APPLICATIONES MATHEMATICAE
22,2 (1994), pp. 181–192

L. S. ZAREMBA (Siedlce)

VISCOSITY SOLUTIONS OF THE ISAACS EQUATION ON AN ATTAINABLE SET

Abstract. We apply a modification of the viscosity solution concept introduced in [8] to the Isaacs equation defined on the set attainable from a given set of initial conditions. We extend the notion of a lower strategy introduced by us in [17] to a more general setting to prove that the lower and upper values of a differential game are subsolutions (resp. supersolutions) in our sense to the upper (resp. lower) Isaacs equation of the differential game. Our basic restriction is that the variable duration time of the game is bounded above by a certain number $T > 0$. In order to obtain our results, we prove the Bellman optimality principle of dynamic programming for differential games.

1. Introduction. This paper may be viewed as a continuation of [8] which concerned optimal control problems and the Bellman equation. Here we concentrate our attention on differential games and Isaacs' equations. By a two-person zero-sum differential game we understand a conflict situation involving two objects whose dynamics are governed by a system

(1) $\dot{x}(t) = f(t, x(t), u, v)\,, \quad x(t_0) = x_0\,,\ (t_0, x_0) \in \Omega_0 \subset \mathbb{R}^{n+1}\,,\ t \geq t_0\,,$

where piecewise continuous control functions $u(t)$, $v(t)$ are chosen by player I and player II, respectively from multivalued sets $U(t)$, $V(t)$. Player I strives to maximize his payoff functional

(2) $$P = P[t_0, x_0, u(\cdot), v(\cdot)] = g(\tau, x(\tau)) + \int_{t_0}^{\tau} h(t, x(t), u(t), v(t))\, dt\,,$$

1991 *Mathematics Subject Classification*: Primary 49L25, 49L20; Secondary 90D25.
Key words and phrases: viscosity solution, Isaacs equation, dynamic programming, differential game.
This work was done when the author was with the Department of Mathematics and Statistics, MUN, St. John's, NF, Canada.

through an appropriate choice of control parameters $u \in U(t)$, while player II tries to minimize his cost (2); here $\tau = \tau[t_0, x_0, x(\cdot)]$ is the first time $t \geq t_0$ for which $(t, x(t)) \in \Gamma$, where Γ is a terminal set. The role of an optimal cost function is replaced in this differential game setting by a value function $W(t, x)$, to be defined later. We study this value function on the set

$$(3) \qquad \Omega = \{(\overline{t}, \overline{x}) : \overline{x} = x(\overline{t}, t_0, x_0, u(\cdot), v(\cdot)),\ (t_0, x_0) \in \Omega_0,\ t_0 \leq \overline{t} \leq \tau\},$$

which is the set of points attainable from Ω_0. Let us note that Ω is invariant under the flow generated by equation (1), which means that each trajectory of (1) starting from Ω will stay in Ω until the terminal time τ, depending on the trajectory.

It has long been known that value functions, which may be defined differently according as one understands strategies, satisfy the Isaacs equation for a given differential game, provided they are differentiable [3, 5, 6, 9, 12]. Since usually this is not the case, the natural question arises how to overcome this difficulty.

An important step forward was made by introducing [2, 11] and developing [1, 4, 7, 13] the idea of a viscosity solution of a PDE; for more references on differential games (and control problems) in the context of viscosity solutions, the reader is referred to [8], from which we borrow the notion of a solution to the Isaacs equation, which is a modification of the viscosity solution.

Independently of those approaches, a different framework for solving the Isaacs equation was proposed by Subbotin [14] who proved later [16] that his approach is equivalent to the viscosity solution, when applied to fixed time duration problems, which the original definition of a viscosity solution was aimed at. Quite recently, in a quite general setting, he developed [15] his earlier joint ideas with Krasovskiĭ [10] to show that the notions of u-stable and v-stable functions introduced some 20 years ago by Krasovskiĭ (see [9] and the references therein) may be used to obtain nonlocal definitions of a viscosity subsolution and supersolution of the Isaacs equation for a suitable differential game.

In this paper, after extending the concept of a lower strategy introduced in [17], we study a general class of differential games which has not been treated so far in the context of viscosity solutions. We prove that the lower and upper values of a differential game are subsolutions (resp. supersolutions) in the sense of the definition of [8] to the corresponding upper (resp. lower) Isaacs equation, assuming the terminal time τ is bounded from above by a certain number $T > 0$. To obtain these results, we derive the Bellman optimality principle of dynamic programming for differential games, which has been proved here under weaker assumptions for more general classes of differential games than in [17]. If the Isaacs condition holds then the lower and upper values of the game are solutions of the Isaacs equation.

2. Assumptions and the value function. Throughout the paper we make assumptions similar to those in [8].

(4) The multivalued function $U : [0,T] \to \mathbb{R}^k$ is continuous (in the Hausdorff metric), all $U(t)$ are closed and lie in a fixed ball $B \subset \mathbb{R}^k$ and, for each $\overline{u} \in U(\overline{t})$, there is a selection $u(t)$ from $U(t)$ that is continuous at $\overline{t}$ and satisfies $u(\overline{t}) = \overline{u}$; a similar condition holds for $V : [0,T] \to \mathbb{R}^m$.

