac{3}{2}-\lambda)P_1^{(\sigma)} = \tfrac{1}{2}(2k+2\lambda+1)P_1^{(-\sigma)}.$$

Using these identities and (2.2), we obtain

$$\begin{aligned} P_1^{(\sigma)} m_k[\boldsymbol{V}_\sigma w] &= QD m_k[\boldsymbol{D}w] + (\tfrac{3}{2} - \lambda) P_1^{(\sigma)} m_k[w] \\ &= \tfrac{1}{2}(2k + 2\lambda + 1) P_1^{(-\sigma)} m_k[w] + QD\varphi_k[w] \\ &= \mu_1(k) P_1^{(-\sigma)} m_k[w] + P_1^{(\sigma)} \varphi_k[(x+\sigma)w]. \end{aligned}$$

Thus, equation (∗) holds with Q_1, M_1 and $\tau_k^{(1)}[w]$ specified in the lemma.

Case 1C. Equation (∗) is a disguised form of (2.2).

The next two lemmas can be proved in a similar manner.

LEMMA 2.3. *Let $\boldsymbol{Q}_2$ be any of the following second-order differential operators*:

$$\boldsymbol{Q}_2 := \begin{cases} \boldsymbol{G} & & (\textit{case } 2\text{A}), \\ \boldsymbol{H}_\sigma & (\sigma = \pm 1) & (\textit{case } 2\text{B}), \\ \boldsymbol{D}^2 & & (\textit{case } 2\text{C}). \end{cases} \tag{2.15}$$

Then the identity

$$Q_2 m_k[\boldsymbol{Q}_2 w] = M_2 m_k[w] + \tau_k^{(2)}[w]$$

holds with

$$Q_2 := \begin{cases} I & (\textit{case } 2\text{A}), \\ P_2^{(\sigma)} & (\textit{case } 2\text{B}), \\ D^2 & (\textit{case } 2\text{C}), \end{cases} \qquad M_2 := \begin{cases} \kappa(k) I & (\textit{case } 2\text{A}), \\ \mu_2(k) E & (\textit{case } 2\text{B}), \\ I & (\textit{case } 2\text{C}), \end{cases}$$

$$\tau_k^{(2)}[w] := \begin{cases} 0 & (\textit{case } 2\text{A}), \\ \mu_2(k) ED\varphi_k[w] + P_2^{(\sigma)} \varphi_k[(x+\sigma)\boldsymbol{D}w] & (\textit{case } 2\text{B}), \\ D\varphi_k[w] + D^2 \varphi_k[\boldsymbol{D}w] & (\textit{case } 2\text{C}). \end{cases}$$

LEMMA 2.4. *Let $\boldsymbol{Q}_3$ be any of the following third-order differential operators*:

$$\boldsymbol{Q}_3 := \begin{cases} \boldsymbol{G}\boldsymbol{U} & & (\textit{case } 3\text{A}), \\ \boldsymbol{V}_\sigma \boldsymbol{G} & (\sigma = \pm 1) & (\textit{case } 3\text{B}), \\ \boldsymbol{D}\boldsymbol{G} & & (\textit{case } 3\text{C}), \\ \boldsymbol{H}_\sigma \boldsymbol{V}_\sigma & (\sigma = \pm 1) & (\textit{case } 3\text{D}), \\ \boldsymbol{D}\boldsymbol{H}_\sigma & (\sigma = \pm 1) & (\textit{case } 3\text{E}), \\ \boldsymbol{D}^3 & & (\textit{case } 3\text{F}). \end{cases} \tag{2.16}$$

Then the identity

$$Q_3 m_k[\boldsymbol{Q}_3 w] = M_3 m_k[w] + \tau_k^{(3)}[w]$$

holds with

$$Q_3 := \begin{cases} I & (\textit{case } 3\mathrm{A}),\\ P_1^{(\sigma)} & (\textit{case } 3\mathrm{B}),\\ D & (\textit{case } 3\mathrm{C}),\\ P_3^{(\sigma)} & (\textit{case } 3\mathrm{D}),\\ P_2^{(\sigma)}D & (\textit{case } 3\mathrm{E}),\\ D^3 & (\textit{case } 3\mathrm{F}).\end{cases} \qquad M_3 := \begin{cases} [\kappa(k)]^2 D & (\textit{case } 3\mathrm{A}),\\ \mu_1(k)P_1^{(-\sigma)}\kappa(k)I & (\textit{case } 3\mathrm{B}),\\ \kappa(k)I & (\textit{case } 3\mathrm{C}),\\ \mu_3(k)EP_1^{(-\sigma)} & (\textit{case } 3\mathrm{D}),\\ \mu_2(k)E & (\textit{case } 3\mathrm{E}),\\ I & (\textit{case } 3\mathrm{F}).\end{cases}$$

