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ON FOURIER COEFFICIENT ESTIMATORS CONSISTENT IN THE MEAN-SQUARE SENSE

Abstract. The properties of two recursive estimators of the Fourier coefficients of a regression function $f \in L^2[a,b]$ with respect to a complete orthonormal system of bounded functions (e_k), $k = 1,2,\ldots$, are considered in the case of the observation model $y_i = f(x_i) + \eta_i$, $i = 1,\ldots,n$, where η_i are independent random variables with zero mean and finite variance, $x_i \in [a,b] \subset \mathbb{R}^1$, $i = 1,\ldots,n$, form a random sample from a distribution with density $\varrho = 1/(b-a)$ (uniform distribution) and are independent of the errors η_i, $i = 1,\ldots,n$. Unbiasedness and mean-square consistency of the examined estimators are proved and their mean-square errors are compared.

1. Introduction. Let y_i, $i = 1,\ldots,n$, be observations at points $x_i \in [a,b] \subset \mathbb{R}^1$, according to the model $y_i = f(x_i) + \eta_i$, where $f : [a,b] \to \mathbb{R}^1$ is an unknown square integrable function ($f \in L^2[a,b]$) and $\eta_i, i = 1,\ldots,n$, are independent identically distributed random variables with zero mean and finite variance $\sigma_\eta^2 > 0$. Let furthermore the points $x_i, i = 1,\ldots,n$, form a random sample from a distribution with density $\varrho = 1/(b-a)$ (uniform distribution), independent of the observation errors η_i, $i = 1,\ldots,n$.

We assume that the functions $(e_k), k = 1,2,\ldots$, constitute a complete orthonormal system in $L^2[a,b]$, and that they are bounded and normalized so that

$$\frac{1}{b-a}\int_a^b e_k^2(x)\,dx = 1, \qquad k = 1,2,\ldots$$

Then f has the representation

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$$f = \sum_{k=1}^{\infty} c_k e_k \,, \qquad \text{where } c_k = \frac{1}{b-a} \int_a^b f(x) e_k(x)\, dx \,, \qquad k = 1, 2, \ldots$$

The first estimator of the Fourier coefficients we shall deal with is well-known and has a simple form

$$\widetilde{c}_k = \frac{1}{n} \sum_{i=1}^{n} y_i e_k(x_i) \,, \qquad k = 1, 2, \ldots, \tag{1.1}$$

so that we easily obtain the following formulae:

$$\begin{aligned} &E\widetilde{c}_k = E_x E_\eta c_k = c_k \,, \\ &E(\widetilde{c}_k - c_k)^2 = \frac{1}{n(b-a)} \int_a^b (f(x) e_k(x) - c_k)^2 dx + \frac{1}{n} \sigma_\eta^2 \,. \end{aligned} \tag{1.2}$$

The estimators $\widetilde{c}_k$, $k = 1, 2, \ldots$, are thus unbiased and consistent in the mean-square sense. If we estimate the Fourier coefficients $c_1, \ldots, c_N$, the number N being fixed, we can write formula (1.1) in the vector form

$$\widetilde{c}(n, N) = \frac{1}{n} \sum_{i=1}^{n} y_i e^N(x_i) \,,$$

where $\widetilde{c}(n, N) = (\widetilde{c}_1, \ldots, \widetilde{c}_N)^T$, $e^N(x) = (e_1(x), \ldots, e_N(x))^T$, which can be rewritten in the recursive form

$$\widetilde{c}(n, N) = \frac{n-1}{n} \widetilde{c}(n-1, N) + \frac{1}{n} y_n e^N(x_n) \,, \qquad \widetilde{c}(0, N) = (0, \ldots, 0)^T \,.$$

In view of (1.2) we also have

$$\begin{aligned} &E\widetilde{c}(n, N) = (c_1, \ldots, c_N)^T = c^N \,, \\ &E\|\widetilde{c}(n, N) - c^N\|^2 \\ &\qquad = \frac{1}{n} \left(\frac{1}{b-a} \int_a^b f^2(x) \|e^N(x)\|^2 \, dx - \|c^N\|^2 \right) + \frac{1}{n} N \sigma_\eta^2 \,. \end{aligned} \tag{1.3}$$

