APPLICATIONES MATHEMATICAE
22,1 (1993), pp. 45–53

L. MUCHE (Freiberg)

AN INCOMPLETE VORONOI TESSELLATION

Abstract. This paper presents distributional properties of a random cell structure which results from a growth process. It starts at the points of a Poisson point process. The growth is spherical with identical speed for all points; it stops whenever the boundaries of different cells have contact. The whole process finally stops after time t. So the space is not completely filled with cells, and the cells have both planar and spherical boundaries. Expressions are given for contact distribution functions, the specific boundary length, the specific surface area, and the mean chord length of this cell structure in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1. Introduction. It is well-known that the Voronoi tessellation (see [3]) can be interpreted as the result of a growth process. There is a point process of so-called nuclei. Each nucleus grows in such a way that at time t it occupies all the previously vacant region within the sphere of radius t centred at its original point. The growth process stops if cells of other nuclei are met. Fig. 1 shows some stages of this growth process. In the classical Voronoi tessellation this process continues until the whole space is divided into cells, which are convex polyhedra.

This paper considers the structure which is given at time t, the "incomplete tessellation". It is assumed that the point process of growth of nuclei is a stationary Poisson process of intensity λ. Clearly the random set of points which are covered by the cells is a Boolean model (see [3], pp. 65–95) with spherical grains. Formulae for it are well-known. More interesting is the structure of the cells. They are random convex sets with boundaries which consist partly of pieces of planes and of pieces of t-spheres. This structure is used in material sciences to describe the nucleation process of metals.

1991 *Mathematics Subject Classification*: 60D05, 52A22.
Key words and phrases: Boolean model, Poisson–Voronoi tessellation, contact distribution function, specific surface area, mean chord length.

