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A MIXED DUEL UNDER ARBITRARY MOTION AND UNCERTAIN EXISTENCE OF THE SHOT

Abstract. The purpose of the paper is to solve a mixed duel in which the numbers of shots given to the players are independent 0-1-valued random variables. The players know their distributions as well as the accuracy function P, the same for both players. It is assumed that the players can move as they like and that the maximal speed of the first player is greater than that of the second player. It is shown that the game has a value, and a pair of optimal strategies is found.

1. Definitions and assumptions. Consider the following *game* $(1,1)$. Two players, say I and II, fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. Players I and II have one bullet each but they can fire the bullets with probability p and q, respectively, $0 < p \leq 1$, $0 < q < 1$. A player knows if he can fire the bullet when he tries to do it. Player II does not hear the shot of Player I, Player I hears the shot of Player II.

At the beginning of the duel the players are at distance 1 from each other. Let $P(s)$ be the probability of succeeding (destroying the opponent) by Player I (II) when the distance between them is $1-s$, $s \leq 1$, and the player can fire his bullet. The function $P(s)$ is called the *accuracy function*. It is assumed that

(i) P is increasing and has a continuous second derivative in $[0,1]$,
(ii) $P(s) = 0$ for $s \leq 0$, $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero-sum game. The game

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is over if at least one of the players succeeds or all bullets are shot. In the other case the duel lasts infinitely long and the payoff is zero. The above facts are known to both players.

Suppose that Player II has had a bullet and fired it. In this case the best what Player I can do, if he has a bullet yet, is to reach Player II in pursuit and to succeed surely provided he can fire his bullet. Since we are looking for optimal strategies we assume this behaviour of Player I in the paper.

Without loss of generality we can suppose that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

For definitions and results in the theory of games of timing see [3]–[8], [10], [15].

2. Auxiliary game of timing. To solve the game $(1,1)$ presented in the previous section we have to determine optimal strategies in the following auxiliary *game* $(1,1)^*$. Consider a silent versus noisy duel with uncertain existence of (at most one) shot and accuracy function $P(s)$, the same for both players. It is assumed that Player I approaches Player II with constant velocity $v = 1$ all the time, even after he has tried to fire his bullet. Player I gains 1 if only he succeeds etc., just as in the duel $(1,1)$ defined in the previous section.

Denote by $K_0(s,t)$ the expected gain of Player I if he tries to fire at time $s \in [0,1]$ and Player II tries to fire at time $t \in [0,1]$. It is assumed that

$$K_0(s,t) = \begin{cases} pP(s) & \text{if } s < t, \\ (p-q)P(s) & \text{if } s = t, \\ -qP(t) + pq(1-P(t)) + p(1-q)P(s) & \text{if } s > t. \end{cases}$$

As is easy to see, $K_0(s,t)$ is the expected payoff in the duel in which Player II is not allowed to fire after Player I has tried to shoot. Player I is allowed to fire after the trial of Player II but he has to act as in the duel $(1,1)$.

Denote by ξ_0^a the strategy of Player I in the game $(1,1)$ in which he tries to fire at a random moment s distributed according to a density $f_1(s)$ in the interval $[a,1)$, $0 < a < 1$, and according to a probability α, $0 < \alpha < 1$, at the point 1. This distribution is chosen in such a way that if $t \in [a,1)$ then

$$(1) \qquad K(\xi_0^a, t) = \int_a^t pP(s) f_1(s)\,ds + \int_t^1 (-qP(t) + pq(1-P(t)) + p(1-q)P(s)) f_1(s)\,ds + (-qP(t) + pq(1-P(t)) + p(1-q))\alpha = \text{const}\,.$$

Here $K(\xi_0^a, t)$ is the expected gain of Player I if he applies the strategy ξ_0^a and Player II tries to fire at time t.

Computing the first and second derivatives of $K(\xi_0^a, t)$ with respect to t and eliminating the integrals from the obtained expressions we obtain

$$\frac{f_1'(t)}{f_1(t)} - \frac{P''(t)}{P'(t)} + \frac{(2+3p)P'(t)}{(1+2p)P(t)-p} = 0\,,$$

the solution of which is

$$f_1(t) = \frac{CP'(t)}{\left(P(t) - \dfrac{p}{1+2p}\right)^E}\,, \tag{2}$$

where

$$E = \frac{2+3p}{1+2p}\,,$$

and C is a constant. Obviously this constant satisfies

$$C\int_a^1 \frac{P'(t)\,dt}{\left(P(t) - \dfrac{p}{1+2p}\right)^E} + \alpha = 1\,. \tag{3}$$

Let η_0^a be the strategy of Player II in the game $(1,1)^*$ in which he chooses at random a moment t to try his shot, according to a density $f_2(t)$ in $[a,1]$, to obtain

$$\begin{aligned} K(s,\eta_0^a) = &\int_a^s (-qP(t) + pq(1-P(t)) + p(1-q)P(s))f_2(t)\,dt \\ &+ \int_s^1 pP(s)f_2(t)\,dt = \text{const} \end{aligned} \tag{4}$$

if $s \in [a,1]$, where $K(s,\eta_0^a)$ is the expected gain of Player I if Player II applies the strategy η_0^a and Player I tries to fire at time s.

