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BESSEL MATRIX DIFFERENTIAL EQUATIONS: EXPLICIT SOLUTIONS OF INITIAL AND TWO-POINT BOUNDARY VALUE PROBLEMS

Abstract. In this paper we consider Bessel equations of the type $t^2X^{(2)}(t) + tX^{(1)}(t) + (t^2I - A^2)X(t) = 0$, where A is an $n \times n$ complex matrix and $X(t)$ is an $n \times m$ matrix for $t > 0$. Following the ideas of the scalar case we introduce the concept of a fundamental set of solutions for the above equation expressed in terms of the data dimension. This concept allows us to give an explicit closed form solution of initial and two-point boundary value problems related to the Bessel equation.

1. Introduction. Numerous problems from chemistry, physics and mechanics, both linear and nonlinear, are related to matrix differential equations of the type $t^2X^{(2)}(t) + tA(t)X^{(1)}(t) + B(t)X(t) = 0$, where $A(t)$, $B(t)$ are matrix-valued functions [8], [10]. This paper is concerned with the Bessel matrix equation

$$t^2X^{(2)}(t) + tX^{(1)}(t) + (t^2I - A^2)X(t) = 0\,, \qquad t > 0\,, \tag{1.1}$$

where A is a matrix in $\mathbb{C}_{n\times n}$, and $X(t)$ is a matrix in $\mathbb{C}_{n\times n}$, for $t > 0$. Note that the matrix problem (1.1) may be regarded as a system of coupled Bessel type equations that cannot be transformed into a set of independent equations if the matrix A is not diagonalizable. Standard techniques to study problems related to (1.1) are based on the consideration of the extended first order system

$$tZ'(t) = M(t)Z(t)$$

where

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$$M(t) = \begin{bmatrix} 0 & I \\ -t^2 I + A^2 & 0 \end{bmatrix}, \qquad Z(t) = \begin{bmatrix} X(t) \\ tX'(t) \end{bmatrix}. \tag{1.2}$$

Then series solutions for (1.2) may be obtained, and the relationship between the solutions $X(t)$ of (1.1) and $Z(t)$ of (1.2) is given by $X(t) = [I, 0]Z(t)$ (see [4], [13] for details). This technique has two basic drawbacks, first of all it involves an increase of the problem dimension and a lack of explicitness derived from the relationship $X(t) = [I, 0]Z(t)$. Secondly, unlike the scalar case it does not provide a pair of solutions of (1.1) which would allow us to give a closed form of the general solution of (1.1) involving a pair of parameters.

This paper is organized as follows. Section 2 is concerned with some preliminaries that will be used in the following sections. In Section 3 we construct series solutions of problem (1.1) and we propose a closed form of the general solution of (1.1) for the case where the matrix A satisfies the spectral condition

(1.3) For every eigenvalue $z \in \sigma(A)$, $2z$ is not an integer, and if z, w belong to $\sigma(A)$ and $z \neq w$, then $z \pm w$ is not an integer.

Here $\sigma(A)$ denotes the set of all eigenvalues of A. Finally, in Section 4 we study the boundary value problem

$$\begin{aligned} &t^2 X^{(2)}(t) + tX^{(1)}(t) + (t^2 I - A^2)X(t) = 0, \qquad 0 < a \le t \le b, \\ &M_{11}X(a) + N_{11}X(b) + M_{12}X^{(1)}(a) + N_{12}X^{(1)}(b) = 0, \\ &M_{21}X(a) + N_{21}X(b) + M_{22}X^{(1)}(a) + N_{22}X^{(1)}(b) = 0, \end{aligned} \tag{1.4}$$

where M_{ij}, N_{ij}, for $1 \le i, j \le 2$, are matrices in $\mathbb{C}_{n\times n}$.

If S is a matrix in $\mathbb{C}_{m\times n}$, we denote by S^+ its Moore–Penrose pseudoinverse and we recall that an account of uses and properties of this concept may be found in [1].

2. Preliminaries. We begin this section with an algebraic result that provides a finite expression for the solution of a generalized algebraic Lyapunov matrix equation

$$A_1 + B_1 X - X D_1 = 0 \tag{2.1}$$

where A_1, B_1, D_1 and the unknown X are matrices in $\mathbb{C}_{n\times n}$.