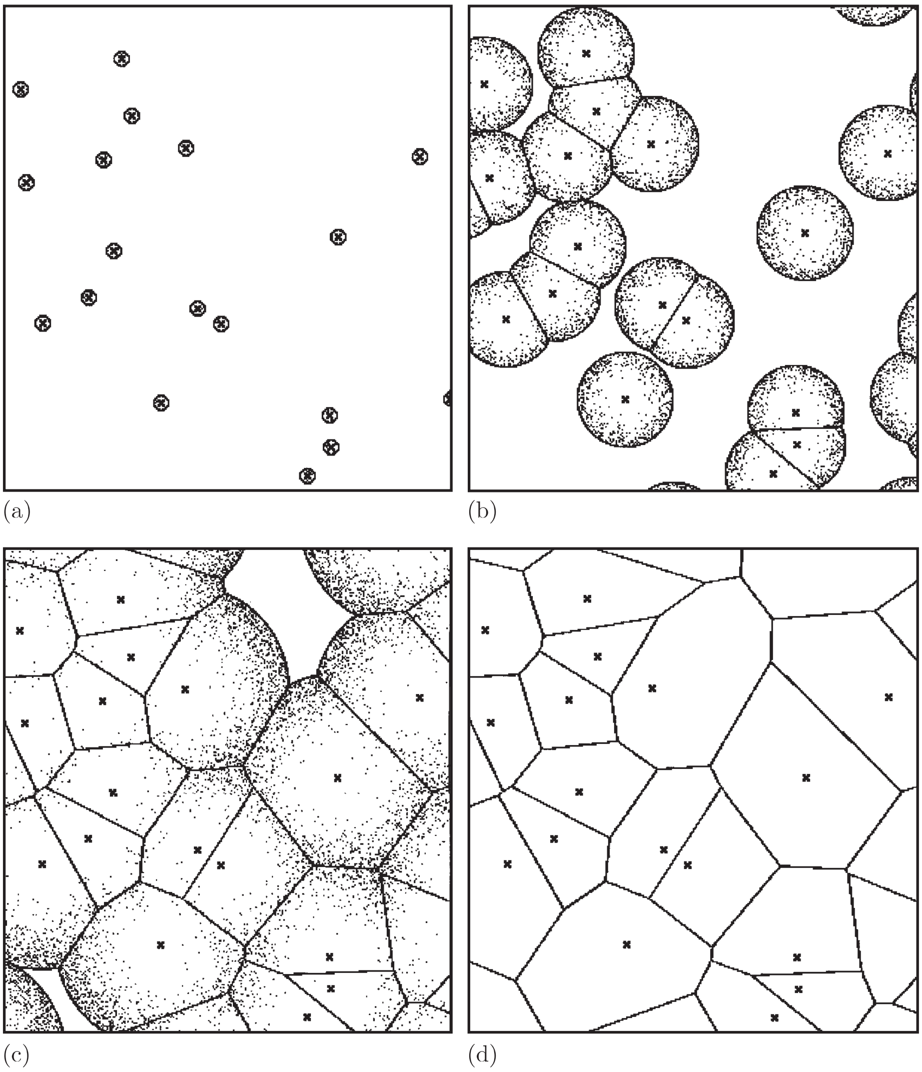

Fig. 1. The growth process which yields a Voronoi (Dirichlet) tessellation. It starts with a Poisson process of nuclei at time $t = 0$. The nuclei grow circularly until contact with the cells of other nuclei. The incomplete tessellations shown in (b) and (c) are studied in this paper. Their cells have both linear and circular boundaries.

This paper gives integral formulae for the spherical and linear contact distribution functions of the cells in the planar case and spatial case. Furthermore, it provides expressions for the specific boundary length in the planar case and the specific surface area in the spatial case. Finally, formulae for the fraction of spherical boundaries and for the mean chord length are given. The complete proofs can be found in [1].

2. Fundamentals. Let Φ be a stationary Poisson process in $\mathbb{R}^d$ of intensity λ. The *cell* C_i corresponding to the point $z_i \in \Phi$ $(i = 0, 1, 2, \ldots)$ can be defined in the following way:

$$C_i = \{x \in \mathbb{R}^d \mid \|x - z_i\| < t \text{ and } \|x - z_i\| < \|x - z_j\| \text{ for all } i \neq j\}\,.$$

The set Ψ is the union of the closures of all cells,

$$\Psi = \bigcup_{i=0}^{\infty} C_i^{\mathrm{cl}} = \bigcup_{i=0}^{\infty} z_i \oplus b(o, t)\,.$$

It is a Boolean model with spherical grains (see [3], p. 66). Its *volume fraction* is given by

$$P(o \in \Psi) = 1 - \exp(-\lambda\omega_d t^d) = P(o \in C_0) = 1 - p\,.$$

Here ω_d is the volume of the unit sphere of $\mathbb{R}^d$. The point $z_0 \in \Phi$ is the nearest neighbour of the origin o and C_0 is the corresponding cell. The distribution function of the distance between z_0 and o is

$$D(r) = P(\|z_0\| \le r) = 1 - \exp(-\lambda\omega_d r^d)\,, \qquad r \ge 0\,.$$

The corresponding density function is denoted by $d(r)$. Furthermore, let X be the union of all cell boundaries,

$$X = \bigcup_{i=0}^{\infty} \partial(C_i^{\mathrm{cl}})\,.$$

For a test set K with $o \in K$, the corresponding *contact distribution function* is defined by

$$H(r) = P(rK \cap X \neq \emptyset \mid o \in \Psi), \qquad r \ge 0\,.$$

An equivalent form is

$$H(r) = 1 - \frac{P(rK \subset C_0)}{P(o \in C_0)}\,, \qquad r \ge 0\,. \tag{2.1}$$

The probability that a set A lies completely in C_0 can be written as

$$P(A \subset C_0) = \frac{1}{d\omega_d} \int\limits_0^{\infty} \int\limits_{S^{d-1}} P_A(\varrho, \sigma) d(\varrho)\, d\sigma\, d\varrho\,. \tag{2.2}$$