$$\tau_k^{(3)}[w] := \begin{cases} 0 & (\textit{case } 3\mathrm{A}),\\ P_1^{(\sigma)}\varphi_k[(x+\sigma)\boldsymbol{G}w] & (\textit{case } 3\mathrm{B}),\\ D\varphi_k[\boldsymbol{G}w] & (\textit{case } 3\mathrm{C}),\\ \mu_2(k)EP_1^{(\sigma)}\{\varphi_k[(x+\sigma)w] + D\varphi_k[\boldsymbol{V}_\sigma w]\} & \\ \qquad + P_3^{(\sigma)}\varphi_k[(x+\sigma)\boldsymbol{D}\boldsymbol{V}_\sigma w] & (\textit{case } 3\mathrm{D}),\\ \mu_2(k)ED\varphi_k[w] & \\ \quad + P_2^{(\sigma)}\{\varphi_k[(x+\sigma)\boldsymbol{D}w] + D\varphi_k[\boldsymbol{H}_\sigma w]\} & (\textit{case } 3\mathrm{E}),\\ D\varphi_k[w] + D^2\varphi_k[\boldsymbol{D}w] + D^3\varphi_k[\boldsymbol{D}^2 w] & (\textit{case } 3\mathrm{F}).\end{cases}$$

Observe that the difference operators Q_1, Q_2, Q_3 given in the above lemmas are *always* of the form

$$P_d^{(\sigma)}D^e$$

with $d, e \geq 0$ and $\sigma \in \{-1, +1\}$. In Section 3.0, we shall need an operator which is a *common multiple* of two operators of the above form. Such an operator is given in Lemma 2.6 below. We must introduce a new family of difference operators first.

For $m = 0, 1, \ldots$ and $\sigma \in \{-1, 1\}$ define the difference operator

$$R_m^{(\sigma)} := (2k + 2\lambda)^{-1}E^{-1} + \sigma\varrho_m(k)I,$$

where

$$\varrho_m(k) := \frac{2k + 2\lambda + 2m + 1}{(2k + 2\lambda + m)_2}.$$

Further, let

$$T_{ij}^{(\sigma)} := R_i^{(\sigma)}R_{i+1}^{(\sigma)}\ldots R_j^{(\sigma)}, \quad U_h^{(\sigma)} := T_{0,h-1}^{(\sigma)} \quad (h = 0, 1, \ldots).$$

By convention, $T_{ij}^{(\sigma)} = I$ for $i > j$.

Lemma 2.5. *The identity*

$$P_v^{(\sigma)}D^r = T_{v,v+r-1}^{(\sigma)}P_{v+r}^{(\sigma)} \tag{2.17}$$

holds for $v, r = 0, 1, \ldots$

Proof. It is easy to verify that

$$R_0^{(\sigma)}A_0^{(\sigma)} = D, \quad R_m^{(\sigma)}A_m^{(\sigma)} = A_{m-1}^{(\sigma)}R_{m-1}^{(\sigma)} \quad (m = 1, 2, \ldots).$$

Hence,

$$D^r = R_0^{(\sigma)} R_1^{(\sigma)} \dots R_{r-1}^{(\sigma)} A_{r-1}^{(\sigma)} A_{r-2}^{(\sigma)} \dots A_0^{(\sigma)} = U_r^{(\sigma)} P_r^{(\sigma)} \qquad (r \geq 0),$$

$$\begin{aligned} P_v^{(\sigma)} U_r^{(\sigma)} &= A^{(\sigma)} v - 1 \dots A_1^{(\sigma)} A_0^{(\sigma)} R_0^{(\sigma)} R_1^{(\sigma)} \dots R_{r-1}^{(\sigma)} \\ &= R_v^{(\sigma)} R_{v+1}^{(\sigma)} \dots R_{v+r-1}^{(\sigma)} A_{v+r-1}^{(\sigma)} \dots A_{r+1}^{(\sigma)} A_r^{(\sigma)} \\ &= T_{v,v+r-1}^{(\sigma)} S_{v+r-1,r}^{(\sigma)} \qquad (v, r \geq 0) \end{aligned}$$

and the result follows in view of $S_{v+r-1,r}^{(\sigma)} P_r^{(\sigma)} = P_{v+r}^{(\sigma)}$ (cf. (2.8), (2.9)).