(5) The functions f, g and h are continuous.

(6) The function $f(t,x,u,v)$ is Lipschitz in x, i.e., for all $x \in \mathbb{R}^n$, $u \in U(t)$ and $v \in V(t)$, one has $\|f(t,x,u,v) - f(t,\overline{x},u,v)\| \le k(t)\|x - \overline{x}\|$, with $k(t)$ being integrable over $[0,T]$.

(7) The terminal set Γ is a closed subset of $\mathbb{R}^{n+1}$.

It is well known that under conditions (4)–(6) equation (1) admits a unique solution on $[t_0,T]$ for each initial point $(t_0,x_0) \in \Omega$ and any pair of controls $u(\cdot)$, $v(\cdot)$; let us denote such a trajectory by $x[\cdot,t_0,x_0,u(\cdot),v(\cdot)]$.

In addition to those rather standard assumptions, we impose the following condition.

(8) There exists a $T > 0$ such that, for each $x[\cdot,t_0,x_0,u(\cdot),v(\cdot)]$ with $(t_0,x_0) \in \Omega_0$, one has $\tau(x[\cdot,t_0,x_0,u(\cdot),v(\cdot)]) \le T$ and $\inf\{t : (t,x) \in \Omega_0\} = 0$.

In order to define the spaces of strategies for both players, denote by $X(t_0,x_0)$ the set of all trajectories (solutions of equation (1)) and by U (resp. V) the space of all piecewise continuous controls for player I (resp. player II). Let Π stand for the set of all finite partitions π of the interval $[t_0,T]$. For any $\alpha : X(t_0,x_0) \to (\Pi,U) \ni (\pi,u(\cdot))$ we shall often denote π by $\alpha_1[x(\cdot)]$ and $u(\cdot)$ by $\alpha_2[x(\cdot)]$ so that $\alpha[x(\cdot)] = (\alpha_1[x(\cdot)],\alpha_2[x(\cdot)])$. Extending the definition of a lower strategy introduced in [17] for systems of the form $\dot{x} = f_1(t,x(t),u(t))$, $\dot{y} = f_2(t,y(t),v(t))$, we propose here the following definition.

Definition 2.1. An operator $\alpha : X(t_0,x_0) \to (\Pi,U)$ is said to be a *strategy of player* I if whenever $x_1(t) = x_2(t)$, $t_0 \le t \le \overline{t}$, then (i) $t_1^1 = t_1^2,\ldots,t_{k+1}^1 = t_{k+1}^2$, where $t_1^1,t_2^1,\ldots,t_{k+1}^1$ are the first $k+1$ points of the partition $\pi_1 = \alpha_1[x_1(\cdot)]$, while $t_1^2,t_2^2,\ldots,t_{k+1}^2$ are the first $k+1$ points of the partition $\pi_2 = \alpha_1[x_2(\cdot)]$, with k being the index for which $t_k^1 \le \overline{t} < t_{k+1}^1$, and (ii) $\alpha_2[x_1(\cdot)](t) = \alpha_2[x_1(\cdot)](t)$, $t_0 \le t \le t_{k+1}$, where $t_{k+1} = t_{k+1}^1 = t_{k+1}^2$.

In an analogous fashion one introduces the concept of a strategy for player II. It is easy to see how both players proceed according to their strategies. Namely, at time t_0 player I (a similar observation applies to player II) chooses his first partition point t_1 (knowing x_0 only) and a control

$u(s)$, $t_0 \le s \le t_1$. At time t_1, knowing $x(s)$ for $t_0 \le s \le t_1$, player I selects t_2 and $u(s)$ for $s \in [t_1, t_2)$ and so on.

Denote by $A(t_0, x_0)$, $B(t_0, x_0)$ the spaces of strategies for player I and player II, respectively. Given a strategy $\alpha \in A(t_0, x_0)$ we say that a trajectory $x(\cdot) \in X(t_0, x_0)$ is an *outcome* of α ($x(\cdot) \in O[\alpha]$ for short) if there are controls $u(\cdot)$, $v(\cdot)$ such that $\dot{x}(t) = f(t, x(t), u(t), v(t))$, $x(t_0) = x_0$, and $\alpha[x(\cdot)](t) = u(t)$, $t \ge t_0$. In a similar fashion we define the notion of an outcome resulting from a strategy $\beta : X(t_0, x_0) \to V$ of player II. When $\alpha[x(\cdot)](t) = u(t)$ and $\beta[x(\cdot)](t) = v(t)$ with $\dot{x}(t) = f(t, x(t), u(t), v(t))$, $x(t_0) = x_0$, we say $x(\cdot)$ is an outcome of the pair (α, β) ($x(\cdot) \in O[\alpha, \beta]$ for short).

