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PERTURBATION OF THE SPECTRUM OF AN ESSENTIALLY SELFADJOINT OPERATOR

Abstract. The aim of this paper is to find estimates of the Hausdorff distance between the spectra of two nonselfadjoint operators. The operators considered are assumed to have their imaginary parts in some normed ideal of compact operators. In the case of the classical Schatten ideals the estimates are given explicitly.

It is well known that $\varrho(\sigma(A),\sigma(B))$, the Hausdorff distance between the spectra of two selfadjoint operators A, B, is bounded by $\|A-B\|$. The same estimate also holds for normal operators. This is related to the fact that the norm of the resolvent of such an operator is equal to $1/d(\lambda,\sigma(A))$.

For nonselfadjoint operators A, B acting in an n-dimensional space an estimate for the distance between their spectra has been obtained by R. Bhatia and K. K. Mukherjea in [1], and L. Elsner [2] improved it to

$$\varrho(\sigma(A),\sigma(B)) \le (\|A\|+\|B\|)^{1-1/n}\|A-B\|^{1/n}\,.$$

The dependence of the bound on the nth root of $\|A-B\|$ is related to the fact that the resolvent norm of an operator can behave in the neighborhood of an eigenvalue as $(1/d(\lambda,\sigma(A)))^n$.

Looking at the form of this estimate one sees that it cannot be generalized to compact operators acting in infinite-dimensional Hilbert space. The class of compact operators is too large. The situation changes if we restrict ourselves to normed ideals of compact operators. In [7, Theorem 3] it was shown that for each normed ideal (with norm not equivalent to the operator norm) there exists an estimate of this kind. In the case of the Schatten ideal $\mathfrak{S}_p$, $1\le p<\infty$, there exists a constant C_p such that

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$$\varrho(\sigma(A),\sigma(B)) \le mC_p\left(\log\left(\frac{3m}{\|A-B\|}\right)\right)^{-1/p},$$

where $m=\max(|A|_p,|B|_p)$.

In this paper we try to find such an estimate for operators which are sums of a selfadjoint operator and a compact one. A result in this direction has been obtained in [6], where an estimate of the distance between the spectra of a selfadjoint operator and its antihermitian perturbation has been found. Moreover, it has been shown that such an estimate is only possible in the case when the antihermitian perturbation is an operator from the Matsaev ideal of compact operators.

In the course of our study we will see how much sensitive to perturbation is the spectrum of an operator from the specified class.

NOTATIONS. Let $\mathfrak{H}$ denote a complex separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $L(\mathfrak{H})$ the algebra of all bounded linear operators acting in $\mathfrak{H}$. $\mathfrak{A}$ stands for the set of all selfadjoint operators acting in $\mathfrak{H}$, not necessarily bounded. We say that an ideal $\mathfrak{S}$ of $L(\mathfrak{H})$ is a (*symmetrically*) *normed ideal* if there exists a norm $|\cdot|$ on $\mathfrak{S}$ satisfying the following conditions:

(i) $\mathfrak{S}$ with norm $|\cdot|$ is a Banach space,

(ii) $|ABC|\le |A|\|B\||C|$, for any $A,C\in L(\mathfrak{H})$, $B\in\mathfrak{S}$,

(iii) $|F|=\|F\|$ for any operator F of rank one,

(iv) $\lim_{n\to\infty}|P_nAP_n|=|A|$, for any $A\in\mathfrak{S}$ and every sequence $\{P_n\}$ of orthogonal projections converging strongly to the identity operator.

The norm $|\cdot|$ is called a *crossnorm*. The crossnorm of an operator A only depends on its singular values $s_j(A)$ $(j=1,2,\ldots)$. The sequence $\{s_j(A)\}$ is the sequence of eigenvalues of the operator $\sqrt{AA^*}$ counted according to multiplicity and arranged in decreasing order of magnitude.

An important role in further considerations is played by the Matsaev ideal $\mathfrak{S}_\omega$ with crossnorm defined by

$$|A|_\omega=\sum_{j=1}^{\infty}\frac{s_j(A)}{2j-1}\,.$$

This ideal contains the classical Schatten ideals $\mathfrak{S}_p$ $(1\le p<\infty)$, where

$$|A|_p=\Big(\sum s_j^p(A)\Big)^{1/p}.$$

The ideal $(\mathfrak{S}_\Omega,|\cdot|_\Omega)$ with crossnorm

$$|A|_\Omega=\sup_n\left(\frac{\sum_{j=1}^n s_j(A)}{\sum_{j=1}^n(2j-1)^{-1}}\right)$$

is adjoint to the Matsaev ideal and $\mathfrak{S}_1 \subset \mathfrak{S}_\Omega \subset \mathfrak{S}_p$ $(1 < p < \infty)$ (cf. [3, Chapter III]).

Resolvent norm estimates. It is well known that the norm of the resolvent of a selfadjoint operator A at a point λ only depends on the distance of λ to the spectrum of A. We consider a class of operators which differ from selfadjoint operators by an operator belonging to some normed ideal and ask for which ideal norms there exists an estimate of the resolvent $(A-\lambda)^{-1}$ of operators A from this class which only depends on the distance of λ to the spectrum of A and on the crossnorm of the imaginary part of A.