LEMMA 1. *Suppose that matrices B_1 and D_1 satisfy the spectral condition*

$$\sigma(B_1) \cap \sigma(D_1) = \emptyset \tag{2.2}$$

and let $p(z) = \sum_{k=0}^{n} a_k z^k$ be such that $p(B_1) = 0$. Then the only solution

X of equation (2.1) *is given by*

$$X = \Big(\sum_{j=1}^{n}\sum_{h=1}^{j} a_j B_1^{h-1} A_1 D_1^{j-h}\Big)\Big(\sum_{j=0}^{n} a_j D_1^j\Big)^{-1}. \tag{2.3}$$

Proof. Under the hypothesis (2.2), equation (2.1) has only one solution [2], [12], and from Corollary 2 of [2], if X is the only solution of (2.1), it follows that

$$\begin{aligned} V &= \begin{bmatrix} B_1 & A_1 \\ 0 & D_1 \end{bmatrix} = W \begin{bmatrix} B_1 & 0 \\ 0 & D_1 \end{bmatrix} W^{-1}, \\ W &= \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}. \end{aligned} \tag{2.4}$$

From (2.4), it follows that

$$\begin{aligned} p(V) &= Wp\left(\begin{bmatrix} B_1 & 0 \\ 0 & D_1 \end{bmatrix}\right) W^{-1} \\ &= W \begin{bmatrix} p(B_1) & 0 \\ 0 & p(D_1) \end{bmatrix} W^{-1} = \begin{bmatrix} 0 & Xp(D_1) \\ 0 & p(D_1) \end{bmatrix} \end{aligned} \tag{2.5}$$

and taking into account the polynomial calculus there exists a matrix M such that

$$p(V) = p\left(\begin{bmatrix} B_1 & A_1 \\ 0 & D_1 \end{bmatrix}\right) = \begin{bmatrix} p(B_1) & M \\ 0 & p(D_1) \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & p(D_1) \end{bmatrix}. \tag{2.6}$$

From (2.5) and (2.6) one sees that $Xp(D_1) = M$ and from the spectral mapping theorem [3, p. 569] and (2.2), the matrix $p(D_1)$ is invertible. Thus we have $X = M(p(D_1))^{-1}$. On the other hand, considering the powers V^j, $j = 0, 1, \ldots, n$, one finds that the $(i, 2)$ block entry of the operator V^j, denoted by $V_{i,2}^j$ for $j = 1, \ldots, n$, $i = 1, 2$, satisfies

$$V_{1,2}^j = B_1 V_{1,2}^{j-1} + A_1 V_{2,2}^{j-1}, \quad V_{2,2}^j = D_1^j, \quad V_{1,2}^0 = 0, \quad V_{2,2}^0 = I.$$

By multiplying the matrix $V_{1,2}^j$ by the coefficient a_j for $j = 0, 1, \ldots, n$ and by addition it follows that the block entry (1, 2) of the block matrix $p(V)$ is given by the expression

$$M = \sum_{j=1}^{n}\sum_{h=1}^{j} a_j B_1^{h-1} A_1 D_1^{j-h}.$$

Hence the result is established.

In accordance with the definition given in [6] for a time invariant regular second order matrix differential equation, we introduce the concept of a fundamental set of solutions for equations of the type

$$Y^{(2)}(t) + P(t)Y^{(1)}(t) + Q(t)Y(t) = 0. \tag{2.7}$$

Definition 1. Consider equation (2.7) where $P(t)$, $Q(t)$ are continuous $\mathbb{C}_{n\times n}$-valued functions on an interval J of the real line, and $Y(t) \in \mathbb{C}_{n\times n}$. We say that a pair of solutions $\{Y_1, Y_2\}$ is a *fundamental set of solutions* of (2.7) in the interval J if for any solution Z of (2.7) defined in J, there exist matrices $C, D \in \mathbb{C}_{n\times n}$, uniquely determined by Z, such that

$$Z(t) = Y_1(t)C + Y_2(t)D\,, \qquad t \in J\,. \tag{2.8}$$

The following result provides a useful characterization of a fundamental set of solutions of (2.7) and it may be regarded as an analogue of Liouville's formula for the scalar case.