Clearly, when $\dot{x}(t) = f(t, x(t), u(t), v(t))$, $x(t_0) = x_0$, then $P[t_0, x_0, x(\cdot)]$ will mean $P[t_0, x_0, u(\cdot), v(\cdot)]$; analogously, since each pair of strategies $\alpha \in A(t_0, x_0)$, $\beta \in B(t_0, x_0)$ gives rise to exactly one outcome $x(\cdot)$, it is natural to understand by $P[t_0, x_0, \alpha, \beta]$ the amount $P[t_0, x_0, x(\cdot)]$. The notions of *lower* and *upper values* of the game are defined in a standard way:

$$\underline{W}(t_0, x_0) = \sup_{\alpha \in A(t_0, x_0)} \inf_{x(\cdot) \in O[\alpha]} P[t_0, x_0, x(\cdot)]\,, \tag{9}$$

$$\overline{W}(t_0, x_0) = \inf_{\beta \in B(t_0, x_0)} \sup_{x(\cdot) \in O[\beta]} P[t_0, x_0, x(\cdot)]\,. \tag{10}$$

These formulas may, clearly, be replaced by the following:

$$\underline{W}(t_0, x_0) = \sup_{\alpha \in A(t_0, x_0)} \inf_{\beta \in B(t_0, x_0)} P[t_0, x_0, \alpha, \beta]\,, \tag{11}$$

$$\overline{W}(t_0, x_0) = \inf_{\beta \in B(t_0, x_0)} \sup_{\alpha \in A(t_0, x_0)} P[t_0, x_0, \alpha, \beta]\,. \tag{12}$$

A game is said to *have a value* $W(t_0, x_0)$ if $W(t_0, x_0) = \underline{W}(t_0, x_0) = \overline{W}(t_0, x_0)$. It follows from our assumptions that all solutions of equation (1) starting from any bounded domain remain uniformly bounded and equicontinuous, which implies the following two properties:

(∗) $\underline{W}(t, x)$ and $\overline{W}(t, x)$ are locally bounded functions, and so they admit both usc and lsc envelopes.

(∗∗) For each $\delta > 0$ there is an $\varepsilon > 0$ such that if $(t, x) \in \Omega$, $\operatorname{dist}[(t, x), \Gamma] \ge \delta$ and $\|(t, x)\| \le 1/\delta$ then $\operatorname{dist}[(t', x[t', t, x, u(\cdot), v(\cdot)]), \Gamma] > \delta/2$, $t \le t' \le t + \varepsilon$, for any pair of control functions $u(\cdot)$, $v(\cdot)$.

3. Concept of a solution of the Isaacs equation and the Bellman optimality principle of dynamic programming. Having defined our differential game (1), (2), we are going to study the *lower Isaacs function*

$$H^-(t, x, p) = \sup_{u \in U(t)} \inf_{v \in V(t)} [f(t, x, u, v)p + h(t, x, u, v)] \tag{13}$$

and the *upper Isaacs function*

$$H^+(t,x,p) = \inf_{v\in V(t)} \sup_{u\in U(t)} [f(t,x,u,v)p + h(t,x,u,v)] \tag{14}$$

on $\overline{\Omega} \times \mathbb{R}^n$, where Ω, given by (3), is the set of points attainable from Ω_0. It follows from conditions (4), (5) that both the lower and upper Isaacs functions are continuous. If they are equal one says *the Isaacs condition holds* and the resulting function is called the *Isaacs function*. The lower Isaacs function leads naturally to the *lower Isaacs equation*

$$w_t(t,x) + \sup_{u\in U(t)} \inf_{v\in V(t)} [f(t,x,u,v)w_x(t,x) + h(t,x,u,v)] = 0 \tag{15}$$

defined on Ω with the boundary condition $\underline{W}(t,x) = g(t,x)$ on $\Gamma \subset \overline{\Omega}$. Similarly, one is led to the *upper Isaacs equation*

$$w_t(t,x) + \inf_{v\in V(t)} \sup_{u\in U(t)} [f(t,x,u,v)w_x(t,x) + h(t,x,u,v)] = 0 \tag{16}$$

with the same boundary condition. Clearly, when the Isaacs condition holds, the equations (15) and (16) coincide and the resulting equation is called the *Isaacs equation*.

All these three PDE are special cases of the equation $w_t(t,x) + H(t,x,w_x(t,x)) = 0$, where $H(t,x,p)$ is a continuous function. Let us now recall the concept of a solution to this PDE introduced in [8].

DEFINITION 3.1. Given a locally bounded function $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, a function $w : \Omega \to \mathbb{R}$ is said to be a *subsolution* of the equation

$$\begin{aligned} w_t(t,x) + H(t,x,w_x(t,x)) = 0\,, &\quad (t,x) \in \overline{\Omega} \setminus \Gamma\,, \\ w(t,x) = g(t,x) &\quad \text{on } \Gamma \subset \overline{\Omega} \end{aligned} \tag{17}$$

if $w(t,x) \le g(t,x)$ on Γ and, for each $C^1(\overline{\Omega}\setminus\Gamma)$ function ϕ, one has $\phi_t(\overline{t},\overline{x}) + H^*(\overline{t},\overline{x},\phi_x(\overline{t},\overline{x})) \ge 0$ at each point $(\overline{t},\overline{x}) \in \overline{\Omega} \setminus \Gamma$ which is a local maximum of the function $w^*(t,x) - \phi(t,x) : \overline{\Omega} \setminus \Gamma \to \mathbb{R}$, where w^* and H^* stand for the usc envelopes of w and H, respectively.