It was shown in [7] that there exists a function α such that

$$\|(A-\lambda)^{-1}\| \le \alpha(|\!|\!| A |\!|\!|, d(\lambda, \sigma(A)))$$

if and only if the crossnorm $|\!|\!| \cdot |\!|\!|$ is not equivalent to the operator norm $\|\cdot\|$. In this paper a similar problem is studied for the class $\mathfrak{A} + \mathfrak{S}$. We say that $\mathfrak{S}$ has the *resolvent estimate property* if for any positive numbers m, d,

$$f(m,d) = \sup\{\|(A-\lambda)^{-1}\| : A \in \mathfrak{A} + \mathfrak{S}, \quad |\!|\!| \operatorname{Im} A |\!|\!| = m,\ d(\lambda, \sigma(A)) = d\} < \infty\,. \tag{1}$$

If $\mathfrak{S}$ has this property then a similar function for quasinilpotent operators

$$f_\nu(m,d) = \sup\{\|(A-\lambda)^{-1}\| : A \in \mathfrak{A} + \mathfrak{S}, \quad |\!|\!| \operatorname{Im} A |\!|\!| = m,\ \sigma(A) = \{0\},\ d = |\lambda|\} \tag{2}$$

is also well defined.

LEMMA 1. *If a normed ideal $\mathfrak{S}$ has the resolvent estimate property then*:

(i) *there exist functions $\beta, \beta_\nu : [0,\infty) \to [1,\infty)$ such that*

$$f(m,d) = \frac{1}{d}\beta\left(\frac{m}{d}\right), \qquad f_\nu(m,d) = \frac{1}{d}\beta_\nu\left(\frac{m}{d}\right), \tag{3}$$

where f, f_ν are defined by (1) *and* (2),

(ii) *the functions β, β_ν are nonincreasing*,

(iii) *the function $\ln(\beta_\nu(e^t))$ is convex.*

Proof. (i) Suppose that $A \in \mathfrak{A} + \mathfrak{S}$, $d = d(\lambda, \sigma(A)) > 0$ and $m = |\!|\!| \operatorname{Im} A |\!|\!| > 0$. Then for any $t > 0$ we have $d(t\lambda, \sigma(tA)) = dt$, $|\!|\!| \operatorname{Im}(tA) |\!|\!| = tm$ and $\|(A-\lambda)^{-1}\| = t\|(tA - t\lambda)^{-1}\| \le tf(tm, td)$. Let $t = x/d$. Then

$$\|(A-\lambda)^{-1}\| \le \frac{1}{d} x f\left(\frac{m}{d}x, x\right).$$

Defining $\beta(s) = \inf_{x>0} x f(sx, x)$ we have

$$\|(A-\lambda)^{-1}\| \le \frac{1}{d}\beta\left(\frac{m}{d}\right) \le f(m,d)\,.$$

This norm estimate holds for every $A \in \mathfrak{A} + \mathfrak{S}$ with $|\operatorname{Im} A| = m$ and any λ such that $d = d(\lambda, \sigma(A))$, therefore (1) implies that $(1/d)\beta(m/d) = f(m, d)$.

(ii) Let again $A \in \mathfrak{A} + \mathfrak{S}$, and $V \neq 0$ be any finite rank nilpotent operator. Then $V \in \mathfrak{S}$ and for any $m' > m = |\operatorname{Im} A|$ there exists $t > 0$ such that $|\operatorname{Im} A \oplus \operatorname{Im}(tV)| = m'$. Since $\operatorname{Im} A$ is a compact operator there exists a real number $\mu \in \sigma(A)$. Let $B = A \oplus (tV + \mu)$. Note that $\sigma(B) = \sigma(A) \cup \sigma(tV + \mu) = \sigma(A)$, $|\operatorname{Im} B| = |\operatorname{Im} A \oplus \operatorname{Im}(Vt)| = m'$ and for any $\lambda \in \sigma(A)$, we have $\|(B-\lambda)^{-1}\| = \max\{\|(A-\lambda)^{-1}\|, \|(tV-\lambda)^{-1}\|\}$; therefore

$$\|(A-\lambda)^{-1}\| \le \|(B-\lambda)^{-1}\| \le \frac{1}{d(\lambda, \sigma(A))}\beta\left(\frac{m'}{d(\lambda, \sigma(A))}\right).$$

Since this holds for any $A \in \mathfrak{A} + \mathfrak{S}$, we see from the definition of f and β that β is a nonincreasing function. In the same way we may define β_ν and show that it satisfies (3) and is also nonincreasing.

(iii) Let V be a quasinilpotent operator with $\operatorname{Im} A \in \mathfrak{S}$, $|\operatorname{Im} A| = m$. For any $k > 0$ the operator-valued function $(V-\lambda)^{-1}(V^* - k^2/\lambda)^{-1}$ is holomorphic in $\mathbb{C}\backslash\{0\}$. Let $d_1^2 \le k^2 = d_1 d_2 \le d_2^2$, $d_1, d_2 > 0$. It follows from the maximum principle and the definition of f_ν that for any λ with $|\lambda| = k$,

$$\begin{aligned}&\left\|(V-\lambda)^{-1}\left(V^* - \frac{k^2}{\lambda}\right)^{-1}\right\| \\ &\qquad \le \max\left\{\left\|(V-\mu)^{-1}\left(V^* - \frac{k^2}{\mu}\right)^{-1}\right\| : |\mu| = d_1 \text{ or } |\mu| = d_2\right\}.\end{aligned}$$

For any μ with $|\mu| = d_i$ we have

$$\begin{aligned}\left\|(V-\mu)^{-1}\left(V^* - \frac{k^2}{\mu}\right)^{-1}\right\| &\le \|(V-\mu)^{-1}\|\left\|\left(V^* - \frac{k^2}{\mu}\right)^{-1}\right\| \\ &\le \frac{1}{d_i}\beta_\nu\left(\frac{m}{d_i}\right)\frac{d_i}{k^2}\beta_\nu\left(\frac{d_i m}{k^2}\right).\end{aligned}$$

If $|\lambda| = k$ then the above inequality implies that

$$\|(V-\lambda)^{-1}\|^2 = \|(V-\lambda)^{-1}(V^* - \bar{\lambda})^{-1}\| \le k^{-2}\beta_\nu\left(\frac{m}{d_1}\right)\beta_\nu\left(\frac{m}{d_2}\right).$$