Lemma 2. *Let* $\{Y_1, Y_2\}$ *be a pair of solutions of* (2.7) *defined on the interval* J *and let* $W(t)$ *be the block matrix function*

$$W(t) = \begin{bmatrix} Y_1(t) & Y_2(t) \\ Y_1^{(1)}(t) & Y_2^{(1)}(t) \end{bmatrix}. \tag{2.9}$$

Then $\{Y_1, Y_2\}$ *is a fundamental set solutions of* (2.7) *on* J *if there exists a point* $t_1 \in J$ *such that* $W(t_1)$ *is nonsingular in* $\mathbb{C}_{2n\times 2n}$. *In this case* $W(t)$ *is nonsingular for all* $t \in J$.

Proof. Since $Y_1(t)$ and $Y_2(t)$ are solutions of (2.7), it follows that $W(t)$ defined by (2.9) satisfies

$$W^{(1)}(t) = \begin{bmatrix} 0 & I \\ -Q(t) & -P(t) \end{bmatrix} W(t)\,, \qquad t \in J\,. \tag{2.10}$$

Thus if $G(t, s)$ is the transition state matrix of (2.10) such that $G(t, t) = I$ [7, p. 598], it follows that $W(t) = G(t, t_1)W(t_1)$ for all $t \in J$. Hence the result is established because $G(t, s)$ is invertible for all t, s in J.

Note that in the interval $0 < t < \infty$, equation (1.1) takes the form (2.7) with $P(t) = I/t$ and $Q(t) = I - (A/t)^2$.

We conclude this section with some recalls concerned with the reciprocal gamma function that may be found in [4, p. 253]. The reciprocal gamma function, denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z, and thus for any matrix $C \in \mathbb{C}_{n\times n}$, the Riesz–Dunford functional calculus shows that $\Gamma^{-1}(C)$ is a well defined matrix (see Chapter 7 of [3]). If C is a matrix in $\mathbb{C}_{n\times n}$ such that

$$C + nI \text{ is invertible for all integer } n \geq 0 \tag{2.11}$$

then from [4, p. 253], it follows that

$$C(C + I)(C + 2I)\ldots(C + nI)\Gamma^{-1}(C + (n + 1)I) = \Gamma^{-1}(C)\,. \tag{2.12}$$

Under the condition (2.11), $\Gamma(C)$ is well defined and it is the inverse matrix of $\Gamma^{-1}(C)$. From the properties of the functional calculus $\Gamma^{-1}(C)$ commutes with C and from [3, p. 557], $\Gamma(C)$ and $\Gamma^{-1}(C)$ are polynomials in C.

In particular, if C is a matrix satisfying (2.11), and $\operatorname{Re}(z) > 0$ for every eigenvalue $z \in \sigma(C)$, then we have

$$\Gamma(C) = \int_0^\infty \exp(-t)\exp((C-I)\ln t)\,dt \tag{2.13}$$

and this representation of $\Gamma(C)$ coincides with the power series expansion, the Riesz–Dunford formula for $\Gamma(C)$ [3, p. 555] and others (see [4, p. 253]). Note that if C satisfies (2.11), from the previous comments and (2.12) we have

$$\Gamma(C+(n+I)) = C(C+I)(C+2I)\ldots(C+nI)\Gamma(C)\,. \tag{2.14}$$

Note that from (2.13) and (2.14), for matrices C satisfying (2.11) the computation of $\Gamma(C)$ may be performed in an analogous way to the scalar case.

3. Bessel matrix differential equations. Suppose that we are looking for solutions of equation (1.1) of the form

$$X(t) = \Big(\sum_{k\geq 0} C_k t^k\Big) t^Z \tag{3.1}$$

where C_k is a matrix in $\mathbb{C}_{n\times n}$, $Z \in \mathbb{C}_{n\times n}$ and $t^Z = \exp(Z\ln t)$, for $t > 0$.

By taking formal derivatives in (3.1), it follows that

$$\begin{aligned} X^{(1)}(t) &= \sum_{k\geq 0} C_k(kI+Z)t^{Z+(k-1)I}\,,\\ X^{(2)}(t) &= \sum_{k\geq 0} C_k(kI+Z)(kI+Z-I)t^{Z+(k-2)I}\,. \end{aligned} \tag{3.2}$$

Assuming the convergence of the series (3.1), (3.2), and substituting into equation (1.1), it follows that

$$\begin{aligned} \Big\{\sum_{k\geq 0}&[C_k(kI+Z)(kI+Z-I) + C_k(kI+Z) - A^2C_k]t^k \\ &+ \sum_{k\geq 2} C_{k-2}t^k\Big\} t^Z = 0\,. \end{aligned} \tag{3.3}$$