Here $P_A(\varrho, \sigma)$ is the probability that C_0 contains A completely under the condition that $z_0 = (\varrho, \sigma)$, where (ϱ, σ) is the point with polar coordinates $\varrho > 0$ and $\sigma \in S^{d-1}$. $P_A(\varrho, \sigma)$ is equal to the probability that a certain set $A_{\varrho,\sigma}$ does not contain a point of Φ. If ν_d denotes the Lebesgue measure in $\mathbb{R}^d$, then

$$P_A(\varrho, \sigma) = \begin{cases} \exp(-\lambda\nu_d(A_{\varrho,\sigma} \setminus b(o, \varrho))) & \text{if } A \subset b(z_0, t), \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

with

$$A_{\varrho,\sigma} = \bigcup_{x \in A} b(x, \|x - (\varrho, \sigma)\|) . \tag{2.4}$$

The subtraction of the ball $b(o, \varrho)$ in (2.3) is necessary, because it does not contain points of Φ under the condition $z_0 = (\varrho, \sigma)$.

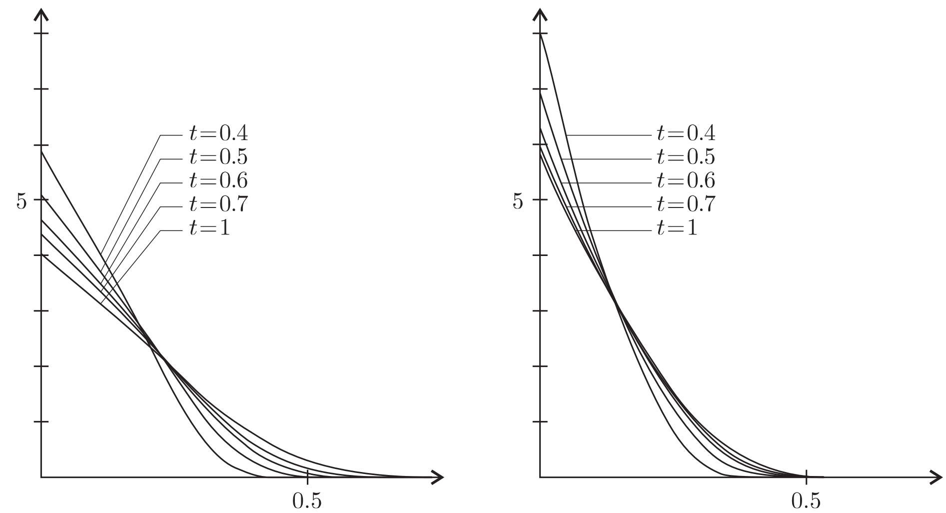

Fig. 2. The densities of the spherical contact distribution functions $H_{\mathrm{s}}(r)$ for different times t in $\mathbb{R}^2$ (a) and in $\mathbb{R}^3$ (b). (The intensity of the generating Poisson process is $\lambda = 1$.)

3. The spherical contact distribution function. The spherical contact distribution function corresponds to the case where K is the unit ball of $\mathbb{R}^d$. It can be determined by means of the formulae (2.1)–(2.4), similarly to the case of the (complete) Voronoi tessellation in [2]. (For details on the corresponding set $A_{\varrho,\sigma}$ and its Lebesgue measure ν_d see [1].) This yields

$$H_{\mathrm{s}}(r) = \tag{3.1}$$
$$1 - \frac{2\pi\lambda}{1-p} \begin{cases} \int_r^{t-r} \varrho \exp\Big(-\lambda\Big(2(2r^2+\varrho^2)\Big(\pi - \arccos\frac{r}{\varrho}\Big) \\ \quad +6r\sqrt{\varrho^2 - r^2}\Big)\Big)\, d\varrho + \frac{1}{4\pi\lambda}\exp(-4\pi\lambda r^2) - \frac{1}{4\pi\lambda}\exp(-6\pi\lambda r^2), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le r < t/2, \\ \frac{1}{4\pi\lambda}\exp(-4\pi\lambda r^2) - \frac{1}{4\pi\lambda}\exp(-2\pi\lambda(3r^2 - 2rt + t^2))\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le r < t, \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad t \le r, \end{cases}$$