Since this estimate holds for any quasinilpotent V it follows from the definition of f_ν and (i) that $(\beta_\nu(m/k))^2 \le \beta_\nu(m/d_1)\beta_\nu(m/d_2)$, and this is equivalent to (iii). ■

Lemma 2. *Suppose that the ideal $\mathfrak{S}$ has the resolvent estimate property. Then*:

(i) *$\mathfrak{S} \subset \mathfrak{S}_\omega$ and there exists a positive number c such that $|A|_\omega \le c|A|$ for any $A \in \mathfrak{S}$*,

(ii) $|P_n|_\omega/|P_n| \to 0$ *as* $n \to \infty$, *where* P_n *denotes an* n*-dimensional orthogonal projection.*

Proof. Suppose (i) is false. Then there exist a sequence $\{A_n\}$ of finite rank selfadjoint operators such that $\pi/2 = |A_n|_\omega > n|A_n|$. It follows from the Matsaev theorem [4, Theorem III.4.2] that there exist compact selfadjoint operators B_n such that $\|B_n\| = 1$ and the operators $V_n = B_n + iA_n$ are quasinilpotent. This implies that there exist unit vectors e_n such that for $\lambda_n = 1$ or $\lambda_n = -1$, $(B_n - \lambda_n)e_n = 0$. Then $(V_n - \lambda_n)e_n = iA_n e_n$ and

$$\begin{aligned}\|(V_n - \lambda_n)e_n\| &\le \|A_n\|\|e_n\| \le |A_n|\|(V_n - \lambda_n)^{-1}(V_n - \lambda_n)e_n\| \\ &\le \frac{\pi}{2n}\|(V_n - \lambda_n)^{-1}\|\|(V_n - \lambda_n)e_n\|.\end{aligned}$$

Thus $\|(V_n - \lambda_n)^{-1}\| \le 2n/\pi$. Since $|\operatorname{Im} V_n| = |A_n|$ we obtain $2n/\pi \le f_\nu(|A_n|) \le \beta_\nu(|A_n|) \le \beta_\nu(\pi)$ with β_ν defined in Lemma 1. This contradicts the fact that β_ν is well defined.

(ii) It follows from the Matsaev theorem that there exists a compact quasinilpotent operator V_n such that $\operatorname{Im} V_n = P_n$, $\|\operatorname{Re} V_n\| = (2/\pi)\|P_n\|_\omega$ and $\lambda_n = \|\operatorname{Re} V_n\|$ is an eigenvalue of $\operatorname{Re} V_n$. Thus for some nonzero vector e_n, $(\operatorname{Re} V_n - \lambda_n)e_n = 0$. This implies further that $(V_n - \lambda_n)e_n = iP_n e_n$ and

$$\begin{aligned}\|e_n\| &= \|(V_n - \lambda_n)^{-1}(V_n - \lambda_n)e_n\| \le \|(V_n - \lambda_n)^{-1}\|\|(V_n - \lambda_n)e_n\| \\ &= \|(V_n - \lambda_n)^{-1}\|\|P_n e_n\| \le \|(V_n - \lambda_n)^{-1}\|\|e_n\|.\end{aligned}$$

Thus $1 \le \|(V_n - \lambda_n)^{-1}\| \le \beta_\nu(|\operatorname{Im} V_n|\lambda_n)/\lambda_n$, or equivalently

$$\frac{2}{\pi}|P_n|_\omega \le \beta_\nu\left(\frac{\pi|P_n|}{2|P_n|_\omega}\right).$$

If (ii) were false then for some $c > 0$ we would have $|P_n|/|P_n|_\omega < c$ for infinitely many n. Since, by Lemma 1, β_ν is an increasing function the previous inequality implies that $(2/\pi)|P_n|_\omega \le \beta_\nu(2c/\pi)$ for infinitely many n; and that is impossible since $|P_n|_\omega \to \infty$ as $n \to \infty$. ∎

LEMMA 3. *Suppose that there exists a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *such that for any operator* A *acting in a finite-dimensional Hilbert space* $\mathfrak{H}$,

$$\|(A - \lambda)^{-1}\| \le \frac{1}{d} f\left(\frac{|\operatorname{Im} A|}{d}\right),$$

where $d = d(\lambda, \sigma(A))$. *Then* $\mathfrak{S}$ *has the resolvent estimate property.*

Proof. Let $A \in \mathfrak{A} + \mathfrak{S}$. It follows from the Weyl–von Neumann theorem [5] that there exists a compact selfadjoint operator K such that $\operatorname{Re} A - K$ is diagonalizable, i.e. there exists an orthonormal basis $\{e_j\}$ of $\mathfrak{H}$ such that each e_j is an eigenvector of $\operatorname{Re} A - K$. Now we define the orthogonal projections $P_n = \sum_{j \le n} (\cdot, e_j)e_j$ and the operators $C_n = P_n A P_n$, $R_n =$

$(I-P_n)A(I-P_n)$, $B=K+i\operatorname{Im}A$. We show that

$$\|A-(C_n+R_n)\|\to 0\,. \tag{4}$$

Indeed, the identity $A-C_n-R_n=(AP_n-P_nA)P_n-P_n(AP_n-P_nA)$ implies that

$$\|A-C_n-R_n\|\le 2\|AP_n-P_nA\|\,. \tag{5}$$

Since $A=(\operatorname{Re}A-K)+B$ and P_n commutes with $\operatorname{Re}A-K$ we see that

$$AP_n-P_nA=BP_n-P_nB=(B-P_nBP_n)P_n-P_n(B-P_nBP_n)\,.$$

This identity and (5) imply that $\|A-(C_n+R_n)\|\le 4\|B-P_nBP_n\|$. Now (4) follows from the compactness of B and the fact that $\{P_n\}$ is a sequence of orthogonal projections converging strongly to the identity operator.