The second estimator of the Fourier coefficients is constructed similarly to the estimators occurring in stochastic approximation methods [1], [2]; namely, it is defined by the recursive formula

$$\widehat{c}(n, N) = \widehat{c}(n-1, N) + \frac{1}{n} \delta_n e^N(x_n) \,, \tag{1.4}$$

where $\delta_n = y_n - \langle \widehat{c}(n-1, N), e^N(x_n) \rangle$, $\widehat{c}(0, N) = (0, \ldots, 0)^T$.

In the sequel we shall use the notation $\Delta_n = \widehat{c}(n,N) - c^N$, $\Delta_0 = -c^N$.

By (1.4) we can write

$$\begin{aligned}\Delta_n &= \widehat{c}(n,N) - c^N\\ &= \widehat{c}(n-1,N) - c^N + \frac{1}{n}(f(x_n) + \eta_n - \langle \widehat{c}(n-1,N), e^N(x_n)\rangle)e^N(x_n)\end{aligned}$$

and, since $f(x) = \sum_{k=1}^N c_k e_k(x) + r_N(x)$, where $r_N = \sum_{k=N+1}^\infty c_k e_k$, we obtain

$$\Delta_n = \Delta_{n-1} - \frac{1}{n}\langle \Delta_{n-1}, e^N(x_n)\rangle e^N(x_n) + \frac{1}{n}(\eta_n + r_N(x_n))e^N(x_n)\,. \tag{1.5}$$

2. Unbiasedness and mean-square consistency of the estimators. We have already remarked that the estimator $\widetilde{c}(n,N)$ is unbiased and consistent in the mean-square sense (see formulae (1.3)). Now we will prove the same for $\widehat{c}(n,N)$. First we prove by induction that $E\Delta_n = 0$ for $n = 1,2,\ldots$ By (1.5) for $n=1$, we have

$$\begin{aligned}E\Delta_1 &= E_x E_\eta \Delta_1 = \Delta_0 - E_x e^N(x_1)e^N(x_1)^T\Delta_0 + E_x r_N(x_1)e^N(x_1)\\ &= \Delta_0 - I\Delta_0 = 0,\end{aligned}$$

since $E_\eta\eta_1 = 0$, $E_x e^N(x_1)e^N(x_1)^T = I$ and $E_x r_N(x_1)e^N(x_1) = 0$.

Assume now that $E\Delta_{n-1} = 0$. Then, by (1.5),

$$E\Delta_n = E\Delta_{n-1} - \frac{1}{n}Ee^N(x_n)e^N(x_n)^T\Delta_{n-1}\,,$$

since $E_\eta\eta_n = 0$ and $E_x r_N(x_n)e^N(x_n) = 0$. Since Δ_{n-1} does not depend on x_n we finally obtain

$$E\Delta_n = E\Delta_{n-1} - \frac{1}{n}E_x e^N(x_n)e^N(x_n)^T E\Delta_{n-1} = \left(1 - \frac{1}{n}\right)E\Delta_{n-1} = 0\,.$$

The unbiasedness of $\widehat{c}(n,N)$ is thus proved. To prove the mean-square consistency of this estimator we need the following two lemmas.

LEMMA 2.1. *The random variables Δ_n, $n = 1,2,\ldots$, satisfy the recursive inequality*

$$\begin{aligned}E\|\Delta_n\|^2 &\le \left(1 - \frac{2}{n} + \frac{1}{n^2}N^2M_N\right)E\|\Delta_{n-1}\|^2\\ &\quad + \frac{1}{n^2}\left(p_N M_N + N\sigma_\eta^2\right),\end{aligned} \tag{2.1}$$

where $p_N = \sum_{k=N+1}^\infty c_k^2$, $M_N = \sup_{a\le x\le b}\|e^N(x)\|^2$.

