ON FOURIER COEFFICIENT ESTIMATORS CONSISTENT IN THE MEAN-SQUARE SENSE

Abstract. The properties of two recursive estimators of the Fourier coefficients of a regression function $f \in L^2[a,b]$ with respect to a complete orthonormal system of bounded functions $(e_k)$, $k = 1, 2, \ldots$, are considered in the case of the observation model $y_i = f(x_i) + \eta_i$, $i = 1, \ldots, n$, where $\eta_i$ are independent random variables with zero mean and finite variance, $x_i \in [a,b] \subset \mathbb{R}$, $i = 1, \ldots, n$, form a random sample from a distribution with density $\varrho = 1/(b-a)$ (uniform distribution) and are independent of the errors $\eta_i$, $i = 1, \ldots, n$. Unbiasedness and mean-square consistency of the examined estimators are proved and their mean-square errors are compared.

1. Introduction. Let $y_i$, $i = 1, \ldots, n$, be observations at points $x_i \in [a,b] \subset \mathbb{R}^1$, according to the model $y_i = f(x_i) + \eta_i$, where $f : [a,b] \rightarrow \mathbb{R}^1$ is an unknown square integrable function ($f \in L^2[a,b]$) and $\eta_i$, $i = 1, \ldots, n$, are independent identically distributed random variables with zero mean and finite variance $\sigma^2 > 0$. Let furthermore the points $x_i, i = 1, \ldots, n$, form a random sample from a distribution with density $\varrho = 1/(b-a)$ (uniform distribution), independent of the observation errors $\eta_i$, $i = 1, \ldots, n$.

We assume that the functions $(e_k)$, $k = 1, 2, \ldots$, constitute a complete orthonormal system in $L^2[a,b]$, and that they are bounded and normalized so that

$$\frac{1}{b-a} \int_a^b e_k^2(x) \, dx = 1, \quad k = 1, 2, \ldots$$

Then $f$ has the representation

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\[ f = \sum_{k=1}^{\infty} c_k e_k, \quad \text{where} \quad c_k = \frac{1}{b-a} \int_{a}^{b} f(x) e_k(x) \, dx, \quad k = 1, 2, \ldots \]

The first estimator of the Fourier coefficients we shall deal with is well-known and has a simple form

\[ \tilde{c}_k = \frac{1}{n} \sum_{i=1}^{n} y_i e_k(x_i), \quad k = 1, 2, \ldots, \]

so that we easily obtain the following formulae:

\[ E\tilde{c}_k = E_x E_y c_k = c_k, \]

\[ E(\tilde{c}_k - c_k)^2 = \frac{1}{n(b-a)} \int_{a}^{b} (f(x) e_k(x) - c_k)^2 \, dx + \frac{1}{n} \sigma_y^2. \]

The estimators \( \tilde{c}_k, k = 1, 2, \ldots, \) are thus unbiased and consistent in the mean-square sense. If we estimate the Fourier coefficients \( c_1, \ldots, c_N, \) the number \( N \) being fixed, we can write formula (1.1) in the vector form

\[ \tilde{c}(n, N) = \frac{1}{n} \sum_{i=1}^{n} y_i e^N(x_i), \]

where \( \tilde{c}(n, N) = (\tilde{c}_1, \ldots, \tilde{c}_N)^T, \) \( e^N(x) = (e_1(x), \ldots, e_N(x))^T, \) which can be rewritten in the recursive form

\[ \tilde{c}(n, N) = \frac{n-1}{n} \tilde{c}(n-1, N) + \frac{1}{n} y_ne^N(x_n), \quad \tilde{c}(0, N) = (0, \ldots, 0)^T. \]

In view of (1.2) we also have

\[ E\tilde{c}(n, N) = (c_1, \ldots, c_N)^T = c^N, \]

\[ E\|\tilde{c}(n, N) - c^N\|^2 = \frac{1}{n} \left( \frac{1}{b-a} \int_{a}^{b} f^2(x) \|e^N(x)\|^2 \, dx - \|c^N\|^2 \right) + \frac{1}{n} N \sigma_y^2. \]