Suppose now that $\lambda\notin\sigma(A)$ and $0<d'<d(\lambda,\sigma(A))$. The disc $K(\lambda,d')$ is contained in the resolvent set of A. Let $m=\max_{\mu\in K(\lambda,d')}\|(A-\mu)^{-1}\|$. It follows from perturbation theory that for sufficiently large n, $K(\lambda,d')$ is contained in the resolvent set of C_n+R_n and that

$$\|(C_n+R_n-\lambda)^{-1}\|\to\|(A-\lambda)^{-1}\|\,. \tag{6}$$

The identity $C_n+R_n=C_n|_{\operatorname{ran}P_n}\oplus R_n|_{\ker P_n}$ implies that

$$\sigma(C_n+R_n)=\sigma(C_n|_{\operatorname{ran}P_n})\cup\sigma(R_n|_{\ker P_n})$$

and

$$\|(C_n+R_n-\lambda)^{-1}\| = \max\{\|((C_n-\lambda)|_{\operatorname{ran}P_n})^{-1}\|,\|((R_n-\lambda)|_{\ker P_n})^{-1}\|\}\,. \tag{7}$$

Therefore $d(\lambda,\sigma(C_n))>d'$ for sufficiently large n. Also, $\|P_n(A-\lambda)^{-1}P_n\|\to\|(A-\lambda)^{-1}\|$ as $n\to\infty$. Together with (6) and (7) this implies that

$$\|((C_n-\lambda)|_{\operatorname{ran}P_n})^{-1}\|\to\|(A-\lambda)^{-1}\|\,. \tag{8}$$

The operator $((C_n-\lambda)|_{\operatorname{ran}P_n})^{-1}$ acts in a finite-dimensional space, thus we have the estimate

$$\|((C_n-\lambda)|_{\operatorname{ran}P_n})^{-1}\|\le\frac{1}{d'}f\biggl(\frac{1}{d'}|\operatorname{Im}C_n|\biggr)\le\frac{1}{d'}f\biggl(\frac{1}{d'}|\operatorname{Im}A|\biggr)\,,$$

and it follows from (8) that also

$$\|(A-\lambda)^{-1}\|\le\frac{1}{d'}f\biggl(\frac{1}{d'}|\operatorname{Im}A|\biggr)\,.$$

This estimate holds for any positive $d'<d(\lambda,\sigma(A))$, thus defining $\beta(s)=\inf_{t>s}f(t)$ we have for any $A\in\mathfrak{A}+\mathfrak{S}$,

$$\|(A-\lambda)^{-1}\|\le\frac{1}{d(\lambda,\sigma(A))}\beta\biggl(\frac{|\operatorname{Im}A|}{d(\lambda,\sigma(A))}\biggr)\,,$$

and this ends the proof. ∎

LEMMA 4. *If* $\sum_{n\geq 1}(n|P_n|)^{-1} < \infty$, *where* P_n *denotes an* n*-dimensional orthogonal projection*, *then* $\mathfrak{S}$ *has the resolvent estimate property.*

Proof. For any $r > 0$, let $k(r)$ be the smallest natural number k such that

$$\sum_{j\geq 1} \frac{1}{(2j-1)|P_{j+k}|} \leq r\,,$$

and let $l(r) = \sum_{j\leq k(r)} |P_j|^{-1}$.

Suppose that $C \in \mathfrak{S}$ is a selfadjoint operator acting in the n-dimensional Hilbert space $\mathfrak{H}_n$ and let $C = \sum_{j\geq 1} \lambda_j(\cdot, f_j)f_j$ be its Schmidt expansion such that $|\lambda_j| = s_j(C)$. Note that $|s_j(C)P_j| \leq |C|$, and hence $s_j(C) \leq |P_j|^{-1}|C|$.

This implies that for any $r > 0$ there exists a selfadjoint operator C_r such that $|C_r|_\omega \leq r|C|$ and $|C - C_r|_1 \leq l(r)|C|$: it suffices to put $C_r = \sum_{j\geq k(r)} \lambda_j(\cdot, f_j)f_j$.

Now let A be any operator acting in an n-dimensional complex space and let $m = |\operatorname{Im} A|$. Let $e_1, \ldots, e_n$ be an orthonormal basis in which A has triangular matrix form, i.e. $A = \sum_{j\geq k} E_j A E_k$, where $E_j = (\cdot, e_j)e_j$. Let $D = \sum_j E_j A E_j$ and $L = A - D$; then $\operatorname{Im} D = \sum_j E_j(\operatorname{Im} A)E_j$, and [3, Theorem III.4.2] implies that $|\operatorname{Im} D| \leq m = |\operatorname{Im} A|$, hence $|\operatorname{Im} L| = |\operatorname{Im} A - \operatorname{Im} D| \leq 2m$.

Let K_r be a selfadjoint operator such that $|K_r|_\omega \leq r|\operatorname{Im} L| \leq 2rm$ and

$$|\operatorname{Im} L - K_r|_1 \leq l(r)|\operatorname{Im} L| \leq 2l(r)m\,.$$

We have the identity $L = 2i\sum_{j>k} E_j(\operatorname{Im} L)E_k$ and we set

$$L_r = 2i\sum_{j\geq k} E_j K_r E_k\,.$$

Then

$$L - L_r = 2i\sum_{j>k} E_j(\operatorname{Im} L - K_r)E_k$$

and it follows from [4, Theorems III.4.1 and III.2.2] that $\|L_r\| \leq 2|K_r|_\omega \leq 4mr$ and

$$|L - L_r|_\Omega \leq 2|\operatorname{Im} L - K_r|_1 \leq 4l(r)m\,. \tag{9}$$

Suppose now that λ is not an eigenvalue of A and that $d = d(\lambda, \sigma(A))$. Then since $\sigma(D) = \sigma(A)$ and D is a normal operator,