By equating to the zero matrix the coefficient of each power t^k appearing in (3.3), it follows that the matrices C_k must satisfy

$$C_0Z(Z-I) + C_0Z - A^2C_0 = C_0Z^2 - A^2C_0 = 0\,, \tag{3.4}$$

$$C_1(Z+I)Z + C_1(Z+I) - A^2C_1 = C_1(Z+I)^2 - A^2C_1 = 0\,, \tag{3.5}$$

$$C_k(kI+Z)^2 - A^2C_k = -C_{k-2}\,, \quad k\geq 2\,. \tag{3.6}$$

Let Z be a matrix in $\mathbb{C}_{n\times n}$ and let C_0 be an invertible matrix in $\mathbb{C}_{n\times n}$ such that

$$Z = C_0^{-1} A C_0 \,. \tag{3.7}$$

Then

$$\sigma(A) = \sigma(Z)\,, \quad Z^2 = C_0^{-1} A^2 C_0\,, \quad C_0 Z^2 - A^2 C_0 = 0\,. \tag{3.8}$$

Given the matrix Z defined by (3.7), from (1.3) and Lemma 1, the only solution for C_1 of the matrix equation

$$C_1(Z+I)^2 - A^2 C_1 = 0$$

is the zero matrix $C_1 = 0$. From (3.6) it follows that $C_{2m+1} = 0$ for $m \geq 0$. In order to determine the matrix coefficients C_{2m}, let $p(z)$ be an annihilating polynomial of the matrix A^2,

$$p(z) = \sum_{j=0}^{n} a_j z^j\,, \qquad p(A^2) = 0\,. \tag{3.9}$$

Under the hypothesis (1.3) it follows that $\sigma((kI+Z)^2) \cap \sigma(A^2) = \emptyset$ for $k \geq 1$, and from Lemma 1, the only solution C_{2m} of the equation

$$A^2 C_{2m} - C_{2m}(2mI+Z)^2 = C_{2m-2}\,, \qquad m \geq 1\,, \tag{3.10}$$

is given by

$$C_{2m} = -\Big(\sum_{j=1}^{n} \sum_{h=1}^{j} a_j A^{2h-2} C_{2m-2} (2mI+Z)^{2(j-h)}\Big)\Big(\sum_{j=1}^{n} a_j (2mI+Z)^{2j}\Big)^{-1}. \tag{3.11}$$

Note that once we choose the matrices C_0 and Z, all the matrix coefficients C_{2m} for $m \geq 1$ are determined by (3.11).

Now we are concerned with the proof of the convergence of the series

$$X(t, Z, C_0) = \Big(\sum_{m \geq 0} C_{2m} t^{2m}\Big) t^Z\,. \tag{3.12}$$

The generalized power series (3.12) is convergent for $t > 0$ if the power series

$$U(t, Z, C_0) = \sum_{m \geq 0} C_{2m} t^{2m}\,, \tag{3.13}$$

is convergent for $t > 0$.

If B is a matrix in $\mathbb{C}_{n\times n}$ and B^H denotes the conjugate transpose of B, we denote by $\|B\|$ its spectral norm, defined to be the maximum of the set $\{|z|^{1/2} : z \in \sigma(B^H H)\}$. Taking norms in (3.10), for large values of m it follows that

$$\begin{aligned} \|C_{2m-2}\| &= \|C_{2m}(2mI+Z)^2 - A^2 C_{2m}\| \\ &\geq \big|\, \|C_{2m}(2mI+Z)^2\| - \|A^2 C_{2m}\| \,\big| \end{aligned} \tag{3.14}$$

$$\geq \|C_{2m}\|(4m^2 - 4m\|Z\| - \|Z^2\| - \|A^2\|)\,.$$

Hence

$$\frac{\|C_{2m}\||t|^{2m}}{\|C_{2m-2}\||t|^{2m-2}} \leq \frac{|t|^2}{4m^2 - 4m\|Z\| - \|Z^2\| - \|A^2\|}$$

and this proves the absolute convergence of the series (3.13) for $t > 0$.