Proof. Taking into account (1.5) and remembering that $E\|\Delta_n\|^2$ can be computed here as $E_{x_1,\ldots,x_{n-1},\eta_1,\ldots,\eta_{n-1}}E_{x_n}E_{\eta_n}\|\Delta_n\|^2$, we can write

$$E\|\Delta_n\|^2 = E_x E_\eta \Big\| \Delta_{n-1} - \frac{1}{n} e^N(x_n) e^N(x_n)^T \Delta_{n-1} + \frac{1}{n}(r_N(x_n)+\eta_n) e^N(x_n)\Big\|^2$$
$$= E\Big\|\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Delta_{n-1} + \frac{1}{n} r_N(x_n) e^N(x_n)\Big\|^2 + \frac{1}{n^2}\sigma_\eta^2 E_x \|e^N(x_n)\|^2 .$$

Since Δ_{n-1} does not depend on x_n and $E\Delta_{n-1} = 0$ we obtain

$$E\|\Delta_n\|^2 = E\Big\|\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Delta_{n-1}\Big\|^2 + \frac{1}{n^2} E_x \|r_N(x_n) e^N(x_n)\|^2 + \frac{1}{n^2}\sigma_\eta^2 E_x \|e^N(x_n)\|^2 .$$

Furthermore, $E_x\|e^N(x_n)\|^2 = E_x \sum_{k=1}^N e_k^2(x_n) = N$, since $E_x e_k^2(x_n) = 1$ for $k = 1, 2, \ldots$, and finally,

$$E\|\Delta_n\|^2 = E\Big\|\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Delta_{n-1}\Big\|^2 + \frac{1}{n^2} E_x \|r_N(x_n) e^N(x_n)\|^2 + \frac{1}{n^2} N\sigma_\eta^2 .$$

For the first term on the right hand side we obtain

$$E\Big\|\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Delta_{n-1}\Big\|^2$$
$$= E\,\mathrm{tr}\Big[\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Delta_{n-1}\Delta_{n-1}^T\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)\Big]$$
$$= E\,\mathrm{tr}\Big[\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)^2 \Delta_{n-1}\Delta_{n-1}^T\Big]$$
$$= \mathrm{tr}\Big[E_x\Big(I - \frac{1}{n} e^N(x_n) e^N(x_n)^T\Big)^2 E\Delta_{n-1}\Delta_{n-1}^T\Big]$$
$$= \mathrm{tr}\Big[\Big(I - \frac{2}{n} I + \frac{1}{n^2} E_x e^N(x_n)\|e^N(x_n)\|^2 e^N(x_n)^T\Big) E\Delta_{n-1}\Delta_{n-1}^T\Big]$$
$$= \Big(1 - \frac{2}{n}\Big)\,\mathrm{tr}\, E\Delta_{n-1}\Delta_{n-1}^T + \frac{1}{n^2}\,\mathrm{tr}[E_x\|e^N(x_n)\|^2 e^N(x_n) e^N(x_n)^T E\Delta_{n-1}\Delta_{n-1}^T]$$
$$= \Big(1 - \frac{2}{n}\Big) E\|\Delta_{n-1}\|^2 + \frac{1}{n^2}\,\mathrm{tr}[E_x\|e^N(x_n)\|^2 e^N(x_n) e^N(x_n)^T E\Delta_{n-1}\Delta_{n-1}^T] .$$

Observe that

$$|E_x\|e^N(x_n)\|^2 e_i(x_n)e_j(x_n)| \le \sup_{a\le x\le b}\|e^N(x)\|^2 E_x|e_i(x_n)e_j(x_n)| \le \sup_{a\le x\le b}\|e^N(x)\|^2 (E_x e_i^2(x_n))^{1/2}(E_x e_j^2(x_n))^{1/2} \equiv M_N$$

for $i,j=1,\ldots,N$. On the other hand, for $\Delta_{n-1}=(\Delta_{n-1,1},\Delta_{n-1,2},\ldots,\Delta_{n-1,N})^T$, we also have

$$|E(\Delta_{n-1,i}\Delta_{n-1,j})| \le E\|\Delta_{n-1}\|^2 \quad \text{for } i,j=1,\ldots,N\,.$$