In the same way as before we obtain

$$f_2(s) = \frac{DP'(s)}{\left(P(s) - \dfrac{p}{1+2p}\right)^F} \tag{5}$$

where D is a constant and

$$F = \frac{1+3p}{1+2p}\,.$$

Obviously we have

$$D\int_a^1 \frac{P'(s)\,ds}{\left(P(s)-\frac{p}{1+2p}\right)^F}=1\,. \tag{6}$$

Moreover, from (1) and (2) we obtain after computing the integral

$$K(\xi_0^a,t)=C\Bigg[-\frac{1+2p}{\left(P(a)-\frac{p}{1+2p}\right)^{E-2}} + \frac{p^2}{1+p}\frac{1}{\left(P(a)-\frac{p}{1+2p}\right)^{E-1}}+(1-q)(1+2p)\left(\frac{1+2p}{1+p}\right)^{E-1}\Bigg]=\text{const} \tag{7}$$

if

$$\alpha=C\left(\frac{1+2p}{1+p}\right)^E. \tag{8}$$

Similarly, from (4) and (5) we obtain

$$K(s,\eta_0^a)=\frac{D(1+2p)qP(a)}{\left(P(a)-\frac{p}{1+2p}\right)^{F-1}}=\text{const} \tag{9}$$

if

$$\frac{1-q}{\left(P(a)-\frac{p}{1+2p}\right)^{F-1}}=\frac{1}{\left(1-\frac{p}{1+2p}\right)^{F-1}}\,. \tag{10}$$

From (3), (6), (8) and (10) we determine the unknown parameters C, D, a, α; $p/(1+2p)<a<1$, $0<\alpha<1$. It is easy to see that if $0<p\le 1$ and $0<q<1$ this solution always exists and is unique.

We now prove that $K(\xi_0^a,t)=K(s,\eta_0^a)$ for $a\le s\le 1$, $a\le t<1$. Computing the integral in (3) and taking into account the equation (8) we obtain

$$C=\frac{1+p}{1+2p}\left(P(a)-\frac{p}{1+2p}\right)^{E-1}. \tag{11}$$

Moreover, computing the integral in (6) and taking into account (10) we get

$$D=\frac{p}{q(1+2p)}\left(P(a)-\frac{p}{1+2p}\right)^{F-1}.$$

Then from (9) we obtain

$$K(s,\eta_0^a)=pP(a)\,. \tag{12}$$

Putting $K(\xi_0^a, t) = K(s, \eta_0^a)$ given by (12) and (7) with C given by (11) we come to the equation

$$(13)\qquad (1-q)\left(P(a) - \frac{p}{1+2p}\right)^{E-2} = \left(1 - \frac{p}{1+2p}\right)^{E-2}.$$

Dividing (10) by (13) and taking into account that $E + F - 3 = 0$ we obtain an identity. Thus $K(\xi_0^a, t) = K(s, \eta_0^a) = pP(a)$ for $a \le s \le 1$, $a \le t < 1$, $0 < p \le 1$, $0 < q < 1$.

LEMMA. *For a being the solution of* (10) *the strategy ξ_0^a is maximin and the strategy η_0^a is minimax in the game* $(1,1)^*$. *The value of the game is $v_{11}^0 = pP(a)$.*

The proof is similar to that in [14] and is omitted.

3. Solution of the duel $(1,1)$**.** We now consider the duel $(1,1)$ defined at the beginning of the paper. For given natural n such that $1/n \le 1-\alpha$ let the constants a_k be defined as follows:

$$a_0 = a\,, \qquad \int\limits_{a_{k-1}}^{a_k} f_1(s)\,ds = \frac{1}{n}\,, \quad k = 1, \ldots, n_0\,, \quad a_{n_0+1} = 1\,,$$

where n_0 is defined from the inequalities $1 - \alpha - 1/n \le n_0/n < 1 - \alpha$.

Define the strategy ξ_ε of Player I in the game $(1,1)$ as follows: Player I moves back and forth with maximal speed in the following manner: at first between 0 and a_1, then between 0 and $a_2, \ldots$, finally between 0 and a_{n_0+1}. At the kth step, $k = 1, \ldots, n_0 + 1$, he can try to fire his shot at random only if he is between the points a_{k-1} and a_k and goes forward, and he tries to fire it with probability density $f_1(s)$. If he has tried it at the kth step, he reaches the point a_k, escapes to 0 and never approaches Player II. If Player I has not tried to fire between points 0 and 1 and survives, he tries when he is at 1, as soon as possible.

The strategy η_0 of Player II is defined as follows: If Player I reaches the point t the first time and his velocity is $v_1(\tau)$, τ being the time, try to fire at random with density $v_1(\tau) f_2(t(\tau))$. Otherwise do not try.

It is assumed that the function $v_1(\tau)$ is piecewise continuous.

THEOREM. *The strategy ξ_ε is ε-maximin and the strategy η_0 is minimax in the game* $(1,1)$. *The value of the game is $v_{11} = pP(a)$ where a is the solution of the equation* (10).

The proof of the Theorem is similar to that in [14] and is omitted.

It is easy to see that there exist ε-maximin strategies of Player I in which he moves with not necessarily maximal speed.

Duels under arbitrary motion, as far as the author knows, have never been considered before except in the papers of the author (see [11]–[14]).

For other results in the theory of duels with uncertain existence of the shots see [1], [2], [9].

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