Now we are going to find a second solution of (1.1) of the form

$$X(t, -Z, C_0) = \Big(\sum_{k\geq 0} C_k^* t^k\Big) t^{-Z} = U(t, -Z, C_0) t^{-Z} \tag{3.15}$$

where C_0 is the matrix satisfying (3.7). In an analogous way to the construction of $X(t, Z, C_0)$, it is straightforward to show that the matrices C_k^* appearing in (3.15), for $k \geq 0$, with $C_0^* = C_0$, must satisfy the equations

$$\begin{gathered} C_0^* Z^2 - A^2 C_0^* = 0\,, \quad C_1^*(I - Z)^2 - A^2 C_1^* = 0\,, \\ C_k^*(kI - Z)^2 - A^2 C_k^* = -C_{k-2}^*\,, \quad k \geq 2\,. \end{gathered} \tag{3.16}$$

From the hyphothesis (1.3), (3.16) and Lemma 1, it follows that $C_1^* = C_{2m+1}^* = 0$, and, for $m \geq 1$,

$$\begin{aligned} &C_{2m}^* \\ &= \Big(\sum_{j=1}^{n} \sum_{h=1}^{j} a_j A^{2h-2} C_{2m-2}^* (2mI - Z)^{2(j-h)}\Big)\Big(\sum_{j=0}^{n} a_j (2mI - Z)^{2j}\Big)^{-1}. \end{aligned} \tag{3.17}$$

The proof of the absolute convergence of the series

$$U(t, -Z, C_0) = \sum_{m\geq 0} C_{2m}^* t^{2m} \tag{3.18}$$

for $t > 0$ is analogous to the previous proof for $U(t, Z, C_0)$.

Now we are going to prove that for any invertible matrices C_0 and Z satisfying (3.7), the pair defined by $X(t, Z, C_0)$ and $X(t, -Z, C_0)$ is a fundamental set of solutions of (1.1) in $0 < t < \infty$. The Wroński block matrix function associated with this pair and defined by (2.9) takes the form

$$\begin{aligned} &W(t) \\ &= \begin{bmatrix} U(t, Z, C_0) t^Z & U(t, -Z, C_0) t^{-Z} \\ U^{(1)}(t, Z, C_0) t^Z + U(t, Z, C_0) Z t^{Z-I} & U^{(1)}(t, -Z, C_0) t^{-Z} - U(t, -Z, C_0) Z t^{-Z-I} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & t^{-1} I \end{bmatrix} T(t) \begin{bmatrix} t^Z & 0 \\ 0 & t^{-Z} \end{bmatrix} \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} &T(t) \\ &= \begin{bmatrix} U(t, Z, C_0) & U(t, -Z, C_0) \\ U^{(1)}(t, Z, C_0) t + U(t, Z, C_0) Z & U^{(1)}(t, -Z, C_0) t - U(t, -Z, C_0) Z \end{bmatrix}. \end{aligned} \tag{3.20}$$

From (3.19) it is clear that $W(t)$ is invertible if and only if $T(t)$ is invertible. Note that $T(t)$ is a continuous $\mathbb{C}_{2n\times 2n}$-valued function defined in the interval $[0,\infty)$. Since $T(0)$ is the matrix

$$T(0) = \begin{bmatrix} C_0 & C_0 \\ C_0 Z & -C_0 Z \end{bmatrix},$$

it is invertible because of the invertibility of C_0, Lemma 1 of [5] and the fact that

$$-C_0 Z - (C_0 Z) C_0^{-1} C_0 = -2C_0 Z \quad \text{is invertible}.$$

From the invertibility of $T(0)$ and the Perturbation Lemma [9, p. 32], there exists a positive number t_1 such that $T(t)$ is invertible in $[0, t_1]$. This proves the invertibility of $W(t_1)$ and from Lemma 2 the pair $\{X(\cdot, Z, C_0), X(\cdot, -Z, C_0)\}$ is a fundamental set of solutions of equation (1.1) in $0 < t < \infty$. From the previous comments the following result has been proved:

THEOREM 1. *Let C_0 and Z be invertible matrices in $\mathbb{C}_{n\times n}$ and let A be a matrix in $\mathbb{C}_{n\times n}$ satisfying* (1.3). *Then the pair $\{X(\cdot, Z, C_0), X(\cdot, -Z, C_0)\}$ defined by* (3.11), (3.12), (3.15), (3.17), (3.18) *is a fundamental set of solutions of the Bessel equation* (1.1) *in $0 < t < \infty$. The general solution of* (1.1) *in $0 < t < \infty$ is given by*

$$X(t) = X(t, Z, C_0)P + X(t, -Z, C_0)Q, \qquad P, Q \in \mathbb{C}_{n\times n}. \tag{3.21}$$

The unique solution of (1.1) *satisfying the initial conditions $X(a) = E$, $X^{(1)}(a) = F$, with $0 < a < \infty$, is given by* (3.21) *where*

$$\begin{bmatrix} P \\ Q \end{bmatrix} = (W(a))^{-1} \begin{bmatrix} E \\ F \end{bmatrix}$$

and $W(a)$ is defined by (3.19).