These estimates yield

$$E\|\Delta_{n-1}\|^2 \le \Big(1-\frac{2}{n}\Big)E\|\Delta_{n-1}\|^2 + \frac{1}{n^2}N^2M_N E\|\Delta_{n-1}\|^2 + \frac{1}{n^2}E_x r_N^2(x_n)\|e^N(x_n)\|^2 + \frac{1}{n^2}N\sigma_\eta^2\,,$$

and since

$$E_x r_N^2(x_n)\|e^N(x_n)\|^2 \le \sup_{a\le x\le b}\|e^N(x)\|^2 E_x r_N^2(x_n) = M_N\sum_{k=N+1}^{\infty} c_k^2 = M_N p_N\,,$$

we finally obtain the estimate

$$E\|\Delta_n\|^2 \le \Big(1-\frac{2}{n}+\frac{1}{n^2}N^2M_N\Big)E\|\Delta_{n-1}\|^2 + \frac{1}{n^2}p_N M_N + \frac{1}{n^2}N\sigma_\eta^2\,. \ ■$$

LEMMA 2.2. *If nonnegative real numbers v_n, $n=0,1,2,\ldots$, satisfy the recursive inequality*

$$v_n \le \Big(1-\frac{2}{n}+\frac{d}{n^2}\Big)v_{n-1} + \frac{b}{n^2}\,, \qquad b>0,\ d>1,\ n=1,2,\ldots,$$

then

$$v_n \le \frac{d-1}{n^2}(v_0+b+b\ln(n-1))\exp(\pi^2(d-1)/6) + \frac{b}{n}\,, \qquad n=1,2,\ldots$$

P r o o f. From the assumptions it follows immediately that

$$v_n \le \Big(1-\frac{2}{n}+\frac{d}{n^2}\Big)\Big(1-\frac{2}{n-1}+\frac{d}{(n-1)^2}\Big)\ldots\Big(1-\frac{2}{1}+\frac{d}{1^2}\Big)v_0 + b\Big(1-\frac{2}{n}+\frac{d}{n^2}\Big)\Big(1-\frac{2}{n-1}+\frac{d}{(n-1)^2}\Big)\ldots\Big(1-\frac{2}{2}+\frac{d}{2^2}\Big)\frac{1}{1^2} + \ldots + b\Big(1-\frac{2}{n}+\frac{d}{n^2}\Big)\frac{1}{(n-1)^2} + b\frac{1}{n^2}\,.$$

Taking into account the identity

$$1-\frac{2}{k}+\frac{d}{k^2}=\frac{k^2-2k+d}{k^2}=\frac{(k-1)^2+d-1}{k^2}$$

we obtain

$$\begin{aligned}v_n \le{}& \frac{(n-1)^2+d-1}{n^2}\cdot\frac{(n-2)^2+d-1}{(n-1)^2}\cdots\frac{(1-1)^2+d-1}{1^2}v_0\\ &+b\frac{(n-1)^2+d-1}{n^2}\cdot\frac{(n-2)^2+d-1}{(n-1)^2}\cdots\frac{(2-1)^2+d-1}{2^2}\cdot\frac{1}{1^2}\\ &+\ldots+b\frac{(n-1)^2+d-1}{n^2}\cdot\frac{1}{(n-1)^2}+b\frac{1}{n^2},\end{aligned}$$

or equivalently,

$$\begin{aligned}v_n \le{}& \frac{1}{n^2}\left(1+\frac{d-1}{(n-1)^2}\right)\left(1+\frac{d-1}{(n-2)^2}\right)\cdots\left(1+\frac{d-1}{1^2}\right)(d-1)v_0\\ &+b\frac{1}{n^2}\left(1+\frac{d-1}{(n-1)^2}\right)\left(1+\frac{d-1}{(n-2)^2}\right)\cdots\left(1+\frac{d-1}{1^2}\right)\\ &+\ldots+b\frac{1}{n^2}\left(1+\frac{d-1}{(n-1)^2}\right)+b\frac{1}{n^2}.\end{aligned}$$