$$A - \lambda = D - \lambda - L = (D - \lambda)(I + (D - \lambda)^{-1}L)$$

and

$$\|(D - \lambda)^{-1}\| = 1/d\,. \tag{10}$$

Setting $r = d/m$ we have $\|(D-\lambda)^{-1}L_r\| \le 1/2$ and hence

$$\|(I+(D-\lambda)^{-1}L_r)^{-1}\| \le 2\,.$$

We set further $N_r = (I+(D-\lambda)^{-1}L_r)^{-1}(D-\lambda)^{-1}(L-L_r)$. Then

$$A-\lambda = (D-\lambda)(I+(D-\lambda)^{-1}L_r)(I+N_r) \tag{11}$$

and

$$|N_r|_\Omega \le \frac{2}{d}|L-L_r|_\Omega < 8ml(m/d)/d\,. \tag{12}$$

In the basis $e_1,\ldots,e_n$, the operators L_r, L have triangular matrix form, and $(D-\lambda)^{-1}$ is represented in this basis by a diagonal matrix. Therefore the operator N_r is nilpotent. It follows from [7, Lemma 2] that there exists a nondecreasing function ψ such that for any finite rank operator B,

$$\|(B-\lambda)^{-1}\| \le \frac{1}{d(\lambda,\sigma(B))}\psi\biggl(\frac{|B|}{d(\lambda,\sigma(B))}\biggr),$$

and ψ does not depend on the dimension of the space. Applying this estimate to N_r we obtain from (9)–(12) the estimate

$$\begin{aligned}\|(A-\lambda)^{-1}\| &= \|(I+N_r)^{-1}(I+(D-\lambda)^{-1}L_r)^{-1}(D-\lambda)^{-1}\|\\ &\le \frac{2}{d}\psi\biggl(\frac{8m}{d}l\biggl(\frac{m}{d}\biggr)\biggr).\end{aligned}$$

The estimate obtained is independent of $A \in L(\mathfrak{H}_n)$ and of the dimension of the space $\mathfrak{H}_n$. Now the assertion of the lemma follows from Lemma 3. ■

It follows from Lemma 4 that the Schatten ideals $\mathfrak{S}_p$ $(1 \le p < \infty)$ have the resolvent estimate property, while Lemma 2 implies that the Matsaev ideal does not have this property.

Remark. *If $A \in \mathfrak{S}_\omega$ then there exists a normed ideal $(\mathfrak{S}, |\cdot|)$ which satisfies the assumption of Lemma 4 and $A \in \mathfrak{S}$.*

Proof. Let $s_1 \ge s_2 \ge \ldots$ be the singular values of A. We define the ideal $\mathfrak{S}$ with crossnorm $|\cdot|$ to be the set of all compact operators B such that

$$|B| \stackrel{\mathrm{df}}{=} s_1 \sup_n \frac{\sum_{j=1}^n s_j(B)}{\sum_{j=1}^n s_j} < \infty.$$

It follows from [3, §14] that $(\mathfrak{S}, |\cdot|)$ is in fact a normed ideal. It is obvious that $A \in \mathfrak{S}$ and $|A| = s_1$. If P_n is an n-dimensional projection then

$|P_n| = s_1 n(\sum_{j=1}^n s_j)^{-1}$ and therefore

$$\sum_{j=1}^{\infty} \frac{1}{(n+1)|P_n|} = s_1^{-1} \sum_{j=1}^{\infty} \frac{1}{n(n+1)} \sum_{j \le n} s_j$$
$$= s_1^{-1} \sum_{j=1}^{\infty} s_j \sum_{n \ge j} \frac{1}{n(n+1)} = s_1^{-1} \sum_{j=1}^{\infty} \frac{s_j}{j} < \infty .$$

This is equivalent to the assumption of Lemma 4. ∎

The theorem stated below shows that the resolvent estimate property is equivalent to some inequalities for crossnorms of the hermitian components of quasinilpotent operators.

THEOREM 1. *An ideal $\mathfrak{S}$ has the resolvent estimate property if and only if there exists a normed ideal $\mathfrak{S}_\nu$ with crossnorm $|\cdot|_\nu$ not equivalent to the operator norm and a constant c such that $|\mathrm{Re}\, V|_\nu \le c|\mathrm{Im}\, V|$ for every quasinilpotent operator $V \in \mathfrak{S}$.*

We precede the proof of this theorem with a simple lemma.

LEMMA 5. *Suppose that $(\mathfrak{S}, |\cdot|)$, $(\mathfrak{S}_\nu, |\cdot|_\nu)$ are symmetrically normed ideals with norms not equivalent to the operator norm and such that there exists a constant c such that $|\mathrm{Re}\, V|_\nu \le c|\mathrm{Im}\, V|$ for any quasinilpotent operator with $\mathrm{Im}\, V \in \mathfrak{S}$. Then $\mathfrak{S} \subset \mathfrak{S}_\nu$ and $|A|_\nu \le 2c|A|$ for any $A \in \mathfrak{S}$.*

Proof. Since the crossnorms are unitarily equivalent it suffices to show the lemma for selfadjoint operators only. Thus assume that $A \in \mathfrak{S}$ is selfadjoint. Consider the operator $W = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$; since it is nilpotent, we have $|\mathrm{Re}\, W|_\nu \le c|\mathrm{Im}\, W|$. To end the proof it suffices to note that $|\mathrm{Im}\, W| \le |A|$ and $2|\mathrm{Re}\, W|_\nu \ge |A|_\nu$. ∎