Remark 1. If we consider the Bessel equation (1.1) with vector-valued unknown $X(t)$, then considering the fundamental set of solutions constructed in Theorem 1, the general solution of the vector problem (1.1) is given by (3.21) upon replacing the matrices P, Q, by arbitrary vectors P, Q in $\mathbb{C}_{n\times 1}$.

Now we are interested in showing that for the case where the matrix A is diagonalizable and satisfies (1.3), the fundamental set of solutions constructed in Theorem 1 coincides with the well known one for the scalar case when $n = 1$, given in terms of the Bessel functions of the first kind.

Let A be a diagonalizable matrix satisfying (1.3) and let C_0 be a basis of $\mathbb{C}_{n\times 1}$ composed of eigenvectors of A. If $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, and $Z = \operatorname{diag}(\lambda_s : 1 \le s \le n)$, then we have

$$Z = C_0^{-1} A C_0.$$

On the other hand, if we denote by $B^{(i)}$ the ith column of the matrix $B \in \mathbb{C}_{n\times n}$, taking the ith column in both members of equation (3.6), it follows that

$$((k+\lambda_s)^2 I - A^2)C_k^{(s)} = -C_{k-2}^{(s)}\,, \quad 1 \le s \le n\,,\ k \ge 2\,. \tag{3.22}$$

Note that we may write the matrix $(m+\frac{1}{2}\lambda_s)^2 I - A^2$ in the form

$$\begin{aligned}(m+\tfrac{1}{2}\lambda_s)^2 I - A^2 &= ((m+\tfrac{1}{2}\lambda_s)I + A)((m+\tfrac{1}{2}\lambda_s)I - A \\ &= (mI + \tfrac{1}{2}(\lambda_s I + A))(mI + \tfrac{1}{2}(\lambda_s I - A)) \\ &= (mI + B_s)(mI + D_s)\,, \\ B_s = \tfrac{1}{2}(\lambda_s I + A)\,, &\quad D_s = \tfrac{1}{2}(\lambda_s I - A)\,.\end{aligned} \tag{3.23}$$

Considering (3.22) for even integers $k = 2m$, we have

$$\begin{aligned} C_{2m}^{(s)} &= \frac{(-1)^m}{2^{2m}} \prod_{j=1}^{m} ((j+\tfrac{1}{2}\lambda_s)^2 I - A^2)^{-1} C_0^{(s)} \\ &= \frac{(-1)^m}{2^{2m}} \prod_{j=1}^{m} (jI + D_s)^{-1}(jI + B_s)^{-1} C_0^{(s)}\,, \quad 1 \le s \le n\,. \end{aligned} \tag{3.24}$$

Now consider the new basis of eigenvectors of A defined by the matrix K_0 whose sth column is given by

$$C_0^{(s)} = 2^{\lambda_s} \Gamma(D_s + I)\Gamma(B_s + I) K_0^{(s)}\,, \quad 1 \le s \le n\,. \tag{3.25}$$

Note that from (1.3) and (3.23), the matrices $\Gamma(D_s + I)$ and $\Gamma(B_s + I)$ are invertible and commute with A. This proves that the columns of K_0 define a basis of eigenvectors of A satisfying

$$Z = K_0^{-1} A K_0\,. \tag{3.26}$$

The corresponding equations (3.24) for $K_{2m}^{(s)}$ satisfy

$$K_{2m}^{(s)} = \frac{(-1)^m}{2^{2m}} \prod_{j=1}^{m} (jI+D_s)^{-1}(jI+B_s)^{-1}\Gamma^{-1}(B_s+I)\Gamma^{-1}(D_s+I) C_0^{(s)} 2^{-\lambda_s}\,.$$