DEFINITION 3.2. A function $w : \Omega \to \mathbb{R}$ is said to be a *supersolution* of (17) if $w(t,x) \ge g(t,x)$ on Γ and, for each $C^1(\overline{\Omega} \setminus \Gamma)$ function ϕ, one has $\phi_t(\overline{t},\overline{x}) + H_*(\overline{t},\overline{x},\phi_x(\overline{t},\overline{x})) \le 0$ at each point $(\overline{t},\overline{x}) \in \overline{\Omega} \setminus \Gamma$ which is a local minimum of $w_*(t,x) - \phi(t,x) : \overline{\Omega} \setminus \Gamma \to \mathbb{R}$, where w_*, H_* stand for lsc envelopes of w and H, respectively.

DEFINITION 3.3. A function $w : \Omega \to \mathbb{R}$ is said to be a *solution* of (17) if it is both a subsolution and a supersolution of (17).

Clearly, this notion of a solution resembles that of the viscosity solution; some differences between those two concepts are pointed out in [8]. Since the only restriction imposed on Γ is condition (7), our framework is general enough to comprise all differential games that satisfy condition (8).

In the remainder of this section, we prove a principle, known as the Bellman optimality principle of dynamic programming, which extends and strengthens the one obtained in [17], where we treated the case of separated dynamics $\dot{x}(t) \in f_1(t, x, U(t), V(t))$, $\dot{y}(t) \in f_2(t, x, U(t), V(t))$ under the assumption that the sets $f_i(t, x, U(t), V(t))$, $i = 1, 2$, are convex. Let us start with the following lemma.

LEMMA 3.1. *Let conditions* (5), (6) *and* (8) *hold and* $(t, x) \in \Omega \setminus \Gamma$. *For each* $\delta > 0$ *with* $t + \delta \le T$, *each control* $u(\cdot) \in U$ *and any* $\varepsilon > 0$ *there exists a control function* $v_\varepsilon(\cdot) \in V$ *such that*, *for the trajectory* $x_\varepsilon(\cdot)$ *satisfying* $\dot{x}_\varepsilon(s) = f(s, x_\varepsilon(s), u(s), v_\varepsilon(s))$, $x_\varepsilon(t) = x$, *one has either* $(t + \overline{\delta}, x_\varepsilon(t + \overline{\delta})) \in \Gamma$ *for some* $0 \le \overline{\delta} \le \delta$, *or else* $\underline{W}(t + \delta, x_\varepsilon(t + \delta)) + r(t + \delta)$ (*the cost incurred on the interval* $[t, t + \delta)$) $\le \underline{W}(t, x) + \varepsilon$, *i.e.*,

$$\sup_{u(\cdot)\in U} \inf_{v(\cdot)\in V} \Big[\underline{W}(t + \delta, x[t + \delta, t, x, u(\cdot), v(\cdot)]) + \int_t^{t+\delta} h(s, x(s), u(s), v(s))\, ds\Big] \le \underline{W}(t, x)\,. \tag{18}$$

Proof. Reasoning by contradiction, we obtain a control $\overline{u}(\cdot) \in U$ and a $\delta > 0$ such that, for each $v(\cdot) \in V$ and for the $x(\cdot)$ satisfying $\dot{x}(s) = f(s, x(s), \overline{u}(s), v(s))$, $x(t) = x$, the points $(t + \overline{\delta}, x(t + \overline{\delta}))$ are in $\Omega \setminus \Gamma$ for all $0 \le \overline{\delta} \le \delta$ and, for $z = x(t + \delta)$, we have

$$\underline{W}(t + \delta, z) + \int_t^{t+\delta} h(s, x(s), \overline{u}(s), v(s))\, ds > \underline{W}(t, x) + \varepsilon\,.$$

By the definition of $\underline{W}(t + \delta, z)$ (cf. (9)), there is a strategy $\alpha^z \in A(t + \delta, z)$ with the property that

$$\inf\{P[t + \delta, x(t + \delta), x(\cdot)] + \varepsilon/2 : x(\cdot) \in O[\alpha^z]\} \ge \underline{W}(t + \delta, x(t + \delta))\,. \tag{19}$$

Using (19), we shall show the existence of a lower strategy $\alpha \in A(t, x)$ for which

$$\inf\{P[t, x, x(\cdot)] : x(\cdot) \in O[\alpha]\} \ge \underline{W}(t, x) + \varepsilon/2\,, \tag{20}$$

a contradiction to (9). We define $\alpha[x(\cdot)] = (\alpha_1[x(\cdot)], \alpha_2[x(\cdot)])$ as follows. Let the first partition point t_1 of $\pi = \alpha_1[x(\cdot)]$ be $t + \delta$; denoting by $x^\delta(\cdot)$ the portion of $x(\cdot)$ on $[t + \delta, T]$, we set

$$\alpha_2[x(\cdot)](s) = \begin{cases} \overline{u}(s) & \text{if } t \le s < t + \delta = t_1, \\ \alpha_2^z[x^\delta(\cdot)](s) & \text{if } x(t + \delta) = z \text{ and } t_1 \le s \le T. \end{cases}$$

Finally, let $\alpha_1[x(\cdot)] = \{t+\delta, t_1^z, \ldots, t_k^z\}$, where $\{t_1^z, \ldots, t_k^z\} = \alpha_1^z[x^\delta(\cdot)]$. With α defined as above, (19) implies (20), as required. ■

The following result may be proved analogously.