Proof of Theorem 1. (*"if" part*) Let A be any operator acting in an n-dimensional complex space and let $m = |\mathrm{Im}\, A|$. Let $e_1, \ldots, e_n$ be an orthonormal basis in which A has triangular matrix form, i.e. $A = \sum_{j \ge k} E_j A E_k$, where $E_j = (\cdot, e_j)e_j$. Let $D = \sum_j E_j A E_j$ and $L = A - D$; then $\sigma(D) = \sigma(A)$. The operator $L(D-\lambda)^{-1}$ is represented in the basis $e_1, \ldots, e_n$ by a triangular matrix with vanishing diagonal, and therefore it is nilpotent. It follows from the identity

$$A - \lambda = D - \lambda + L = (I + L(D-\lambda)^{-1})(D - \lambda)$$

and the equality

$$\|(D-\lambda)^{-1}\| = (d(\lambda, \sigma(D)))^{-1} = (d(\lambda, \sigma(A)))^{-1} \tag{13}$$

that

$$\|(A-\lambda)^{-1}\| \le \frac{\|(I + L(D-\lambda)^{-1})^{-1}\|}{d(\lambda, \sigma(A))} .$$

The equality $\operatorname{Im} D = \sum_{j=1}^{n} E_j(\operatorname{Im} A)E_j$ and [3, Theorem III.4.2] imply that $|\operatorname{Im} D| \le |\operatorname{Im} A|$ and since $\operatorname{Im} L = \operatorname{Im} A - \operatorname{Im} D$ we have $|\operatorname{Im} L| \le 2|\operatorname{Im} A|$. Now if the suitable ideal $\mathfrak{S}_\nu$ and constant c exist then

$$|L|_\nu \le |\operatorname{Re} L|_\nu + |\operatorname{Im} L|_\nu \le c|\operatorname{Im} L| + 2c|\operatorname{Im} L| \le 6c|\operatorname{Im} A| .$$

This inequality and (13) imply that for the nilpotent operator $L(D-\lambda)^{-1}$ we have the estimate

$$|L(D-\lambda)^{-1}|_\nu \le \|(D-\lambda)^{-1}\| |L|_\nu \le 6c\frac{|\operatorname{Im} A|}{d(\lambda, \sigma(A))} .$$

It is shown in [7, Lemma 2] that there exists an increasing function $\psi : (0,\infty) \to (0,\infty)$ such that $\|(I+N)^{-1}\| \le \psi(|N|_\nu)$ for any finite rank nilpotent operator N. This implies that

$$\|(I + L(D-\lambda)^{-1})^{-1}\| \le \psi(|L(D-\lambda)^{-1}|) \le \psi\left(6c\frac{|\operatorname{Im} A|}{d(\lambda, \sigma(A))}\right).$$

Thus setting $\beta(t) = \psi(6ct)$ we have the estimate

$$\|(A-\lambda)^{-1}\| \le \frac{1}{d(\lambda, \sigma(A))}\beta\left(\frac{|\operatorname{Im} A|}{d(\lambda, \sigma(A))}\right)$$

for any finite rank operator. The "if" part now follows from Lemma 3.

(*"only if" part*) It follows from Lemma 2 and [4, Theorem III.4.2] that there exists a positive number s such that $\|\operatorname{Re} V\| \le 2|\operatorname{Im} V|_\omega \le s|\operatorname{Im} V|$ for any quasinilpotent operator V with $\operatorname{Im} V \in \mathfrak{S}$. Assume that $|\operatorname{Im} V| = 1$ and that $\operatorname{Re} V = \sum \lambda_j(\cdot, f_j)f_j$ is the eigenvalue expansion of $\operatorname{Re} V$. Let Γ_r be the positively oriented closed polynomial path connecting the points $r-i$, $r+i$, $s+1+i$, $s+1-i$.

Note that:

(i) $$\sum_{\lambda_j > r} (\lambda_j - r)(\cdot, f_j)f_j = \frac{-1}{2\pi i}\int_{\Gamma_r} (\lambda - r)(\operatorname{Re} V - \lambda)^{-1} d\lambda$$
$$= \frac{-1}{2\pi i}\int_{\Gamma_r} ((\operatorname{Re} V - \lambda)^{-1} - (V-\lambda)^{-1})\, d\lambda .$$

(ii) If $\operatorname{Re}\lambda = r$ then

$$|\lambda - r| = |\operatorname{Im}\lambda| \le d(\lambda, \sigma(\operatorname{Re} V)) = \|(\operatorname{Re} V - \lambda)^{-1}\|^{-1} ,$$

and it follows from the identity

$$(\operatorname{Re} V - \lambda)^{-1} - (V-\lambda)^{-1} = i(\operatorname{Re} V - \lambda)^{-1}(\operatorname{Im} V)(V-\lambda)^{-1}$$

and Lemma 1 that

$$|(\lambda - r)((\operatorname{Re} V - \lambda)^{-1} - (V-\lambda)^{-1})| \le \frac{1}{r}\beta_\nu\left(\frac{1}{r}\right).$$

(iii) If $\lambda \in \Gamma_r$ and $\operatorname{Re}\lambda > r$ then $|\lambda| > r$, $d(\lambda, \sigma(V)) \geq 1$ and hence

$$\left|(\lambda - r)((\operatorname{Re} V - \lambda)^{-1} - (V - \lambda)^{-1})\right| \leq |\lambda - r| \frac{1}{r} \beta_\nu\left(\frac{1}{r}\right).$$

These facts imply that there exists $c > 0$ such that for all $r \in (0, s+1)$,

$$\Big| \sum_{\lambda_j > r} (\lambda_j - r)(\cdot, f_j) f_j \Big| \leq \frac{c}{r} \beta_\nu\left(\frac{1}{r}\right).$$