Taking into account (2.14) and the fact that $jI + B_s$ and $jI + D_s$ commute shows that

$$K_{2m}^{(s)} = \frac{(-1)^m}{2^{2m}} \Gamma^{-1}(D_s+(m+1)I)\Gamma^{-1}(B_s+(m+1)I) C_0^{(s)} 2^{-\lambda_s}\,, \quad 1 \le s \le n.$$

In matrix form the above expression may be written as

$$\begin{aligned} K_{2m} &= \frac{(-1)^m}{2^{2m}} L_{2m} 2^{-Z}\,, \\ L_{2m}^{(s)} &= \Gamma^{-1}(D_s + (m+1)I)\Gamma^{-1}(B_s + (m+1)I) C_0^{(s)}\,, \end{aligned} \tag{3.27}$$

and $X(t, Z, K_0)$ takes the form

$$X(t, Z, K_0) = \Big(\sum_{m\geq 0} K_{2m} t^{2m}\Big) t^Z = \Big(\sum_{m\geq 0} \frac{(-1)^m}{2^{2m}} L_{2m} t^{2m}\Big)(t/2)^Z\,, \qquad t > 0\,. \tag{3.28}$$

In an analogous way, if we denote by B_s^* and D_s^* the matrices

$$\begin{aligned} B_s^* &= \tfrac{1}{2}(A - \lambda_s I)\,, \\ D_s^* &= \tfrac{1}{2}(-A - \lambda_s I)\,, \end{aligned} \qquad 1 \leq s \leq n\,, \tag{3.29}$$

and

$$\begin{gathered} K_{2m}^* = \frac{(-1)^m}{2^{2m}} L_{2m}^* 2^Z\,, \\ L_{2m}^{*(s)} = \Gamma^{-1}(D_s^* + (m+1)I)\Gamma^{-1}(B_s^* + (m+1)I)C_0^{(s)}\,, \end{gathered} \tag{3.30}$$

then

$$X(t, -Z, K_0) = \Big(\sum_{m\geq 0} K_{2m}^* t^{2m}\Big) t^{-Z} = \Big(\sum_{m\geq 0} \frac{(-1)^m}{2^{2m}} L_{2m}^* t^{2m}\Big)(t/2)^{-Z}\,, \qquad t > 0\,. \tag{3.31}$$

Thus for the case where A is diagonalizable and $\sigma(A) = \{\lambda_s : 1 \leq s \leq n\}$, Theorem 1 provides the fundamental set of solutions in $0 < t < \infty$, defined by $X(\cdot, Z, K_0)$ and $X(\cdot, -Z, K_0)$.

Now we show that for the scalar case, when $A = \nu$ is a complex number such that 2ν is not an integer, which is the condition (1.3) for the case $n = 1$, the fundamental set of solutions of (1.1) given by (3.28) and (3.31) coincides with the Bessel functions of the first kind $J_\nu(x)$ and $J_{-\nu}(x)$, respectively. Note that for the scalar case we have

$$\begin{gathered} A = Z = \nu\,, \qquad C_0 = 1\,, \\ B_1 = \tfrac{1}{2}(\nu + \nu) = \nu\,, \qquad B_1^* = \tfrac{1}{2}(\nu - \nu) = 0\,, \\ D_1 = \tfrac{1}{2}(\nu - \nu) = 0\,, \qquad D_1^* = \tfrac{1}{2}(-\nu - \nu) = -\nu\,, \\ \Gamma^{-1}(B_1 + (m+1)I) = \Gamma^{-1}(\nu + m + 1)\,, \\ \Gamma^{-1}(D_1 + (m+1)I) = \Gamma^{-1}(m+1) = 1/m!\,, \\ L_{2m} = \frac{1}{m!\Gamma(\nu + m + 1)}\,, \\ L_{2m}^* = \frac{1}{m!\Gamma(-\nu + m + 1)}\,, \end{gathered}$$

$$
(3.32)\qquad
\begin{aligned}
K_0 &= K_0^{(1)} = 2^{-Z}\Gamma^{-1}(B_1+I)\Gamma^{-1}(D_1+I)\\
&= 2^{-\nu}\Gamma^{-1}(\nu+1)0! = 2^{-\nu}\Gamma^{-1}(\nu+1)\,,\\
K_0^* &= K_0^{*(1)} = 2^{Z}\Gamma^{-1}(B_1^*+I)\Gamma^{-1}(D_1^*+I)\\
&= 2^{\nu}\Gamma^{-1}(1)\Gamma^{-1}(-\nu+1) = 2^{\nu}\Gamma^{-1}(-\nu+1)\,.
\end{aligned}
$$

Hence for the scalar case with $A=\nu$ such that 2ν is not an integer, taking K_0 and K_0^* defined by (3.32), it follows that the fundamental set of solutions of (1.1) given by (3.28), (3.31) is

$$X(t,\nu,K_0) = J_\nu(t)\,,\qquad X(t,-\nu,K_0) = J_{-\nu}(t)\,,\qquad t>0\,,$$

where $J_\nu(t)$ and $J_{-\nu}(t)$ denote the Bessel functions of the first kind of order ν.