in $\mathbb{R}^2$ and

(3.2) $H_{\mathrm{s}}(r) =$

$$1-\frac{4\pi\lambda}{1-p}\begin{cases} \int\limits_0^r \varrho^2\exp(-\frac{32}{3}\pi\lambda r(r^2+\varrho^2))\,d\varrho \\ \qquad + \int\limits_r^{t-r}\varrho^2\exp\left(-\frac{4\pi}{3\varrho}\lambda(r+\varrho)^4\right)d\varrho, & 0\le r< t/2,\\ \int\limits_0^{t-r}\varrho^2\exp(-\frac{32}{3}\pi\lambda r(r^2+\varrho^2))\,d\varrho, & t/2\le r<t,\\ 0, & t\le r, \end{cases}$$

in $\mathbb{R}^3$. The corresponding densities are plotted in Fig. 2.

4. The linear contact distribution function. More complicated is the determination of the linear contact distribution function $H_\ell(r)$. The test set K is here a segment of unit length, $rK = s(o,r)$. Since X is isotropic, $H_\ell(r)$ does not depend on the direction of the segment. Therefore the segment with endpoints at o and the point $\underline{r}$ with coordinate r on the x_1-axis may be used. $A_{\varrho,\alpha}$ is here the union of two spheres with radii ϱ and $\sqrt{\varrho^2-2\varrho r\cos\alpha+r^2}$ and with distance r of midpoints,

$$A_{\varrho,\alpha} = \bigcup_{x\in s(o,r)} b(x,\|x-z_0\|) = b(o,\|z_0\|)\cup b(\underline{r},\|\underline{r}-z_0\|)\,,$$

where α is the angle between $s(o,r)$ and the line connecting o and (ϱ,σ). Since both radii have to be smaller than t, the integration with respect to ϱ in (2.2) goes from $\varrho_- = r\cos\alpha-\sqrt{t^2-r^2\sin^2\alpha}$ to $\varrho_+ = r\cos\alpha+\sqrt{t^2-r^2\sin^2\alpha}$. Then similarly to [2] the linear contact distribution function in $\mathbb{R}^d$ is obtained as

(4.1) $H_\ell(r) =$

$$1-\frac{C_d\lambda}{1-p}\begin{cases} \Big(\int\limits_0^{\arccos(r/(2t))}\int\limits_0^t + \int\limits_{\arccos(r/(2t))}^{\pi}\int\limits_0^{\varrho_+}\Big)\varrho^{d-1}\sin^{d-2}\alpha \\ \qquad\times\exp(-\lambda\nu_d(A_{\varrho,\alpha}))\,d\varrho\,d\alpha, & 0\le r<t,\\ \Big(\int\limits_0^{\arccos(r/(2t))}\int\limits_{\varrho_-}^t + \int\limits_{\arccos(r/(2t))}^{\arcsin(t/r)}\int\limits_{\varrho_-}^{\varrho_+}\Big)\varrho^{d-1}\sin^{d-2}\alpha \\ \qquad\times\exp(-\lambda\nu_d(A_{\varrho,\alpha}))\,d\varrho\,d\alpha, & t\le r<\sqrt{2}t, \end{cases}$$

(4.1 *cont.*) $H_\ell(r) =$

$$1-\frac{C_d\lambda}{1-p}\begin{cases}\displaystyle\int_0^{\arccos(r/(2t))}\int_{\varrho_-}^{t}\varrho^{d-1}\sin^{d-2}\alpha \\ \qquad\qquad \times\exp(-\lambda\nu_d(A_{\varrho,\alpha}))\,d\varrho\,d\alpha, & \sqrt{2}t\le r<2t,\\ 0, & 2t\le r,\end{cases}$$

with $C_d = 2\pi^{(d-1)/2}/\Gamma(\frac{d-1}{2})$. It can be shown that the corresponding density function is continuous in r.