LEMMA 3.2. *Let conditions* (5), (6) *and* (8) *hold and* $(t,x) \in \Omega\setminus\Gamma$. *Then for each* $\delta > 0$ *with* $t+\delta < T$, *each control function* $v(\cdot) \in V$ *and* $\varepsilon > 0$ *there is a control function* $u_\varepsilon(\cdot) \in U$ *such that for the trajectory* $x_\varepsilon(\cdot)$ *satisfying* $\dot{x}_\varepsilon(t) = f(s, x_\varepsilon(s), u_\varepsilon(s), v(s))$, $x_\varepsilon(t) = x$, *one has either* $(t+\overline{\delta}, x_\varepsilon(t+\overline{\delta})) \in \Gamma$ *for some* $\overline{\delta}$, $0 \le \overline{\delta} \le \delta$, *or else* $\overline{W}(t+\delta, x_\varepsilon(t+\delta)) + r(t+\delta) \ge \overline{W}(t,x) - \varepsilon$, *i.e.*,

$$\inf_{v(\cdot)\in V} \sup_{u(\cdot)\in U} (\overline{W}(t+\delta, x[t+\delta, t, x, u(\cdot), v(\cdot)]) + r(t+\delta)) \ge \overline{W}(t,x)\,,$$

where

$$r(t+\delta) = \int_t^{t+\delta} h(s, x(s), u(s), v(s))\,ds\,.$$

LEMMA 3.3. *For any* $(t_0, x_0) \in \Omega$, *one has*

$$\underline{W}(t_0,x_0) \le \sup_{\alpha\in A(t_0,x_0)} \inf_{x(\cdot)\in O[\alpha]} \Big[\underline{W}(t_0+\varepsilon, x(t_0+\varepsilon)) + \int_{t_0}^{t_0+\varepsilon} h(s,x(s),u(s),v(s))\,ds\Big]\,. \tag{21}$$

Proof. Given $\varepsilon > 0$, denote by $X_\varepsilon(t_0,x_0)$ the set of all portions $x_\varepsilon(\cdot)$ of trajectories $x(\cdot) \in X(t_0,x_0)$ on $[t_0, t_0+\varepsilon)$ and by $x^\varepsilon(\cdot)$ the portion of $x(\cdot)$ on $[t_0+\varepsilon, T]$ so that $x(\cdot) = [x_\varepsilon(\cdot), x^\varepsilon(\cdot)]$. We may assume that for each lower strategy $\alpha = (\alpha_1, \alpha_2) \in A(t_0,x_0)$, the point $t_0+\varepsilon$ appears in the partition $\alpha_1[x(\cdot)]$ for each $x(\cdot) \in X(t_0,x_0)$; this can be readily accomplished without changing the values of $\alpha_2[x(\cdot)]$, $x(\cdot) \in X(t_0,x_0)$. Now we can "divide" any strategy $\alpha = (\alpha_1,\alpha_2)$ into two "pieces" $\alpha_\varepsilon = (\alpha_{\varepsilon 1}, \alpha_{\varepsilon 2})$ and $\alpha^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon)$, which are the restrictions of α to $[t_0, t_0+\varepsilon)$ (resp. $[t_0+\varepsilon, T]$). Strictly speaking, we have

$$\alpha_{\varepsilon 1}[x(\cdot)] = \alpha_1[x(\cdot)] \cap [t_0, t_0+\varepsilon)\,, \quad \alpha_{\varepsilon 2}[x(\cdot)](t) = \alpha_2[x(\cdot)](t)\,, \quad t_0 \le t < t_0+\varepsilon\,,$$

for all $x(\cdot) \in X(t_0,x_0)$; moreover, $\alpha_1^\varepsilon[x(\cdot)] = \alpha_1[x(\cdot)] \cap [t_0+\varepsilon, T]$, while α_2^ε represents a family of strategies $\alpha_{x(\cdot)}^\varepsilon \in A(t_0+\varepsilon, x(t_0+\varepsilon))$ uniquely determined via the formula

$$\alpha_2[x(\cdot)](t) = \begin{cases} \alpha_{\varepsilon 2}[x(\cdot)](t), & t_0 \le t < t_0+\varepsilon, \\ \alpha_{x(\cdot)}^\varepsilon[x(\cdot)](t), & t_0+\varepsilon \le t \le T, \end{cases} \tag{22}$$

for $x(\cdot)\in X(t_0,x_0)$. Observe that if $\overline{x}_\varepsilon(\cdot)$ is fixed and $x^\varepsilon(\cdot)$ varies over $X(t_0+\varepsilon,\overline{x}(t_0+\varepsilon))$, then (22) determines exactly one strategy $\alpha^\varepsilon_{\overline{x}(\cdot)}\in A(t_0+\varepsilon,\overline{x}(t_0+\varepsilon))$. Clearly, $\alpha^\varepsilon_{\overline{x}(\cdot)}$ means, in general, something different than $\alpha^\varepsilon_{x(\cdot)}$ even though the equality $\overline{x}(t_0+\varepsilon)=x(t_0+\varepsilon)$ may hold. These notational simplifications lead to the obvious identity