The second estimator of the Fourier coefficients is constructed similarly to the estimators occurring in stochastic approximation methods [1], [2]; namely, it is defined by the recursive formula

\[ \hat{c}(n, N) = \hat{c}(n-1, N) + \frac{1}{n} \delta_n e^N(x_n), \]

where \( \delta_n = y_n - \langle \hat{c}(n-1, N), e^N(x_n) \rangle, \hat{c}(0, N) = (0, \ldots, 0)^T. \)
In the sequel we shall use the notation $\Delta_n = \hat{c}(n, N) - c^N$, $\Delta_0 = -c^N$.
By (1.4) we can write
$$\Delta_n = \hat{c}(n, N) - c^N = \hat{c}(n - 1, N) - c^N + \frac{1}{n} \langle f(x_n) + \eta_n - \langle \hat{c}(n - 1, N), e^N(x_n) \rangle \rangle e^N(x_n)$$
and, since $f(x) = \sum_{k=1}^{N} c_k e_k(x) + r_N(x)$, where $r_N = \sum_{k=N+1}^{\infty} c_k e_k$, we obtain
$$\Delta_n = \Delta_{n-1} - \frac{1}{n} \langle \Delta_{n-1}, e^N(x_n) \rangle e^N(x_n) + \frac{1}{n} \langle \eta_n + r_N(x_n) \rangle e^N(x_n).$$

2. Unbiasedness and mean-square consistency of the estimators. We have already remarked that the estimator $\hat{c}(n, N)$ is unbiased and consistent in the mean-square sense (see formulae (1.3)). Now we will prove the same for $\hat{c}(n, N)$. First we prove by induction that $E\Delta_n = 0$ for $n = 1, 2, \ldots$ By (1.5) for $n = 1$, we have
$$E\Delta_1 = E_x E_0 \Delta_1 = E_0 - E_x e^N(x_1)e^N(x_1)^T \Delta_0 + E_x r_N(x_1)e^N(x_1)$$
$$= \Delta_0 - I \Delta_0 = 0,$$ since $E_0 \eta_1 = 0$, $E_x e^N(x_1)e^N(x_1)^T = I$ and $E_x r_N(x_1)e^N(x_1) = 0$.
Assume now that $E\Delta_{n-1} = 0$. Then, by (1.5),
$$E\Delta_n = E\Delta_{n-1} - \frac{1}{n} E_x e^N(x_n)e^N(x_n)^T \Delta_{n-1},$$ since $E_0 \eta_1 = 0$ and $E_x r_N(x_n)e^N(x_n) = 0$. Since $\Delta_{n-1}$ does not depend on $x_n$ we finally obtain
$$E\Delta_n = E\Delta_{n-1} - \frac{1}{n} E_x e^N(x_n)e^N(x_n)^T E\Delta_{n-1} = \left(1 - \frac{1}{n}\right) E\Delta_{n-1} = 0.$$ The unbiasedness of $\hat{c}(n, N)$ is thus proved. To prove the mean-square consistency of this estimator we need the following two lemmas.

**Lemma 2.1.** The random variables $\Delta_n, n = 1, 2, \ldots$, satisfy the recursive inequality

$$E\|\Delta_n\|^2 \leq \left(1 - \frac{2}{n} + \frac{1}{n^2} N^2 M_N\right) E\|\Delta_{n-1}\|^2 + \frac{1}{n^2} \left(p_N M_N + N\sigma_y^2\right),$$

where $p_N = \sum_{k=N+1}^{\infty} c_k^2$, $M_N = \sup_{0 \leq x \leq b} \|e^N(x)\|^2$.