If $\lambda_j \geq r$ then $\lambda_j \leq 2(\lambda_j - r/2)$ and therefore

$$\begin{aligned}\Big| \sum_{\lambda_j \geq r} \lambda_j (\cdot, f_j) f_j \Big| &\leq 2 \Big| \sum_{\lambda_j \geq r} \left(\lambda_j - \frac{r}{2}\right)(\cdot, f_j) f_j \Big| \\ &\leq 2 \Big| \sum_{\lambda_j \geq r/2} \left(\lambda_j - \frac{r}{2}\right)(\cdot, f_j) f_j \Big| \leq 2 \frac{2c}{r} \beta_\nu\left(\frac{2}{r}\right).\end{aligned}$$

In a similar way we obtain the estimate

$$\Big| \sum_{\lambda_j \leq -r} \lambda_j (\cdot, f_j) f_j \Big| \leq \frac{4c}{r} \beta_\nu\left(\frac{2}{r}\right).$$

The last two estimates imply that

$$\Big| \sum_{|\lambda_j| \geq r} \lambda_j (\cdot, f_j) f_j \Big| \leq \frac{8c}{r} \beta_\nu\left(\frac{2}{r}\right). \tag{14}$$

It follows from Lemma 1 that the function $\psi(r) = (8c/r)\beta_\nu(2/r)$ is decreasing and continuous, $\psi(r) \to \infty$ as $r \to 0$ and $\psi(r) \to 0$ as $r \to \infty$. Let P_n denote an n-dimensional orthogonal projection and τ_n be a positive number such that $|P_n| = \psi(\tau_n)$. The sequence $\tau_1, \tau_2, \ldots$ is nonincreasing and $\tau_n \to 0$ as $n \to \infty$. Note also that $|P_n| \leq \psi(\tau)$ if and only if $\tau \leq \tau_n$.

Now we define the ideal $\mathfrak{S}_\nu$ to be the set of all those compact operators A for which

$$|A|_\nu = \tau_1 \sup_n \frac{\sum_{j=1}^n s_j(A)}{\sum_{j=1}^n \tau_j} < \infty.$$

It follows from [3, Lemma III.4.1 and Theorem III.4.2] that $\mathfrak{S}_\nu$ is a symmetrically normed ideal with crossnorm $|\cdot|_\nu$ not equivalent to the operator norm.

Now let $s_k = s_k(\operatorname{Re} V)$ and suppose that for some $l \leq k$, $s_l = s_k > s_{k+1}$. Then it follows from (14) that

$$s_k |P_k| \leq \Big| \sum_{|\lambda_j| \geq s_k} \lambda_j (\cdot, f_j) f_j \Big| \leq s_k \psi(s_k)$$

and therefore $s_l = s_k \le \tau_k \le \tau_l$. This shows that if V is a quasinilpotent operator such that $\operatorname{Im} V \in \mathfrak{S}$ then $\operatorname{Re} V \in \mathfrak{S}_\nu$ and $|\operatorname{Re} V|_\nu \le \tau_1 |\operatorname{Im} V|$. ■

Effective resolvent norm estimates. In order to find effective resolvent norm estimate for operators A with $\operatorname{Im} A \in \mathfrak{S}_p$ we have to check the hypothesis of Theorem 1. The Matsaev theorem [4, Theorem III.6.2] states that if N is a quasinilpotent operator with $\operatorname{Im} N \in \mathfrak{S}_p$ $(1 < p < \infty)$ then

$$|\operatorname{Re} N|_p \le c_p |N|_p, \quad \text{where} \quad c_p = \begin{cases} \dfrac{1}{e^{3/2}\ln 2}\, p, & 2 \le p < \infty, \\ \dfrac{1}{e^{3/2}\ln 2} \cdot \dfrac{p}{p-1}, & 1 < p \le 2, \end{cases}$$

while for $p = 1$ [4, Theorem III.2.1] we have

$$|N|_\Omega \le \frac{4}{\pi} |N|_1 .$$

Following the proof of Theorem 1 we have to estimate $\|(I-N)^{-1}\|$; the method of obtaining such an estimate is given in [7, Lemma 2], and gives

$$\|(I-N)^{-1}\| \le \frac{3}{2} \exp\left(19.5 |N| \tau\left(\frac{1}{3|N|}\right)\right),$$

where

$$\tau(r) = |P_{n(r)}|_*, \quad n(r) = \max\{n : r|P_n| \le 1\},$$

P_n stands for an n-dimensional orthogonal projection, and $|\cdot|_*$ for the adjoint crossnorm. It is easy to see that in the case of the crossnorms $|\cdot|_p$ $(1 < p < \infty)$ we have

$$n(r) = n_p(r) \le r^{-p}, \quad \tau(r) = \tau_p(r) = |P_{n(r)}|_{p/(p-1)} \le r^{1-p} .$$

The respective estimation of n_Ω, τ_Ω is a little harder. The crossnorm $|\cdot|_\omega$ is adjoint to $|\cdot|_\Omega$ and

$$|P_n|_\Omega = \frac{n}{\sum_{j=1}^n \frac{1}{2j-1}}, \quad |P_n|_\omega = \sum_{j=1}^n \frac{1}{2j-1} .$$

Using the elementary inequality $\sum_{j=1}^n \frac{1}{2j-1} < \ln(e\sqrt{n})$ we see that

$$n_\Omega(r) \le \frac{1}{r}\ln\frac{1}{r}, \quad \tau_\Omega(r) \le \ln\left(e\sqrt{\frac{1}{r}\ln\frac{1}{r}}\right) \quad \text{for } r < 0.0430156\ldots$$

Now inspecting the proof of Theorem 1 we find that if $\operatorname{Im} A \in \mathfrak{S}_p$ then with $d = d(\lambda, \sigma(A))$ and $a = |\operatorname{Im} A|_p$,

$$\|(A-\lambda I)^{-1}\| \le \frac{3}{2d} \exp\left(6.5\left(18 c_p \frac{a}{d}\right)^p\right), \quad 1 < p < \infty, \tag{15}$$

$$(16) \quad \|(A-\lambda I)^{-1}\| \leq \frac{3}{2d}\exp\left(\frac{156}{\pi}\frac{a}{d}\ln\left(e\sqrt{\frac{24a}{\pi d}\ln\left(\frac{24a}{\pi d}\right)}\right)\right), \quad p=1\,.$$

The last inequality holds if $a > d\cdot 3.043\ldots$ The constant $3.043\ldots$ has been evaluated with the help of Mathematica [8].