LEMMA 2.6. *Let*

$$Q_1 := P_v^{(\sigma)} D^r, \qquad Q_2 := P_u^{(\tau)} D^s, \tag{2.18}$$

where $v, r, u, s \geq 0$, $v + r \geq u + s$ *and* $\sigma, \tau \in \{-1, 1\}$. *Set* $q := 0$ *when* $\sigma = \tau$, *and* $q := u$ *otherwise. Then the operator* $Q := P_d^{(\sigma)} D^e$, *where* $e := \max\{r, q + s\}$ *and* $d := v + r - e$, *is a common multiple of* Q_1 *and* Q_2,

$$Q = Y_i Q_i \qquad (i = 1, 2), \tag{2.19}$$

where

$$Y_1 := T_{d,v-1}^{(\sigma)}, \qquad Y_2 := \begin{cases} S_{d-1,h}^{(\sigma)} T_{h,u-1}^{(\sigma)} & (\sigma = \tau), \\ P_d^{(\sigma)} D^{-h} U_u^{(\tau)} & (\sigma = -\tau), \end{cases} \tag{2.20}$$

and where $h := u + s - e$.

Proof. Let $\sigma = \tau$. Substitute the expressions for Y_i and Q_i, given in (2.18) and (2.20), into the right-hand side of (2.19), and use Lemma 2.5. We obtain

$$Y_1 Q_1 = T_{d,v-1}^{(\sigma)} P_v^{(\sigma)} D^r = P_d^{(\sigma)} D^{v-d} D^r = P_d^{(\sigma)} D^e = Q,$$

$$Y_2 Q_2 = S_{d-1,h}^{(\sigma)} T_{h,u-1}^{(\sigma)} P_u^{(\sigma)} D^s = S_{d-1,h}^{(\sigma)} P_h^{(\sigma)} D^{u-h} D^s = P_d^{(\sigma)} D^e = Q.$$

The case $\tau = -\sigma$ can be treated in an analogous way.

3. Formulae

3.0. *Introduction.* Let $\boldsymbol{P}_n$ be a differential operator of the form

$$\boldsymbol{P}_n = \sum_{i=0}^{n} p_{ni}(x) \boldsymbol{D}^i \tag{3.1}$$

of order $n \leq 3$, with polynomial coefficients p_{ni} $(i = 0, 1, \dots, n)$. Let the operator $\boldsymbol{P}_{n-1}$ of order $n - 1$ be defined by

$$\boldsymbol{P}_{n-1} w := \boldsymbol{P}_n w - \boldsymbol{Q}_n (q_n w),$$

where $\boldsymbol{Q}_n$ is an nth-order operator, of the form given in formula (2.$\{13+n\}$), and q_n is a polynomial such that $\boldsymbol{Q}_n(q_n w) = p_{nn} \boldsymbol{D}^n w + \dots$ and that the corresponding difference operator Q_n (given in Lemma 2.$\{1+n\}$) has the least order. Repeating this process with n replaced by $n-1, n-2$ etc. we obtain the representation

$$\boldsymbol{P}_n w = \sum_{i=0}^{n} \boldsymbol{Q}_i(q_i w), \tag{3.2}$$

where we set $\boldsymbol{Q}_0 = \boldsymbol{I}$ (identity operator), for convenience.