**Proof.** Taking into account (1.5) and remembering that $E\|\Delta_n\|^2$ can be computed here as $E_{x_1, \ldots, x_{n-1}, \eta_1, \ldots, \eta_{n-1}} E_{\eta_n} \|\Delta_n\|^2$, we can write
Furthermore, since $\Delta_{-1}$ does not depend on $x_n$ and $E\Delta_{-1} = 0$ we obtain

$$E\|\Delta_n\|^2 = E\left\| \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right) \Delta_{n-1} \right\|^2 + \frac{1}{n^2}E_x\|r_N(x_n)e^N(x_n)\|^2 + \frac{1}{n^2}\sigma_n^2 E_x\|e^N(x_n)\|^2.$$  

Furthermore, $E_x\|e^N(x_n)\|^2 = E_x\sum_{k=1}^N e_k^2(x_n) = N$, since $E_xe_k^2(x_n) = 1$ for $k = 1, 2, \ldots$, and finally,

$$E\|\Delta_n\|^2 = E\left\| \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right) \Delta_{n-1} \right\|^2 + \frac{1}{n^2}E_x\|r_N(x_n)e^N(x_n)\|^2 + \frac{1}{n^2}N\sigma_n^2.$$  

For the first term on the right hand side we obtain

$$E\left\| \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right) \Delta_{n-1} \right\|^2$$

$$= E\text{tr}\left[ \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right) \Delta_{n-1}\Delta_{n-1}^T \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right) \right]$$

$$= E\text{tr}\left[ \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right)^2 \Delta_{n-1}\Delta_{n-1}^T \right]$$

$$= \text{tr}\left[ E_x \left( I - \frac{1}{n}e^N(x_n)e^N(x_n)^T \right)^2 E\Delta_{n-1}\Delta_{n-1}^T \right]$$

$$= \text{tr}\left[ \left( I - \frac{2}{n}I + \frac{1}{n^2}E_xe^N(x_n)|e^N(x_n)|^2e^N(x_n)^T \right) E\Delta_{n-1}\Delta_{n-1}^T \right]$$

$$= \left( 1 - \frac{2}{n} \right) \text{tr} E\Delta_{n-1}\Delta_{n-1}^T$$

$$+ \frac{1}{n^2} \text{tr}[E_x|e^N(x_n)|^2e^N(x_n)e^N(x_n)^TE\Delta_{n-1}\Delta_{n-1}^T]$$

$$= \left( 1 - \frac{2}{n} \right) E\|\Delta_{n-1}\|^2 + \frac{1}{n^2} \text{tr}[E_x|e^N(x_n)|^2e^N(x_n)e^N(x_n)^TE\Delta_{n-1}\Delta_{n-1}^T].$$
Observe that
\[ |E_x| |e^N(x_n)||i^2e_i(x_n)e_j(x_n)| \]
\[ \leq \sup_{a \leq x \leq b} |e^N(x)||i^2E_xe_i(x_n)e_j(x_n)| \]
\[ \leq \sup_{a \leq x \leq b} |e^N(x)||i^2(E_xe_i^2(x_n))^{1/2}(E_xe_j^2(x_n))^{1/2} = M_n \]
for \( i, j = 1, \ldots, N \). On the other hand, for \( \Delta_{n-1} = (\Delta_{n-1,1}, \Delta_{n-1,2}, \ldots, \Delta_{n-1,N})^T \), we also have
\[ |E(\Delta_{n-1,1}, \Delta_{n-1,2})| \leq E|\Delta_{n-1}|^2 \text{ for } i, j = 1, \ldots, N. \]
These estimates yield
\[ E|\Delta_{n-1}|^2 \leq \left( 1 - \frac{2}{n} \right) E|\Delta_{n-1}|^2 + \frac{1}{n^2} N^2 M_n E|\Delta_{n-1}|^2 \]
\[ + \frac{1}{n^2} E_x r_N^2(x_n)||e^N(x_n)||^2 + \frac{1}{n^2} N^2 \eta^2, \]
and since
\[ E_x r_N^2(x_n)||e^N(x_n)||^2 \leq \sup_{a \leq x \leq b} ||e^N(x)||^2 E_x r_N^2(x_n) \]
\[ = M_N \sum_{k=N+1}^{\infty} c_k^2 = M_N p_N, \]
we finally obtain the estimate
\[ E|\Delta_n|^2 \leq \left( 1 - \frac{2}{n} + \frac{d}{n^2} \right) E|\Delta_{n-1}|^2 + \frac{1}{n^2} p_N M_N + \frac{1}{n^2} N^2 \eta^2. \]