5. Characteristics connected with the contact distribution functions. When the spherical contact distribution function $H_{\rm s}(r)$ is known, then the specific surface area can be determined. Clearly the random closed set X consists of two parts, the spherical one, $X_1 = \partial\Psi$, and the planar one, $X_2 = X\backslash X_1$. So

$$S_V^{(d)} = S_{V,1}^{(d)} + S_{V,2}^{(d)}$$

with

$$S_{V,1}^{(d)} = E\nu_{d-1}(X_1\cap[0,1]^d) = \lambda p\nu_{d-1}(\partial b(o,t))$$

(see [3], p. 79) and

$$S_{V,2}^{(d)} = E\nu_{d-1}(X_2\cap[0,1]^d)\,.$$

The spherical contact distribution function and the specific surface area are connected by

$$\frac{d}{dr}H_{\rm s}(0)(1-p) = S_{V,1}^{(d)} + 2S_{V,2}^{(d)}$$

(see [3], p. 179). The duplication of $S_{V,2}^{(d)}$ is the result of the fact that planar boundaries are almost surely boundaries of two cells. Using (3.1) and (3.2) this yields

$$S_V^{(2)} = L_A = 4\lambda\int_0^t \exp(-\lambda\pi\varrho^2)\,d\varrho + (2\pi-4)\lambda t\exp(-\lambda\pi t^2)$$

for a cell structure in $\mathbb{R}^2$ and

$$S_V^{(3)} = S_V = \tfrac{16}{3}\pi\lambda\int_0^t \varrho\exp(-\tfrac{4}{3}\pi\lambda\varrho^3)\,d\varrho + \tfrac{4}{3}\pi\lambda t^2\exp(-\tfrac{4}{3}\pi\lambda t^3)$$

for one in $\mathbb{R}^3$. Fig. 3 shows L_A and S_V as functions of t.

Another important characteristic of the incomplete Voronoi tessellation is the mean chord length. The term "mean typical chord length" is defined as in [3], p. 180. It corresponds to the segments on a test line $\mathbf{g}$ which

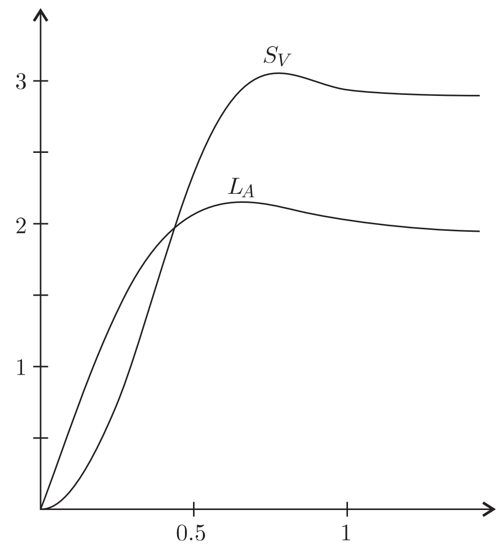

Fig. 3. The specific boundary length $L_A(t)$ and the specific surface area $S_V(t)$ of an incomplete Voronoi tessellation in $\mathbb{R}^2$ and $\mathbb{R}^3$. (The intensity of the generating Poisson process is $\lambda = 1$.)

intersects the stationary and isotropic cell structure. There are chords in Ψ and outside Ψ, and the corresponding mean lengths are denoted by $\overline{l}_{\text{int}}$ and $\overline{l}_{\text{ext}}$. Obviously P_L, the mean number of points $x_i \in X \cap \mathbf{g}$ per unit length, is connected with the mean chord lengths by

$$P_L = \frac{1}{\frac{q}{2}\overline{l}_{\text{ext}} + \left(1 - \frac{q}{2}\right)\overline{l}_{\text{int}}}, \tag{5.1}$$