$$\underline{W}(t_0,x_0)=\sup_{\alpha_\varepsilon}\sup_{\alpha^\varepsilon}\inf_{x_\varepsilon(\cdot)}\inf_{x^\varepsilon(\cdot)}\{P[t_0,x_0,x(\cdot)]:x(\cdot)\in O[\alpha]\,,\ \alpha\in A(t_0,x_0)\}\,,$$

implying

$$\underline{W}(t_0,x_0)\le\sup_{\alpha_\varepsilon}\inf_{x_\varepsilon(\cdot)}\sup_{\alpha^\varepsilon}\inf_{x^\varepsilon(\cdot)}\{P[t_0,x_0,x(\cdot)]:x(\cdot)\in O[\alpha]\,,\ \alpha\in A(t_0,x_0)\}\,,$$

which is easy to prove and obvious in elementary game theory. Using the last inequality and the definition of $\underline{W}(t_0+\varepsilon,x(t_0+\varepsilon))$, one readily obtains (21). ■

Arguing similarly, one can also show (cf. (10))

$$\overline{W}(t_0,x_0)\ge\inf_{\beta\in B(t_0,x_0)}\sup_{x(\cdot)\in O[\beta]}\Big[\overline{W}(t_0+\varepsilon,x(t_0+\varepsilon))+\int_{t_0}^{t_0+\varepsilon}h(s,x(s),u(s),v(s))\,ds\Big]\,. \tag{23}$$

Theorem 3.1 (optimality principle of dynamic programming for differential games). *Let conditions* (5), (6), (8) *hold and* $(t,x)\in\Omega\setminus\Gamma$. *Then there is an* $\varepsilon>0$ *such that*, *for all* $0\le\delta\le\varepsilon$, *the identities below hold*:

$$\sup_{\alpha\in A(t,x)}\inf_{x(\cdot)\in O[\alpha]}\Big[\underline{W}(t+\delta,x(t+\delta))+\int_t^{t+\delta}h(s,x(s),u(s),v(s))\,ds\Big]=\underline{W}(t,x)\,, \tag{24}$$

$$\inf_{\beta\in B(t,x)}\sup_{x(\cdot)\in O[\beta]}\Big[\overline{W}(t+\delta,x(t+\delta))+\int_t^{t+\delta}h(s,x(s),u(s),v(s))\,ds\Big]=\overline{W}(t,x)\,, \tag{25}$$

where $\dot{x}(s)=f(s,x(s),u(s),v(s))$, $x(t)=x$.

Proof. It follows from property $(**)$ that there is an $\varepsilon=\varepsilon_{(t,x)}$ such that, for all $x(\cdot)\in O[\alpha]$, one has $(t+\delta,x(t+\delta))\in\Omega\setminus\Gamma$. We have already proved (21); the reverse inequality may be deduced from (18), by the definition of a lower strategy. The second identity is proved analogously, using inequality (23) and Lemma 3.2. ■

4. Existence of solutions to the lower and upper Isaacs equations

THEOREM 4.1. *If assumptions* (4)–(7) *hold then* $\underline{W}(t,x)$ *and* $\overline{W}(t,x)$ (*given by* (9) *and* (10), *resp.*) *are subsolutions of equation* (17) *in the sense of Definition* 3.1, *with* $H(t,x,p)$ *being the upper Isaacs function.*

Proof. Obviously, $\underline{W}(t,x)=\overline{W}(t,x)=g(t,x)$ for $(t,x)\in\Gamma$. To show $\underline{W}(t,x)$ is a subsolution of (16), assume $\underline{W}^*(t,x)-\phi(t,x):\overline{\Omega}\setminus\Gamma\to\mathbb{R}$, with $\phi\in C^1(\overline{\Omega}\setminus\Gamma)$, attains a finite local maximum at some point $(\overline{t},\overline{x})\in\overline{\Omega}\setminus\Gamma$. By the definition of $\underline{W}^*(t,x)$, we obtain a sequence $(t_k,x_k)\in\Omega\setminus\Gamma$ such that

$$(26)\qquad \underline{W}(t_k,x_k)-\phi(t_k,x_k)+\frac{1}{k}>\underline{W}(t_k+\varepsilon,x(t_k+\varepsilon))-\phi(t_k+\varepsilon,x(t_k+\varepsilon))$$

for each $x(\cdot)\in X(t_k,x_k)$, with $(t_k+\varepsilon,x(t_k+\varepsilon))$ belonging to $\Omega\setminus\Gamma$ for all ε smaller than some $\overline{\varepsilon}>0$ (by $(**)$), and $k=1,2,\ldots$ We infer from Lemma 3.3 that, for each positive integer k, one can find $\alpha_k\in A(t_k,x_k)$ for which

$$(27)\qquad \underline{W}(t_k,x_k)<\underline{W}(t_k+\varepsilon,x_k(t_k+\varepsilon))+\int_{t_k}^{t_k+\varepsilon}h(s,x_k(s),u(s),v(s))\,ds+\frac{1}{k}\,,$$