Since $\exp(x)>1+x$ for $x>0$, we have

$$\begin{aligned}v_n \le{}& \frac{1}{n^2}(d-1)v_0\exp\left((d-1)\sum_{k=1}^{n-1}\frac{1}{k^2}\right)\\ &+\frac{1}{n^2}b\left[\exp\left((d-1)\sum_{k=1}^{n-1}\frac{1}{k^2}\right)+\ldots+\exp\left((d-1)\frac{1}{(n-1)^2}\right)+1\right].\end{aligned}$$

Since $\sum_{k=1}^{\infty}1/k^2$ is known to be equal to $\pi^2/6$, and clearly

$$\exp(x)\le 1+Mx\,,\quad M=\exp(\pi^2(d-1)/6),\quad \text{for } x\in[0,\pi^2(d-1)/6]\,,$$

we have

$$\begin{aligned}v_n \le{}& \frac{1}{n^2}(d-1)v_0M\\ &+\frac{1}{n^2}b\Big[1+(d-1)M\sum_{k=1}^{n-1}\frac{1}{k^2}+1+(d-1)M\sum_{k=2}^{n-1}\frac{1}{k^2}\\ &\qquad+\ldots+1+(d-1)M\frac{1}{(n-1)^2}+1\Big]\\ \le{}& \frac{(d-1)M}{n^2}\left(v_0+b\left[\sum_{k=1}^{n-1}\frac{1}{k^2}+\sum_{k=2}^{n-1}\frac{1}{k^2}+\ldots+\frac{1}{(n-1)^2}\right]\right)+\frac{b}{n}.\end{aligned}$$

Summing the terms in square brackets we get

$$v_n \le \frac{(d-1)M}{n^2}\left(v_0 + b\left[\frac{n-1}{(n-1)^2} + \frac{n-2}{(n-2)^2} + \ldots + \frac{1}{1^2}\right]\right) + \frac{b}{n}$$
$$= \frac{(d-1)M}{n^2}\left(v_0 + b\sum_{k=1}^{n-1}\frac{1}{k}\right) + \frac{b}{n}\,.$$

Since $\ln(1+x) \ge x/(1+x)$ for $x > 0$, putting $x = 1/k$ we obtain

$$\ln\left(\frac{k+1}{k}\right) \ge \frac{1}{k+1} \qquad \text{for } k = 1, 2, \ldots,$$

and consequently

$$\sum_{k=1}^{n-1}\frac{1}{k} \le 1 + \sum_{k=1}^{n-2}\ln\left(\frac{k+1}{k}\right) = 1 + \sum_{k=1}^{n-2}(\ln(k+1) - \ln(k)) = 1 + \ln(n-1)\,,$$

which completes the proof. ∎

Inequality (2.1) assures that the sequence $v_n = E\|\Delta_n\|^2$, $n = 0, 1, 2, \ldots,$ satisfies the assumptions of Lemma 2.2 ($\sup_{a\le x\le b}\|e^N(x)\|^2 > 1$ for $N > 1$ since $E\|e^N(x)\|^2 = N$) so that we have the estimate

$$\begin{aligned} E\|\Delta_n\|^2 \le{}& \frac{1}{n^2}(N^2M_N - 1)\exp(\pi^2(N^2M_N - 1)/6) \\ &\times [E\|\Delta_0\|^2 + (p_NM_N + N\sigma_\eta^2)(1 + \ln(n-1))] \\ &+ \frac{1}{n}(p_NM_N + N\sigma_\eta^2) \end{aligned}$$

and putting $C = \exp(-\pi^2/6)$ we can write

$$\begin{aligned} E\|\Delta_n\|^2 \le{}& \frac{1}{n^2}CN^2M_N\exp(\pi^2N^2M_N/6) \\ &\times [\|c^N\|^2 + (p_NM_N + N\sigma_\eta^2)(1 + \ln n)] \\ &+ \frac{1}{n}(p_NM_N + N\sigma_\eta^2)\,. \end{aligned} \tag{2.2}$$

This implies that, for fixed N, the estimator $\widehat{c}(n, N)$ is consistent in the mean-square sense.