Let w be a solution of the differential equation

$$\boldsymbol{P}_n w = q \qquad (n \le 3),$$

where the differential operator $\boldsymbol{P}_n$ is of the form given in (3.2). Using Lemmas 2.2–2.4 we obtain difference operators Q_i, M_i and functionals $\tau_k^{(i)}[\cdot]$ such that

$$Q_i m_k[\boldsymbol{Q}_i w] = M_i m_k[w] + \tau_k^{(i)}[w] \qquad (i = 1, \ldots, n). \tag{3.3}$$

Now, using Lemma 2.6, a common multiple of the operators Q_i can be obtained in the form

$$P = P_d^{(\sigma)} D^e, \tag{3.4}$$

where $\sigma \in \{-1, +1\}$ and $d, e \ge 0$ are integers; let the difference operators Z_i be such that

$$P = Z_i Q_i \qquad (i = 1, \ldots, n). \tag{3.5}$$

Multiplying both sides of the equation (3.3) on the left by Z_i, and using Lemma 2.6, (3.5) and (2.6), we obtain the result summarized in the following theorem.

Theorem 3.1. *Let w be a solution of the differential equation*

$$\boldsymbol{P}_n w = q \qquad (n \le 3),$$

where the differential operator $\boldsymbol{P}_n$ is of the form given in (3.2) *and q is a known function, and suppose the moments $m_k[w^{(i)}]$ $(i = 0, 1, \ldots, n)$ and $m_k[q]$ exist. Then we have the recurrence relation*

$$L m_k[w] = \varrho(k) \tag{3.6}$$

with

$$L := \sum_{i=0}^{n} Z_i M_i q_i(X), \tag{3.7}$$

$$\varrho(k) := P m_k[q] - \sum_{i=1}^{n} Z_i \tau_k^{(i)}[q_i w]. \tag{3.8}$$

The order of the recurrence (3.6) *is*

$$\mathrm{ord}(P) + 2 \max_{0\le i\le n,\, p_{ni}\not\equiv 0} (\deg p_{ni} - i),$$

where $\mathrm{ord}(P) = d + 2e$ *is the order of the difference operator* P.

The last part of the theorem follows from [6, Eq. (80)].

Now, the form of the operators $\boldsymbol{Q}_i$ $(i = 1, \dots, n)$ in the representation (3.2) can be deduced from the coefficients p_{ni} of the operator (3.1). Thus, we can actually obtain closed-form expressions for P, L and ϱ, at least for small n. In the next subsections we give such formulae for $n = 1$ and $n = 2$, and describe the way the case $n = 3$ may be treated with a small effort.

3.1. *First-order differential equation.* Assume that w satisfies the equation

$$\boldsymbol{P}_1 w \equiv p_{11}(x)w'(x) + p_{10}(x)w(x) = q(x).$$

Case 3.1.1. $p_{11}(\pm 1) = 0$:

$$P := I, \quad L := \kappa(k)Dq_1(X) + q_0(X), \quad \varrho(k) := m_k[q],$$

where

$$q_1(x) := p_{11}(x)/(x^2 - 1), \qquad q_0(x) := p_{10}(x) - \boldsymbol{U}q_1(x).$$

Case 3.1.2. $p_{11}(\sigma) \neq 0$, $p_{11}(-\sigma) = 0$ for $\sigma = -1$ or $\sigma = 1$:

$$P := P_1^{(\sigma)}, \quad L := \mu_1(k)P_1^{(-\sigma)}q_1(X) + P_1^{(\sigma)}q_0(X),$$
$$\varrho(k) := P\{m_k[q] - \varphi_k[p_{11}w]\},$$

where

$$q_1(x) := p_{11}(x)/(x + \sigma), \qquad q_0(x) := p_{10}(x) - \boldsymbol{V}_\sigma q_1(x).$$

Case 3.1.3. $p_{11}(\pm 1) \neq 0$:

$$P := D, \quad L := q_1(X) + Dq_0(X), \quad \varrho(k) := P\{m_k[q] - \varphi_k[p_{11}w]\},$$

where

$$q_1 := p_{11}, \qquad q_0 := p_{10} - \boldsymbol{D}q_1.$$

3.2. *Second-order differential equation.* Assume that w satisfies the equation

$$\boldsymbol{P}_2 w \equiv p_{22}w''(x) + p_{21}(x)w'(x) + p_{20}(x)w(x) = q(x).$$

Define

$$l_\sigma := p_{21}(\sigma) - \tfrac{1}{2}(3 - 2\lambda)p_{22}'(\sigma). \tag{3.9}$$