where $\dot{x}_k(s)=f(s,x_k(s),u(s),v(s))$, $x_k(t_k)=x_k$, $x_k(\cdot)\in O[\alpha_k]$. Fixing temporarily $\overline{v}\in V(\overline{t})$, let us choose a sequence $v_k\in V(t_k)$ converging to $\overline{v}$ and controls $v_k(\cdot)$ that are continuous at t_k and satisfy $v_k(t_k)=v_k$ (see assumption (4)). If we substitute $v_k(\cdot)$ for $v(\cdot)$ in (27), which is valid for any $v(\cdot)\in V$, then we get a control $u_k(\cdot)\in U$, which together with $x_k(\cdot)\in O[\alpha_k]$ and $v_k(\cdot)$ satisfies (27). Subtracting (26) from (27), and taking into account that $u(t_k)$ converges to some $\overline{u}\in U(\overline{t})$, we obtain

$$(28)\qquad -\frac{2}{k}\le\phi(t_k+\varepsilon,x_k(t_k+\varepsilon))-\phi(t_k,x_k)+\int_{t_k}^{t_k+\varepsilon}h(s,x_k(s),u_k(s),v_k(s))\,ds\,.$$

Dividing (28) by ε and passing with k to infinity, we arrive at $\phi_t(\overline{t},\overline{x})+H^+(\overline{t},\overline{x},\phi_x(\overline{t},\overline{x}))\ge 0$ because $\overline{v}\in V(\overline{t})$ was chosen arbitrarily and $\overline{u}$ was determined later; here H^+ is the upper Isaacs function given by (14). In this way we have shown that $\underline{W}(t,x)$ is a subsolution of (16) because H^+ is its own usc envelope.

To show $\overline{W}(t,x)$ is a subsolution of (16), we proceed as at the beginning of this proof (with $\underline{W}(\cdot)$ replaced by $\overline{W}(\cdot)$) to arrive at inequality (26). As previously, after fixing (temporarily) $\overline{v}$, we choose $v_k\in V(t_k)$ converging to $\overline{v}$ and controls $v_k(\cdot)$, continuous at t_k, with $v_k(t_k)=v_k$. This time we use Lemma 3.2 (instead of Lemma 3.3) to find controls $u_k(\cdot)$ such that with

$x_k(\cdot)$ satisfying $\dot{x}_k(t) = f(t, x_k(t), u_k(t), v_k(t))$, $x_k(t_k) = x_k$, we arrive at

$$(29)\quad \overline{W}(t_k, x_k) < \overline{W}(t_k+\varepsilon, x_k(t_k+\varepsilon)) + \int\limits_{t_k}^{t_k+\varepsilon} h(s, x_k(s), u_k(s), v_k(s))\, ds + \frac{1}{k},$$

playing the role of inequality (27), from which we subtract (26) (with $\underline{W}(\cdot)$ replaced by $\overline{W}(\cdot)$, of course), to obtain (28) and consequently $\phi_t(\overline{t}, \overline{x}) + H^+(\overline{t}, \overline{x}, \phi_x(\overline{t}, \overline{x})) \ge 0$, as required. ∎

THEOREM 4.2. *If assumptions* (4)–(7) *hold then* $\underline{W}(t, x)$ *and* $\overline{W}(t, x)$ *are supersolutions of equation* (17), *with* $H(t, x, p)$ *being the lower Isaacs function.*

Proof. We start to argue as at the beginning of the previous proof to arrive at

$$(30)\quad \underline{W}(t_k, x_k) - \phi(t_k, x_k) - \frac{1}{k} < \underline{W}(t_k+\varepsilon, x(t_k+\varepsilon)) - \phi(t_k+\varepsilon, x(t_k+\varepsilon))$$

for each $x(\cdot) \in X(t_k, x_k)$ with $(t_k + \varepsilon, x(t_k + \varepsilon)) \in \Omega \setminus \Gamma$ for all $\varepsilon \le \overline{\varepsilon}$ and $k = 1, 2, \ldots$ Now we change our reasoning a little by fixing $\overline{u} \in U(\overline{t})$, choosing a sequence $u_k \in U(t_k)$ converging to $\overline{u}$ and controls $u_k(\cdot)$, continuous at t_k, for which $u_k(t_k) = u_k$. Next we use Lemma 3.1 to find controls $v_k(\cdot)$ for which, with $x_k(\cdot)$ satisfying $\dot{x}_k(t) = f(t, x_k(t), u_k(t), v_k(t))$, $x_k(t_k) = x_k$, the inequality

$$(31)\quad \underline{W}(t_k, x_k) > \underline{W}(t_k+\varepsilon, x(t_k+\varepsilon)) + \int\limits_{t_k}^{t_k+\varepsilon} h(s, x_k(s), u_k(s), v_k(s))\, ds$$

holds. Based on (4), we may assume $v_k(t_k)$ converges to some $\overline{v} \in V(\overline{t})$; subtracting (30) from (31), we obtain

$$(32)\quad \frac{2}{k} > \phi(t_k+\varepsilon, x_k(t_k+\varepsilon)) - \phi(t_k, x_k) + \int\limits_{t_k}^{t_k+\varepsilon} h(s, x_k(s), u_k(s), v_k(s))\, ds$$

and consequently, after dividing (32) by ε, and passing with k to infinity, we arrive at $\phi_t(\overline{t}, \overline{x}) + H^-(\overline{t}, \overline{x}, \phi_x(\overline{t}, \overline{x})) \le 0$ because $\overline{u}$ was chosen arbitrarily and $\overline{v}$ was determined later; here H^- is the lower Isaacs function, which is equal to its lsc envelope since it is a continuous function.