Now we shall compare the mean-square errors of $\widehat{c}(n, N)$ and $\widetilde{c}(n, N)$ in the case when $f \in L^2(0, 2\pi)$. The system

$$e_1(x) = 1\,, \quad e_{2m}(x) = \sqrt{2}\sin(mx)\,,$$
$$e_{2m+1}(x) = \sqrt{2}\cos(mx)\,, \quad m = 1, 2, \ldots,$$

is a complete orthogonal system in $L^2(0, 2\pi)$ and $(2\pi)^{-1}\int_0^{2\pi} e_k^2(x)\,dx = 1$,

$k = 1, 2, \ldots$ For this system we also have

$$\|e^N(x)\|^2 = \sum_{k=1}^{2m+1} e_k^2(x) = 2m + 1 = N \qquad \text{for } N = 2m+1,\ m \ge 0$$

so that the estimates for the mean-square errors considered (see (1.3) and (2.2)) take the form

$$\begin{aligned}
E\|\widetilde{c}(n,N) - c^N\|^2 &= \frac{1}{n} N(p_N + \sigma_\eta^2) + \frac{1}{n}(N-1)\|c^N\|^2\,, \\
E\|\widehat{c}(n,N) - c^N\|^2 & \\
&\le \frac{1}{n^2} C N^3 \exp(\pi^2 N^3/6)[\|c^N\|^2 + N(p_N + \sigma_\eta^2)(1 + \ln n)] \\
&\quad + \frac{1}{n} N(p_N + \sigma_\eta^2)\,,
\end{aligned} \tag{2.3}$$

where $N = 2m + 1$, $m > 0$ and $C = \exp(-\pi^2/6)$.

From (2.3) we see that for $N > 1$ and $\|c^N\|^2 > 0$ we have

$$E\|\widehat{c}(n,N) - c^N\|^2 < E\|\widetilde{c}(n,N) - c^N\|^2 \tag{2.4}$$

for sufficiently large n, so that $\widehat{c}(n,N)$, although more complicated in form, has a smaller mean-square error for large values of n than $\widetilde{c}(n,N)$.

3. Conclusions. We now assume that $f \in L^2(0, 2\pi)$. Having determined the estimators $\overline{c}^N = (\overline{c}_1, \ldots, \overline{c}_N)^T$ of Fourier coefficients we can form an estimator of the regression function f, called a projection type estimator [3]:

$$\overline{f}_N(x) = \sum_{k=1}^{N} \overline{c}_k e_k(x) = \langle \overline{c}^N, e^N(x)\rangle, \tag{3.1}$$

$N = 2m + 1$, $m > 0$, $e^N(x) = (1, \sqrt{2}\sin(x), \sqrt{2}\cos(x), \ldots, \sqrt{2}\sin(mx), \sqrt{2}\cos(mx))^T$.

In case $\overline{c}^N = \widetilde{c}(n,N)$ this estimator is also a kernel type estimator [3], since then formula (3.1) takes the form

$$\overline{f}_N(x) = \frac{1}{n} \sum_{i=1}^{n} y_i \sum_{k=1}^{N} e_k(x_i) e_k(x)\,.$$

For such an estimator the following formula for the integrated mean-square error is valid:

$$\begin{aligned}
E\frac{1}{2\pi}\int_0^{2\pi} (f(x) - \overline{f}_N(x))^2\,dx &= E\|c^N - \overline{c}^N\|^2 + \sum_{k=N+1}^{\infty} c_k^2 \\
&= E\|\overline{c}^N - c^N\|^2 + p_N\,.
\end{aligned} \tag{3.2}$$

In view of the inequality

$$\|c^N\|^2 = \sum_{k=1}^{N} c_k^2 \le \sum_{k=1}^{\infty} c_k^2 = \frac{1}{2\pi}\|f\|^2$$

and (2.3) we can obtain the following estimates for the mean-square errors:

$$\begin{aligned}(3.3)\qquad & E\|\widetilde{c}(n,N) - c^N\|^2 \le \frac{1}{n}N(p_N + \sigma_\eta^2) + \frac{1}{n}\frac{N}{2\pi}\|f\|^2,\\ & E\|\widehat{c}(n,N) - c^N\|^2\\ & \quad \le \frac{1}{n^2}CN^3\exp(\pi^2N^3/6)\left[\frac{1}{2\pi}\|f\|^2 + N(p_N+\sigma_\eta^2)(1+\ln n)\right]\\ & \qquad + \frac{1}{n}N(p_N+\sigma_\eta^2)\,,\end{aligned}$$

where $N = 2m+1$, $m > 0$ and $C = \exp(-\pi^2/6)$.