4. Boundary value problems. Under the hypotheses and notation of Section 3, let $X(t,Z,C_0)$, $X(t,-Z,C_0)$ be a fundamental set of solutions of (1.1), constructed for matrices Z and C_0 satisfying (3.7). Taking into account the expression (3.21) for the general solution of (1.1) in $t>0$, its derivative is

$$
(4.1)\qquad
\begin{aligned}
X^{(1)}(t) &= X^{(1)}(t,Z,C_0)P + X^{(1)}(t,-Z,C_0)Q\\
&= (U^{(1)}(t,Z,C_0)t^Z + U(t,Z,C_0)Zt^{Z-I})P\\
&\quad + (U^{(1)}(t,-Z,C_0)t^{-Z} - U(t,-Z,C_0)Zt^{-Z-I})Q\,,
\end{aligned}
$$

where $U(t,Z,C_0)$, $U(t,-Z,C_0)$ are defined by (3.13) and (3.18), respectively, and P, Q are arbitrary matrices in $\mathbb{C}_{n\times n}$.

If we impose on the general solution $X(t)$ of (1.1), described by (3.21), the boundary value conditions of (1.4), then from (3.21) and (4.1), it follows that problem (1.4) is solvable if and only if the algebraic system

$$(4.2)\qquad S\begin{bmatrix}P\\Q\end{bmatrix}=0$$

is compatible, where $S=(S_{ij})_{1\le i,\,j\le 2}$ is the block matrix whose entries are

$$
(4.3)\qquad
\begin{aligned}
S_{i1} &= M_{i1}U(a,Z,C_0)a^Z + N_{i1}U(b,-Z,C_0)b^Z\\
&\quad + M_{i2}(U^{(1)}(a,Z,C_0)a^Z + U(a,Z,C_0)Za^{Z-I})\\
&\quad + N_{i2}(U^{(1)}(b,Z,C_0)b^Z + U(b,Z,C_0)Zb^{Z-I})\,,\qquad i=1,2,
\end{aligned}
$$

$$
(4.4)\qquad
\begin{aligned}
S_{i2} &= M_{i1}U(a,-Z,C_0)a^{-Z} + N_{i1}U(b,-Z,C_0)b^{-Z}\\
&\quad + M_{i2}(U^{(1)}(a,-Z,C_0)a^{-Z} - U(a,-Z,C_0)Za^{-Z-I})\\
&\quad + N_{i2}(U^{(1)}(b,-Z,C_0)b^{-Z} - U(b,-Z,C_0)Zb^{-Z-I})\,,\quad i=1,2.
\end{aligned}
$$

Thus the boundary value problem (4.1) is solvable if and only if the matrix S defined by (4.3)–(4.4) is singular. Under this condition, from Theorem 2.3.2

of [11, p. 24], the general solution of the algebraic system (4.2) is given by

$$\begin{bmatrix} P \\ Q \end{bmatrix} = S^{+}SG\,, \qquad G \in \mathbb{C}_{2n\times n}\,. \tag{4.5}$$

Hence the general solution of problem (1.3), under the hypothesis of singularity for the matrix S, is given by (3.21) where the matrices P, Q are given by (4.5) for an arbitrary matrix G in $\mathbb{C}_{2n\times n}$.

Hence the following result has been established:

THEOREM 2. *Under the hypotheses and notation of Theorem* 1, *let S be the block matrix defined by* (4.3)–(4.4) *and associated with the fundamental set* $\{X(\cdot, Z, C_0), X(\cdot, -Z, C_0)\}$. *Then the boundary value problem* (1.3) *is solvable if and only if S is singular. Under this condition the general solution of* (1.3) *is given by* (3.21), *where P, Q are matrices in $\mathbb{C}_{n\times n}$ given by* (4.5).

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