**Lemma 2.2.** If nonnegative real numbers \( v_n, n = 0, 1, 2, \ldots \), satisfy the recursive inequality
\[ v_n \leq \left( 1 - \frac{2}{n} + \frac{d}{n^2} \right) v_{n-1} + \frac{b}{n^2}, \quad b > 0, \quad d > 1, \quad n = 1, 2, \ldots, \]
then
\[ v_n \leq \frac{d-1}{n^2} (v_0 + b + b \ln(n-1)) \exp(\pi^2(d-1)/6) + \frac{b}{n}, \quad n = 1, 2, \ldots. \]

**Proof.** From the assumptions it follows immediately that
\[ v_n \leq \left( 1 - \frac{2}{n} + \frac{d}{n^2} \right) \left( 1 - \frac{2}{n-1} + \frac{d}{(n-1)^2} \right) \ldots \left( 1 - \frac{2}{1} + \frac{d}{1^2} \right) v_0 \]
\[ + b \left( 1 - \frac{2}{n} + \frac{d}{n^2} \right) \left( 1 - \frac{2}{n-1} + \frac{d}{(n-1)^2} \right) \ldots \left( 1 - \frac{2}{2} + \frac{d}{2^2} \right) \frac{1}{1^2} \]
\[ + \ldots + b \left( 1 - \frac{2}{n} + \frac{d}{n^2} \right) \frac{1}{(n-1)^2} + b \frac{1}{n^2}. \]
Taking into account the identity
\[ 1 - \frac{2}{k} + \frac{d}{k^2} = \frac{k^2 - 2k + d}{k^2} = \frac{(k-1)^2 + d - 1}{k^2} \]
we obtain
\[
v_n \leq \frac{(n-1)^2 + d - 1}{n^2} \cdot \frac{(n-2)^2 + d - 1}{(n-1)^2} \cdots \frac{(1-1)^2 + d - 1}{1^2} v_0
\]
\[ + b \frac{(n-1)^2 + d - 1}{n^2} \cdot \frac{(n-2)^2 + d - 1}{(n-1)^2} \cdots \frac{(2-1)^2 + d - 1}{2^2} \cdot \frac{1}{1^2} \]
\[ + \ldots + b \frac{(n-1)^2 + d - 1}{n^2} \cdot \frac{1}{(n-1)^2} + b \frac{1}{n^2}, \]
or equivalently,
\[
v_n \leq \frac{1}{n^2} \left( 1 + \frac{d-1}{(n-1)^2} \right) \left( 1 + \frac{d-1}{(n-2)^2} \right) \cdots \left( 1 + \frac{d-1}{1^2} \right) (d-1)v_0
\]
\[ + \frac{b}{n^2} \left( 1 + \frac{d-1}{(n-1)^2} \right) \left( 1 + \frac{d-1}{(n-2)^2} \right) \cdots \left( 1 + \frac{d-1}{1^2} \right)
\]
\[ + \ldots + \frac{b}{n^2} \left( 1 + \frac{d-1}{(n-1)^2} \right) + b \frac{1}{n^2}. \]

Since \( \exp(x) > 1 + x \) for \( x > 0 \), we have
\[
v_n \leq \frac{1}{n^2} (d-1)v_0 \exp \left( (d-1) \sum_{k=1}^{n-1} \frac{1}{k^2} \right)
\]
\[ + \frac{1}{n^2} b \left[ \exp \left( (d-1) \sum_{k=1}^{n-1} \frac{1}{k^2} \right) + \ldots + \exp \left( (d-1) \frac{1}{(n-1)^2} \right) + 1 \right]. \]