where q is the fraction of spherical boundaries,

$$q = S_{V,1}^{(d)}/S_V^{(d)}.$$

Expressions for the mean chord length $\overline{l}_{\text{ext}}$ outside Ψ can be obtained using the formulae for the Boolean model (see [3], p. 82). We have $\overline{l}_{\text{ext}} = 1/(2\lambda t)$ in $\mathbb{R}^2$ and $\overline{l}_{\text{ext}} = 1/(\pi\lambda t^2)$ in $\mathbb{R}^3$. With (5.1) and the stereological relations $P_L = 2L_A/\pi$ and $P_L = S_V/2$ in the two- and three-dimensional case, the mean typical chord length inside a cell is given by

$$\overline{l}_{\text{int}} = \frac{\frac{\pi}{2}(1 - \exp(-\pi\lambda t^2))}{4\lambda \int_0^t \exp(-\pi\lambda\varrho^2)\, d\varrho - (4-\pi)\lambda t \exp(-\pi\lambda t^2)} \tag{5.2}$$

in $\mathbb{R}^2$ and

$$\bar{l}_{\mathrm{int}} = \frac{3(1-\exp(-\frac{4}{3}\pi\lambda t^3))}{\pi\lambda\Big(8\int_0^t \varrho\exp(-\frac{4}{3}\pi\lambda\varrho^3)\,d\varrho - t^2\exp(-\frac{4}{3}\pi\lambda t^3)\Big)} \tag{5.3}$$

in $\mathbb{R}^3$ (see Fig. 4).

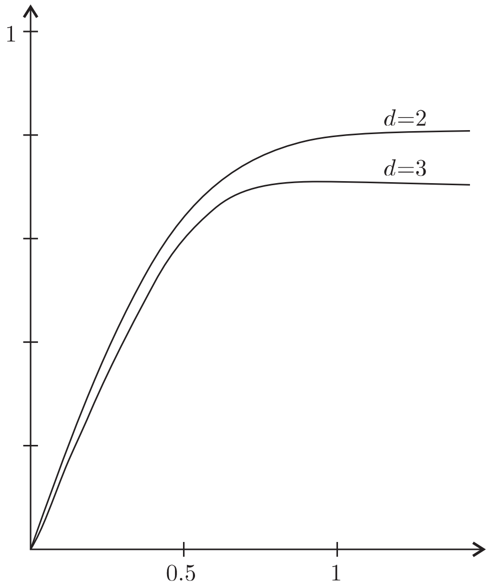

Fig. 4. The mean chord length $\bar{l}_{\mathrm{int}}$ inside a cell in the planar (d =2) and spatial (d =3) case. The intensity of the generating Poisson process is λ =1. Note that $\bar{l}_{\mathrm{int}}$ does not increase monotonically, but it takes a maximum value for $t \approx 2.045$ in the planar case and $t \approx 0.936$ in the spatial case.

The mean chord length $\bar{l}_{\mathrm{int}}$ could be also obtained from the relation

$$H_\ell(r) = \frac{1}{\bar{l}_{\mathrm{int}}} \int_0^r [1 - L(l)]\,dl$$

between the linear contact distribution function $H_\ell(r)$ and the chord length distribution function $L(r)$, which implies

$$\frac{d}{dr} H_\ell(0) = \frac{1}{\bar{l}_{\mathrm{int}}}\,.$$

Differentiation of (4.1) leads to (5.2) and (5.3).

Acknowledgements. The author thanks Prof. D. Stoyan for the inspiration and for helpful discussions.

References

[1] L. Muche, *Untersuchung von Verteilungseigenschaften des Poisson–Voronoi-Mosaiks*, Technical Report, Freiberg, 1992.

[2] L. Muche and D. Stoyan, *Contact and chord length distributions of the Poisson–Voronoi tessellation*, J. Appl. Probab. 29 (1992), 467–471.

[3] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic Geometry and Its Applications*, Wiley, Chichester, 1987.

LUTZ MUCHE
BERGAKADEMIE FREIBERG
FACHBEREICH MATHEMATIK
D-O-9200 FREIBERG, GERMANY

Received on 2.9.1992