To show $\overline{W}(t, x)$ is a supersolution of (15), we assume $\overline{W}_*(t, x) - \phi(t, x)$ attains a local minimum at $(\overline{t}, \overline{x}) \in \overline{\Omega} \setminus \Gamma$, to arrive at

$$(33)\quad \overline{W}(t_k, x_k) - \phi(t_k, x_k) - \frac{1}{k} < \overline{W}(t_k+\varepsilon, x(t_k+\varepsilon)) - \phi(t_k+\varepsilon, x(t_k+\varepsilon))$$

for each $x(\cdot) \in X(t_k, x_k)$, $k = 1, 2, \ldots$ By inequality (23), there are strategies

$\beta_k \in B(t_k, x_k)$ such that, for all $x(\cdot) \in O[\beta_k]$, we have

$$(34) \qquad \overline{W}(t_k, x_k) \geq \overline{W}(t_k + \varepsilon, x(t_k + \varepsilon)) + \int\limits_{t_k}^{t_k+\varepsilon} h(s, x(s), u(s), v(s))\, ds - \frac{1}{k}\,,$$

where $\dot{x}(s) = f(s, x(s), u(s), v(s))$, $x(t_k) = x_k$. Now we fix $\overline{u} \in U(\overline{t})$, select points u_k and controls $u_k(\cdot)$ with appropriate properties, and next determine $v_k(\cdot)$ so that (34) holds (with $x(\cdot)$, $u(\cdot)$, $v(\cdot)$ replaced by $x_k(\cdot)$, $u_k(\cdot)$, $v_k(\cdot)$). If we subtract (33) from (34), then we obtain (32) and $\phi_t(\overline{t}, \overline{x}) + H^-(\overline{t}, \overline{x}, \phi_x(\overline{t}, \overline{x})) \leq 0$, as required. ■

COROLLARY 4.1. *If*, *in addition to assumptions* (4)–(7), *the Isaacs condition holds*, *then* $\underline{W}(t, x)$ *and* $\overline{W}(t, x)$ *are solutions of the Isaacs equation.*

Based on the uniqueness result from [8], obtained under additional assumptions, one can deduce the existence of a value to our differential game; a suitable formulation is left to the reader.

References

[1] E. Barron, L. Evans and R. Jensen, *Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls*, J. Differential Equations 53 (1984), 213–233.

[2] M. Crandall and P. Lions, *Viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc. 277 (1983), 1–42.

[3] R. Elliott and N. Kalton, *Cauchy problems for certain Isaacs–Bellman equations and games of survival*, ibid. 198 (1974), 45–72.

[4] L. Evans and H. Ishii, *Differential games and nonlinear first order PDE on bounded domains*, Manuscripta Math. 49 (1984), 109–139.

[5] A. Friedman, *Differential Games*, Wiley, New York, 1971.

[6] W. Fleming, *The Cauchy problem for degenerate parabolic equations*, J. Math. Mech. 13 (1964), 987–1008.

[7] H. Ishii, *Remarks on existence of viscosity solutions of Hamilton–Jacobi equations*, Bull. Fac. Sci. Engrg. Chuo Univ. 26 (1983), 5–24.

[8] H. Ishii, J.-L. Menaldi and L. Zaremba, *Viscosity solutions of the Bellman equation on an attainable set*, Problems Control Inform. Theory 20 (1991), 317–328.

[9] N. Krasovskiĭ and A. Subbotin, *Positional Differential Games*, Nauka, Moscow, 1974 (in Russian).

[10] —, —, *An alternative for the game problem of convergence*, J. Appl. Math. Mech. 34 (1971), 948–965.

[11] P. Lions, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, Boston, 1982.

[12] O. Oleĭnik and S. Kruzhkov, *Quasi-linear second order parabolic equations with several independent variables*, Uspekhi Mat. Nauk 16 (5) (1961), 115–155 (in Russian).

[13] P. Souganidis, *Existence of viscosity solutions of Hamilton–Jacobi equations*, J. Differential Equations 56 (1985), 345–390.

[14] A. Subbotin, *A generalization of the fundamental equation of the theory of differential games*, Dokl. Akad. Nauk SSSR 254 (1980), 293–297 (in Russian).
[15] —, *Existence and uniqueness results for Hamilton–Jacobi equation*, Nonlinear Anal., to appear.
[16] A. Subbotin and A. Taras'ev, *Stability properties of the value function of a differential game and viscosity solutions of Hamilton–Jacobi equations*, Problems Control Inform. Theory 15 (1986), 451–463.
[17] L. S. Zaremba, *Optimality principles of dynamic programming in differential games*, J. Math. Anal. Appl. 138 (1989), 43–51.

LESZEK S. ZAREMBA
INSTITUTE OF MATHEMATICS AND PHYSICS
AGRICULTURAL AND PEDAGOGICAL UNIVERSITY
08-110 SIEDLCE, POLAND
E-mail: LZAREMBA@PLEARN.BITNET

Received on 4.11.1992