Since \( \sum_{k=1}^{\infty} 1/k^2 \) is known to be equal to \( \pi^2/6 \), and clearly
\( \exp(x) \leq 1 + Mx \), \( M = \exp(\pi^2(d-1)/6) \), for \( x \in [0, \pi^2(d-1)/6] \),
we have
\[
v_n \leq \frac{1}{n^2} (d-1)v_0 M
\]
\[ + \frac{1}{n^2} b \left[ 1 + (d-1)M \sum_{k=1}^{n-1} \frac{1}{k^2} + 1 + (d-1)M \sum_{k=2}^{n-1} \frac{1}{k^2}
\]
\[ + \ldots + 1 + (d-1)M \frac{1}{(n-1)^2} + 1 \right]
\]
\[ \leq \frac{(d-1)M}{n^2} \left( v_0 + b \left[ \sum_{k=1}^{n-1} \frac{1}{k^2} + \sum_{k=2}^{n-1} \frac{1}{k^2} + \ldots + \frac{1}{(n-1)^2} \right] \right) + b \frac{1}{n}. \]
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Summing the terms in square brackets we get
\[ v_n \leq \frac{(d - 1)M}{n^2} \left( v_0 + b \frac{n - 1}{(n - 1)^2} + \frac{n - 2}{(n - 2)^2} + \ldots + \frac{1}{1^2} \right) + \frac{b}{n} \]
\[ = \frac{(d - 1)M}{n^2} \left( v_0 + b \sum_{k=1}^{n-1} \frac{1}{k} \right) + \frac{b}{n}. \]

Since \( \ln(1 + x) \geq \frac{x}{1 + x} \) for \( x > 0 \), putting \( x = \frac{1}{k} \) we obtain
\[ \ln \left( \frac{k + 1}{k} \right) \geq \frac{1}{k+1} \] for \( k = 1, 2, \ldots \),
and consequently
\[ \sum_{k=1}^{n-1} \frac{1}{k} \leq 1 + \sum_{k=1}^{n-2} \ln \left( \frac{k + 1}{k} \right) = 1 + \sum_{k=1}^{n-2} (\ln(k + 1) - \ln(k)) = 1 + \ln(n - 1), \]
which completes the proof. \( \blacksquare \)

Inequality (2.1) assures that the sequence \( v_n = E\|\Delta_n\|^2, n = 0, 1, 2, \ldots \), satisfies the assumptions of Lemma 2.2 (sup \( a \leq x \leq b \|e_N^N(x)\|^2 > 1 \) for \( N > 1 \) since \( E\|e_N^N(x)\|^2 = N \)) so that we have the estimate
\[ E\|\Delta_n\|^2 \leq \frac{1}{n^2} (N^2 M_N - 1) \exp(\pi^2 (N^2 M_N - 1)/6) \]
\[ \times \left[ E\|\Delta_0\|^2 + (p_N M_N + N \sigma_0^2)(1 + \ln(n - 1)) \right] \]
\[ + \frac{1}{n} (p_N M_N + N \sigma_0^2) \]
and putting \( C = \exp(-\pi^2/6) \) we can write
\[ (2.2) \quad E\|\Delta_n\|^2 \leq \frac{1}{n^2} C N^2 M_N \exp(\pi^2 N^2 M_N/6) \]
\[ \times \left[ \|e_N^N\|^2 + (p_N M_N + N \sigma_0^2)(1 + \ln n) \right] \]
\[ + \frac{1}{n} (p_N M_N + N \sigma_0^2) . \]

This implies that, for fixed \( N \), the estimator \( \hat{c}(n, N) \) is consistent in the mean-square sense.