Formula (3.2) and the estimates in (3.3) imply that if we put $N(n) = 2m(n)+1$, $\overline{c}^{N(n)} = \widehat{c}(n, N(n))$ and if

$$\lim_{n\to\infty} N(n) = \infty\,, \qquad \limsup_{n\to\infty} N(n)/(\ln n)^{1/3} < (12/\pi^2)^{1/3}\,,$$

then $\lim_{n\to\infty} E\|f - \overline{f}_{N(n)}\|^2 = 0$. The same is true if we put $\overline{c}^{N(n)} = \widetilde{c}(n, N(n))$ with $\lim_{n\to\infty} N(n) = \infty$ and $\lim_{n\to\infty} N(n)/n = 0$.

In this way we have obtained sufficient conditions for convergence to zero of the integrated mean-square error of the estimator $\overline{f}_N$.

If the estimator $\overline{c}^N$ is unbiased then

$$\begin{aligned}E(f(x) - \overline{f}_N(x))^2 &= E\langle c^N - \overline{c}^N, e^N(x)\rangle^2\\ &\quad + 2r_N(x)E\langle c^N - \overline{c}^N, e^N(x)\rangle + Er_N^2(x)\\ &= E\langle c^N - \overline{c}^N, e^N(x)\rangle^2 + r_N^2(x)\,,\end{aligned}$$

where $r_N = \sum_{k=N+1}^{\infty} c_k e_k$. From the Cauchy–Schwarz inequality it follows that

$$E(f(x) - \overline{f}_N(x))^2 \le E\|\overline{c}^N - c^N\|^2\|e^N(x)\|^2 + r_N^2(x)$$

and since $\|e^N(x)\|^2 = N$ for $N = 2m+1$, $m \ge 0$, we finally have

$$(3.4)\qquad E(f(x) - \overline{f}_N(x))^2 \le NE\|\overline{c}^N - c^N\|^2 + r_N^2(x)\,.$$

If the Fourier series of f converges at a point $x \in [0, 2\pi]$ to $f(x)$ then, of course, $\lim_{n\to\infty} r_{N(n)}(x) = 0$ if $\lim_{n\to\infty} N(n) = \infty$. The estimates in (3.3) and (3.4) imply that if we put $N(n) = 2m(n)+1$, $\overline{c}^{N(n)} = \widehat{c}(n, N(n))$ and if

$$\lim_{n\to\infty} N(n) = \infty\,, \qquad \limsup_{n\to\infty} N(n)/(\ln n)^{1/3} < (12/\pi^2)^{1/3}\,,$$

then $\lim_{n\to\infty} E(f(x) - \overline{f}_{N(n)}(x))^2 = 0$. The same is true if we put $\overline{c}^{N(n)} =$

$\widetilde{c}(n, N(n))$ and

$$\lim_{n\to\infty} N(n) = \infty\,, \qquad \lim_{n\to\infty} N(n)^2/n = 0\,.$$

Sufficient conditions for the point convergence of the Fourier series are described in [4], [5] and together with the conditions for the sequence $N(n)$ given above they are sufficient for the point convergence in the mean-square sense of the regression function estimator $\bar{f}_N$.

The theory presented above can be extended to the case of functions $f \in L^2(A, \mu)$ defined on subsets $A \subset \mathbb{R}^m, m > 1$, satisfying the conditions $0 < \mu(A) < \infty$, and inequality (2.4) is then also true for certain orthogonal systems of functions (for example, spherical harmonics), if n is large enough.

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