Case 3.2.1. $p_{22}(\pm 1) = 0$ and $l_{-1} = l_1 = 0$:

$$P := I, \quad L := \kappa(k)\{q_2 + Dq_1(X)\} + q_0(X), \quad \varrho(k) := m_k[q],$$

where

$$q_2(x) := p_{22}(x)/(x^2-1),$$
$$q_1(x) := [p_{21}(x) - (3-2\lambda)xq_2(x)]/(x^2-1) - 2\boldsymbol{D}q_2(x),$$
$$q_0 := p_{20} - \boldsymbol{G}q_2 - \boldsymbol{U}q_1.$$

Case 3.2.2. $p_{22}(\pm 1) = 0$, $l_{-\sigma} = 0$, $l_\sigma \neq 0$ for $\sigma = -1$ or $\sigma = 1$:

$$P := P_1^{(\sigma)}, \qquad L := P_1^{(\sigma)}\{\kappa(k)q_2(X) + q_0(X)\} + \mu_1(k)P_1^{(-\sigma)}q_1(X),$$
$$\varrho(k) := P\{m_k[q] - \varphi_k[(x+\sigma)q_1 w]\},$$

where

$$q_2(x) := p_{22}(x)/(x^2-1),$$
$$q_1(x) := [p_{21}(x) - (3-2\lambda)xq_2(x)]/(x+\sigma) - 2(x-\sigma)\boldsymbol{D}q_2(x),$$
$$q_0 := p_{20} - \boldsymbol{V}_\sigma q_1 - \boldsymbol{G}q_2.$$

Case 3.2.3. $p_{22}(\pm 1) = 0$, $l_{-1} \neq 0$, $l_1 \neq 0$:

$$P := D, \quad L := q_1(X) + D\{\kappa(k)q_2(X) + q_0(X)\},$$
$$\varrho(k) := D\{m_k[q] - \varphi_k[q_1 w]\},$$

where

$$q_2(x) := p_{22}(x)/(x^2-1),$$
$$q_1(x) := p_{21}(x) - (3-2\lambda)xq_2(x) - 2(x^2-1)\boldsymbol{D}q_2(x),$$
$$q_0 := p_{20} - \boldsymbol{D}q_1 - \boldsymbol{G}q_2.$$

Case 3.2.4. $p_{22}(\sigma) \neq 0$, $p_{22}(-\sigma) = 0$, $l_{-\sigma} = 0$ for $\sigma = -1$ or $\sigma = 1$:

$$P := P_2^{(\sigma)}, \qquad L := A_1^{(\sigma)}\{\mu_1(k)P_1^{(-\sigma)}q_1(X) + P_1^{(\sigma)}q_0(X)\} + \mu_2(k)Eq_2(X),$$
$$\varrho(k) := P\{m_k[q] - \varphi_k[(x+\sigma)((q_2w)' + q_1 w)]\} - \mu_2(k)ED\varphi_k[q_2 w],$$

where

$$q_2(x) := p_{22}(x)/(x+\sigma),$$
$$q_1(x) := [p_{21}(x) - \tfrac{1}{2}(3-2\lambda)q_2(x)]/(x+\sigma) - 2\boldsymbol{D}q_2(x),$$
$$q_0 := p_{20} - \boldsymbol{V}_\sigma q_1 - \boldsymbol{H}_\sigma q_2.$$

Case 3.2.5. $p_{22}(\sigma) \neq 0$, $p_{22}(-\sigma) = 0$, $l_{-\sigma} \neq 0$ for $\sigma = -1$ or $\sigma = 1$:

$$P := P_1^{(\sigma)}D, \qquad L := Wq_2(X) + P_1^{(\sigma)}\{q_1(X) + Dq_0(X)\},$$
$$\varrho(k) := P\{m_k[q] - \varphi_k[q_1 w + p_{22}w']\} - WD\varphi_k[q_2 w],$$

where $W := R_1^{(\sigma)}\mu_2(k)E = \mu_1(k-1)I + \sigma\mu_1(k+1)E$, and

$$q_2(x) := p_{22}(x)/(x+\sigma),$$
$$q_1(x) := p_{21}(x) - \tfrac{1}{2}(3-2\lambda)q_2(x) - 2(x+\sigma)\boldsymbol{D}q_2(x),$$
$$q_0 := p_{20} - \boldsymbol{D}q_1 - \boldsymbol{H}_\sigma q_2.$$