Perturbation of spectra

THEOREM 2. *If $\mathfrak{S}$ is an ideal with the resolvent estimate property then there exists an increasing function $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(t) \to \infty$ as $t \to 0$ and for any two operators A, B with $\operatorname{Im} A, \operatorname{Im} B \in \mathfrak{S}$,*

$$\operatorname{dist}(\sigma(A),\sigma(B)) \leq m\alpha\left(\frac{\|A-B\|}{m}\right),$$

where $m = \max\{|\operatorname{Im} A|, |\operatorname{Im} B|\}$.

Proof. Since the spectrum of A is an approximate spectrum, for any $\lambda \in \sigma(A)$ and any $\varepsilon > 0$ there exists a unit vector x such that $\|(A-\lambda)x\| \leq \varepsilon$. Then $\|(B-\lambda)x\| \leq \|B-A\| + \varepsilon$ and since $1 = \|(B-\lambda)^{-1}(B-\lambda)x\| \leq \|(B-\lambda)^{-1}\|\|(B-\lambda)x\|$ we see that

$$\|(B-\lambda)^{-1}\| \geq \frac{1}{\|(B-\lambda)x\|} \geq \frac{1}{\|B-A\|+\varepsilon}\,.$$

Hence it follows from Lemma 1 that with the appropriate function β we have

$$\frac{1}{\|B-A\|} \leq \frac{1}{d(\lambda,\sigma(B))}\beta\left(\frac{|\operatorname{Im} B|}{d(\lambda,\sigma(B))}\right).$$

This inequality holds for any $\lambda \in \sigma(A)$ and therefore we have

$$\frac{1}{\|B-A\|} \leq \frac{1}{\widehat{\varrho}}\beta\left(\frac{|\operatorname{Im} B|}{\widehat{\varrho}}\right) \quad \text{where} \quad \widehat{\varrho} = \sup_{\lambda\in\sigma(A)} d(\lambda,\sigma(B))\,.$$

Interchanging A and B we see that

$$\frac{m}{\|A-B\|} \leq \frac{m}{d}\beta\left(\frac{m}{d}\right),$$

where $m = \max\{|\operatorname{Im} A|, |B|\}$, $d = \operatorname{dist}(\sigma(A),\sigma(B))$. The function $g(t) = t\beta(t)$ is decreasing and $g(t) \to \infty$ as $t \to \infty$, while $g(t) \to 0$ as $t \to 0$, therefore there exists an increasing function $\alpha : [0,\infty) \to [0,\infty)$ such that $g(t) > 1/r$ implies $t \leq \alpha(r)$ and $\alpha(s) \to 0$ as $s \to 0$. This function is as desired. ∎

Effective estimates of distances between spectra. The effective estimates of the distance between the spectra of operators A, B with $\operatorname{Im} A$, $\operatorname{Im} B \in \mathfrak{S}_p$ may be given in terms of the inverse functions to those appearing on the right-hand sides of (15) and (16). These functions cannot be expressed by elementary functions, therefore we content ourselves with a little worse estimates but given in a simple form. We set

$$m = \max\{|\operatorname{Im} A|_p, |\operatorname{Im} B|_p\}, \quad t = \frac{m}{\operatorname{dist}(\sigma(A), \sigma(B))}, \quad s = \frac{m}{\|A - B\|}.$$

It follows from the proof of Theorem 2 and (15) that $s \le c_1 t e^{c_2 t^p} \le c_1 e^{c_3 t_p}$ for $t \ge 1$ and appropriate constants c_1, c_2, c_3. This implies that

$$t \ge \left(\frac{1}{c_3} \ln \frac{s}{c_1}\right)^{1/p}$$

and hence

$$\operatorname{dist}(\sigma(A), \sigma(B)) \le c_0 m \left(\ln \frac{m}{c_1 \|A - B\|}\right)^{-1/p}$$

provided that $\|A - B\|/m$ is sufficiently small.

Similarly, for $p = 1$ it follows from (16) that

$$\begin{aligned} s &\le a_1 t \exp(a_2 t \ln(e\sqrt{a_3 t \ln(a_3 t)})) \\ &\le a_1 \exp(t(1 + a_2 \ln(e\sqrt{a_3})) + a_2 t \ln t) \\ &= a_1 \exp(a_4 t + a_5 t \ln t) \le a_1 \exp(a_0 t \ln t) \end{aligned}$$

with some constants $a_0, a_2, \ldots$ if $\ln t > a_4/a_5$, hence

$$\frac{1}{a_0} \ln \frac{s}{a_1} \le t \ln t\,.$$

Using the inequality

$$u \ge \frac{u}{\ln u} \ln \frac{u}{\ln u}$$

(valid for $u \ge e$) with $u = \frac{1}{a_0} \ln \frac{s}{a_1}$ we see that for s sufficiently large we have $u \ln u \ge t \ln t$, and since the function $t \ln t$ is increasing we have further

$$\frac{1}{a_0} \ln \frac{s}{a_1} = u \ge t\,.$$

This implies that

$$\operatorname{dist}(\sigma(A), \sigma(B)) \le a_0 m \left(\ln \frac{m}{a_1 \|A - B\|}\right)^{-1}$$

if $\|A - B\|/m$ is sufficiently small.

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