Now we shall compare the mean-square errors of \( \hat{c}(n, N) \) and \( \hat{c}(n, N) \) in the case when \( f \in L^2(0, 2\pi) \). The system
\[ e_1(x) = 1, \quad e_{2m}(x) = \sqrt{2} \sin(mx), \]
\[ e_{2m+1}(x) = \sqrt{2} \cos(mx), \quad m = 1, 2, \ldots, \]
is a complete orthogonal system in \( L^2(0, 2\pi) \) and \( (2\pi)^{-1} \int_0^{2\pi} e_k^2(x) \, dx = 1, \)
\[ k = 1, 2, \ldots \] For this system we also have
\[ \|e^N(x)\|_2^2 = \sum_{k=1}^{2m+1} e_k^2(x) = 2m + 1 = N \quad \text{for } N = 2m + 1, \ m \geq 0 \]
so that the estimates for the mean-square errors considered (see (1.3) and (2.2)) take the form
\[
E\|\hat{c}(n, N) - c^N\|^2 = \frac{1}{n} N(p_N + \sigma^2_n) + \frac{1}{n} (N - 1)\|c^N\|^2,
\]
(2.3)
\[
E\|\tilde{c}(n, N) - c^N\|^2 \leq \frac{1}{n^2} C N^3 \exp(\pi^2 N^3/6)\|c^N\|^2 + N(p_N + \sigma^2_n)(1 + \ln n)
+ \frac{1}{n} N(p_N + \sigma^2_n),
\]
where \( N = 2m + 1, \ m > 0 \) and \( C = \exp(-\pi^2/6) \).

From (2.3) we see that for \( N > 1 \) and \( \|c^N\|^2 > 0 \) we have
(2.4)
\[ E\|\hat{c}(n, N) - c^N\|^2 \leq E\|\tilde{c}(n, N) - c^N\|^2 \]
for sufficiently large \( n \), so that \( \tilde{c}(n, N) \), although more complicated in form, has a smaller mean-square error for large values of \( n \) than \( \hat{c}(n, N) \).

3. Conclusions. We now assume that \( f \in L^2(0, 2\pi) \). Having determined the estimators \( \tau^N = (\tau_1, \ldots, \tau_N)^T \) of Fourier coefficients we can form an estimator of the regression function \( f \), called a projection type estimator [3]:
(3.1)
\[ \tilde{f}_N(x) = \sum_{k=1}^{N} \tau_k e_k(x) = \langle \tau^N, e^N(x) \rangle, \]
\[ N = 2m + 1, \ m > 0, \ e^N(x) = (1, \sqrt{2}\sin(x), \sqrt{2}\cos(x), \ldots, \sqrt{2}\sin(mx), \sqrt{2}\cos(mx))^T. \]

In case \( \tau^N = \hat{c}(n, N) \) this estimator is also a kernel type estimator [3], since then formula (3.1) takes the form
\[ \tilde{f}_N(x) = \frac{1}{n} \sum_{i=1}^{n} y_i \sum_{k=1}^{N} e_k(x_i)e_k(x). \]
For such an estimator the following formula for the integrated mean-square error is valid:
(3.2)
\[ E\frac{1}{2\pi} \int_0^{2\pi} (f(x) - \tilde{f}_N(x))^2 \, dx = E\|c^N - \tau^N\|^2 + \sum_{k=N+1}^{\infty} c_k^2 \]
\[ = E\|\tau^N - c^N\|^2 + p_N. \]
In view of the inequality
\[
\|c^N\|^2 = \sum_{k=1}^{N} c_k^2 \leq \sum_{k=1}^{\infty} c_k^2 = \frac{1}{2\pi} \|f\|^2
\]
and (2.3) we can obtain the following estimates for the mean-square errors:
\[
E\|\hat{c}(n, N) - c^N\|^2 \leq \frac{1}{n} N(p_N + \sigma_n^2) + \frac{1}{n} \frac{N}{2\pi} \|f\|^2, 
\]
(3.3)
\[
E\|\hat{c}(n, N) - c^N\|^2 
\leq \frac{1}{n^2} CN^3 \exp(\pi^2 N^3/6) \left[ \frac{1}{2\pi} \|f\|^2 + N(p_N + \sigma_n^2)(1 + \ln n) \right] 
+ \frac{1}{n} N(p_N + \sigma_n^2),
\]
where \(N = 2m + 1, m \geq 0\) and \(C = \exp(-\pi^2/6)\).