Case 3.2.6. $p_{22}(\pm 1) \neq 0$:

$$P := D^2, \quad L := q_2(X) + Dq_1(X) + D^2 q_0(X),$$
$$\varrho(k) := P\{m_k[q] - \varphi_k[\boldsymbol{D}(q_2 w) + q_1 w]\} - D\varphi_k[q_2 w],$$

where

$$q_2 := p_{22}, \quad q_1 := p_{21} - 2\boldsymbol{D}q_2, \quad q_0 := p_{20} - \boldsymbol{D}q_1 - \boldsymbol{D}^2 q_2.$$

3.3. *Third-order differential equation.* Assume that w satisfies the equation

$$\boldsymbol{P}_3 w \equiv p_{33} w'''(x) + p_{32} w''(x) + p_{31}(x) w'(x) + p_{30}(x) w(x) = q(x). \tag{3.10}$$

The list of explicit formulae for the difference operators P, L and the function ϱ covers ten cases. Thus, it seems too long to be given here. However, one may obtain a recurrence for the modified moments of w with a small effort, using the results of Section 3.2. To this end, represent the left-hand side of (3.10) in the form $\boldsymbol{P}_3 w = \boldsymbol{Q}_3(q_3 w) + \boldsymbol{P}_2 w$, where $\boldsymbol{Q}_3$ is of the form (2.16) (cf. Section 3.0). Then apply (formally) the results of Section 3.2 to the equation $\boldsymbol{P}_2 w = q^*$ with $q^* := q - \boldsymbol{Q}_3(q_3 w)$, which will yield operators P, L and a function ψ such that

$$Lm_k[w] = \varrho(k) \tag{3.11}$$

with $\varrho(k) = Pm_k[q^*] - \psi(k)$. Next, using Lemma 2.4, obtain operators Q_3, M_3 and a functional $\tau_k^{(3)}[\cdot]$ satisfying $Q_3 m_k[\boldsymbol{Q}_3 w] = M_3 m_k[w] + \tau_k^{(3)}[w]$. Using Lemma 2.6 obtain a common multiple P^* of P and Q_3, i.e. find difference operators W and Z such that $P^* = WP = ZQ_3$. Then multiply (3.11) from the left by W and observe that

$$\begin{aligned} WPm_k[q^*] &= ZQ_3 m_k[q^*] = P^* m_k[q] - ZQ_3 m_k[\boldsymbol{Q}_3(q_3 w)] \\ &= P^* m_k[q] - Z\{M_3 q_3(X) m_k[w] + \tau_k^{(3)}[q_3 w]\}. \end{aligned}$$

Now, it is easy to see that the recurrence relation in question has the form

$$L^* m_k[w] = \varrho^*(k)$$

with

$$L^* := WL + ZM_3 q_3(X), \qquad \varrho^*(k) := P^* m_k[q] - W\psi(k) - Z\tau_k^{(3)}[q_3 w].$$

4. An example. Consider the numerical evaluation of the integral

$$I = \int_0^1 f(x) x^\alpha J_p(\omega x)\, dx, \tag{4.1}$$

where J_p is the Bessel function of the first kind and of order p, and where $\alpha > -p-1$, $\omega > 0$ are real numbers. We assume that f is a smooth function. If ω is large, the use of standard integration rules is not efficient in view of the highly oscillatory character of the integrand. Therefore, special methods should be used, as, e.g. the method given in [8] or the modified Clenshaw–Curtis method (see [10] where the case $\alpha = 0$ is discussed). The latter method is based on the approximation of f by a polynomial

$$p(x) = {\sum_{k=0}^{n}}' c_k T_k^*(x) \qquad (0 \le x \le 1).$$

Here the symbol $\sum'$ denotes the sum with the first term halved, and T_k^* is the kth shifted Chebyshev polynomial, $T_k^*(x) = T_k(2x-1)$. Replacing f by p in (4.1), we obtain

$$\begin{aligned} I &\approx {\sum_{k=0}^{n}}' c_k \int_0^1 x^\alpha J_p(\omega x) T_k^*(x)\,dx \\ &= 2^{-\alpha-1} {\sum_{k=0}^{n}}' c_k \int_{-1}^{1} (1+x)^\alpha J_p(\tfrac{1}{2}\omega(1+x)) T_k(x)\,dx \\ &= 2^{-\alpha-1} {\sum_{k=0}^{n}}' c_k \tau_k[w], \end{aligned}$$

where

$$w(x) = (1+x)^\alpha J_p(a(1+x)) \qquad (a = \tfrac{1}{2}\omega).$$

We show that the Chebyshev moments $\tau_k[w]$ obey a sixth-order recurrence relation.