Formula (3.2) and the estimates in (3.3) imply that if we put \(N(n) = 2m(n) + 1, \pi^{N(n)} = \hat{c}(n, N(n))\) and if
\[
\lim_{n \to \infty} N(n) = \infty, \quad \limsup_{n \to \infty} N(n)/(\ln n)^{1/3} < (12/\pi^2)^{1/3},
\]
then \(\lim_{n \to \infty} E\|f - \tilde{f}_{N(n)}\|^2 = 0\). The same is true if we put \(\pi^{N(n)} = \hat{c}(n, N(n))\) with \(\lim_{n \to \infty} N(n) = \infty\) and \(\lim_{n \to \infty} N(n)/n = 0\).

In this way we have obtained sufficient conditions for convergence to zero of the integrated mean-square error of the estimator \(\tilde{f}_N\).

If the estimator \(\pi^N\) is unbiased then
\[
E(f(x) - \tilde{f}_N(x))^2 = E(c^N - \pi^N, c^N(x))^2 
+ 2r_N(x)E(c^N - \pi^N, c^N(x)) + Er_N^2(x) 
= E(c^N - \pi^N, c^N(x))^2 + r_N^2(x),
\]
where \(r_N = \sum_{k=N+1}^{\infty} c_k e_k\). From the Cauchy–Schwarz inequality it follows that
\[
E(f(x) - \tilde{f}_N(x))^2 \leq E\|\pi^N - c^N\|^2 E\|c^N(x)\|^2 + r_N^2(x)
\]
and since \(\|c^N(x)\|^2 = N\) for \(N = 2m + 1, m \geq 0\), we finally have
\[
E(f(x) - \tilde{f}_N(x))^2 \leq NE\|\pi^N - c^N\|^2 + r_N^2(x).
\]
(3.4)

If the Fourier series of \(f\) converges at a point \(x \in [0, 2\pi]\) to \(f(x)\) then, of course, \(\lim_{n \to \infty} r_{N(n)}(x) = 0\) if \(\lim_{n \to \infty} N(n) = \infty\). The estimates in (3.3) and (3.4) imply that if we put \(N(n) = 2m(n) + 1, \pi^{N(n)} = \hat{c}(n, N(n))\) and if
\[
\lim_{n \to \infty} N(n) = \infty, \quad \limsup_{n \to \infty} N(n)/(\ln n)^{1/3} < (12/\pi^2)^{1/3},
\]
then \(\lim_{n \to \infty} E(f(x) - \tilde{f}_{N(n)}(x))^2 = 0\). The same is true if we put \(\pi^{N(n)} = \hat{c}(n, N(n))\) and if
\( \tilde{c}(n, N(n)) \) and
\[
\lim_{n \to \infty} N(n) = \infty, \quad \lim_{n \to \infty} N(n)^2/n = 0.
\]

Sufficient conditions for the point convergence of the Fourier series are described in [4], [5] and together with the conditions for the sequence \( N(n) \) given above they are sufficient for the point convergence in the mean-square sense of the regression function estimator \( \tilde{f}_N \).

The theory presented above can be extended to the case of functions \( f \in L^2(A, \mu) \) defined on subsets \( A \subset \mathbb{R}^m, m > 1 \), satisfying the conditions \( 0 < \mu(A) < \infty \), and inequality (2.4) is then also true for certain orthogonal systems of functions (for example, spherical harmonics), if \( n \) is large enough.

References


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