The second-order differential equation for the Bessel function J_p implies

$$(1+x)^2 w'' + (1-2\alpha)(1+x)w' + [a^2(1+x)^2 + \alpha^2 - p^2]w = 0.$$

We have

$$l_\varepsilon = -2(1+\alpha)(1+\varepsilon) \qquad (\varepsilon = \pm 1)$$

(cf. (3.9)). It is easy to see that we have here case 3.2.4 with $\sigma = 1$, $P = P_2^{(1)}$, and

$$\begin{aligned} L &= P_2^{(1)} q_0(X) + \mu_2(k) E[q_1 D + q_2(X)], \\ \varrho(k) &= -P_2^{(1)} \varphi_k[(1+x)^2 w' - (2\alpha + \tfrac{3}{2})(1+x)w] \\ &\quad - \mu_2(k) E D \varphi_k[(1+x)w], \end{aligned}$$

where

$$\begin{aligned} q_2(x) &= 1+x, \qquad q_1(x) = -2\alpha - \tfrac{5}{2}, \\ q_0(x) &= a^2(1+x)^2 + (\alpha + \tfrac{3}{2})^2 - p^2. \end{aligned}$$

Using (2.3), (2.4) and (2.11), we get

$$L = \frac{1}{4} a^2 \frac{k-2}{k} E^{-2} + a^2 \frac{(k+3)(k-1)}{k(k+2)} E^{-1}$$

$$+\frac{1}{4}\left\{2[b+2(k-1)(k-2\alpha-1)]-a^2\frac{k(14k+31)}{(k+2)(2k+3)}\right\}I$$
$$+\frac{2(k+1)}{2k+3}\left\{3(1-2\alpha)-b+2(k-1)(k+3)+a^2\frac{2k^2+4k+3}{k(k+2)}\right\}E$$
$$+\frac{1}{4(2k+3)}\left\{2(2k+1)[b+2(k+3)(k+2\alpha+3)]\right.$$
$$\left.-a^2\frac{(k+2)(14k-3)}{k}\right\}E^2$$
$$+a^2\frac{(k-1)(k+3)}{(k+2)(2k+3)}E^3+\frac{1}{4}a^2\frac{(k+4)(2k+1)}{(k+2)(2k+3)}E^4,$$

where

$$b := 3a^2+2\alpha^2-2p^2.$$

From (2.5) we obtain

$$\varrho(k) = -\frac{2^{\alpha+3}}{k(k+2)(2k+3)}\{[2(k+1)^2-3\alpha-5]J_p(2a)+6aJ_p'(2a)\}.$$

Now, the Gegenbauer moments $m_k^0[w]$ obey the recurrence relation

$$Lm_k^0[w] = \varrho(k).$$

Substituting $m_k^0[w] = \frac{2}{k}\tau_k[w]$ (cf. (1.2)), replacing k by $k-1$ and multiplying the resulting equation by $2(k^2-1)(2k+1)$, we obtain the desired recurrence relation

$$\sum_{j=0}^{6} C_j(k)\tau_{k-3+j}[w] = 2^{\alpha+4}\{(5+3\alpha-2k^2)J_p(2a)-6aJ_p'(2a)\},$$

where

$$C_0(k) = C_6(-k) = a^2(k+1)(2k+1),$$
$$C_1(k) = C_5(-k) = 4a^2(k+1)(k+2),$$
$$C_2(k) = C_4(-k)$$
$$= 2(k+1)(2k+1)[b+2(k-2)(k-2\alpha-2)]-a^2(k-1)(14k+17),$$
$$C_3(k) = 8(k^2-1)[2(k^2-4)-b+3(1-2\alpha)]+8a^2(2k^2+1).$$

Note that in [10] an eighth-order homogeneous recurrence relation is given for the special case $\alpha = 0$.

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STANISŁAW LEWANOWICZ
INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
UL. PRZESMYCKIEGO 20
51-151 WROCŁAW, POLAND
E-mail: SLE@II.UNI.WROC.PL

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