APPLICATIONES MATHEMATICAE
22,1 (1993), pp. 103–115

A. L. RUKHIN (Baltimore)

ESTIMATING NORMAL DENSITY AND NORMAL DISTRIBUTION FUNCTION: IS KOLMOGOROV'S ESTIMATOR ADMISSIBLE?

Abstract. The statistical estimation problem of the normal distribution function and of the density at a point is considered. The traditional unbiased estimators are shown to have Bayes nature and admissibility of related generalized Bayes procedures is proved. Also inadmissibility of the unbiased density estimator is demonstrated.

1. Introduction. In this paper we study the classical statistical estimation problems of the normal distribution function and of the normal density evaluated at a given point. Our main goal is to investigate the admissibility condition in this problem for generalized conjugate priors.

Estimation of the normal distribution function and of the normal density is discussed in [14], Examples 3.1, 3.2, 3.8 and 3.10.

Let $x_1, \ldots, x_n$ be a normal random sample with an unknown mean ξ and an unknown standard deviation σ. Clearly (X, S), where

$$X = \sum_{i=1}^{n} x_i/\sqrt{n}, \qquad S^2 = \sum_{j=1}^{n} (x_j - X/\sqrt{n})^2 ,$$

is a version of the complete sufficient statistic such that X and S are independent; the distribution of X is normal, say $N(\mu, \sigma), \mu = \sqrt{n}\xi$ and S^2/σ^2 has χ^2-distribution with $n-1$ degrees of freedom. For a given x_0 on the basis of observed X and S^2 one has to find a good estimator of the distribution

1991 *Mathematics Subject Classification*: Primary 62C15; Secondary 62C10, 62F10, 62N05.

Key words and phrases: admissibility, Bayes estimator, normal density, normal distribution function, point estimation, quadratic loss.

function evaluated at x_0,

$$\theta(\mu,\sigma) = P(x_1 \le x_0) = \Phi\left(\frac{x_0-\xi}{\sigma}\right),$$

or of the density at x_0,

$$\varphi(\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x_0-\xi)^2}{2\sigma^2}\right\}.$$

The traditional method of estimating these functions is based on the uniformly minimum variance unbiased estimator (UMVUE) which can be derived from the Rao–Blackwell Theorem and the Basu Lemma (see [11]). For θ this estimator δ_U has the form

$$\begin{aligned}\delta_U(X,S;x_0) = \delta_U(X,S) &= P(x_1 \le x_0 \mid X,S)\\ &= P\left(\frac{x_1-X}{S} \le \frac{x_0-X}{S} \,\middle|\, X,S\right).\end{aligned}$$

Since (X,S) and $T=(x_1-X)/S$ are independent,

$$\delta_U(X,S) = P\left(T \le \frac{x_0-X}{S}\right).$$

By using the distribution of T one obtains for $n\ge 3$ the expression of the unbiased estimator in terms of incomplete beta-function:

$$\delta_U(X,S) = \begin{cases} 1, & W \le -1\,,\\ \int_W^1 (1-u^2)^{n/2-2}\,du/B(n/2-1,1/2), & |W|<1\,,\\ 0, & W\ge 1\,,\end{cases}$$

where $W=(X-\sqrt{n}x_0)/(\sqrt{n-1}S)$. First this representation was obtained by Kolmogorov [10] in 1950 (see also [12]).

The best unbiased estimator of φ has the form

$$\phi_U(X,S) = \frac{d}{dx_0}\delta_U(X,S;x_0) = \frac{\sqrt{n}}{\sqrt{n-1}S}g(W)$$

with a beta density g,

$$g(w) = \frac{(1-w^2)^{n/2-2}}{B(n/2-1,1/2)} = \frac{[(1-w)/2]^{n/2-2}[(1+w)/2]^{n/2-2}}{2B(n/2-1,n/2-1)}$$

if $|w|<1$ and $g(w)=0$ otherwise.

Several interesting optimality implications between the best unbiased density estimator and unbiased estimators of other parametric functions are given in [9]. Various forms of δ_U and its characteristics are discussed in [1, 2, 4, 6, 8, 13 and 15]. They are surveyed in [7]. A characteristic feature of these estimators is that ϕ_U vanishes outside the interval $|W|<1$ while δ_U takes extreme values 1 and 0. In particular, neither of these estimators is an analytic function and this fact makes their admissibility doubtful.

Indeed, for an exponential family any admissible rule of the mean is the generalized Bayes estimator with respect to some σ-finite measure (see [5], Theorems 4.16, 4.23 and references there). The latter is an analytic function of sufficient statistic. However, as we show in this paper, the fact that our parametric functions have exponential type leads to admissibility of some nonsmooth estimators analogous to δ_U and ϕ_U.

In Section 2 it is proven that δ_U is a pointwise limit of proper Bayes estimators such that for the limiting generalized prior density the "marginal" density of (X, S) is finite if and only if $|W| < 1$. This fact explains the structure of δ_U.

The admissibility of a related family of distribution function estimators within the class of all procedures depending only on W (the so-called scale equivariant estimators) is established in Section 3 where also the inadmissibility of ϕ_U is demonstrated. This suggests the inadmissibility of Kolmogorov's estimator as well.

2. Bayes estimators for conjugate priors and scale equivariant procedures. By shifting the original sample one can assume that $x_0 = 0$. Under this assumption we look here at the estimation of more general parametric functions

$$\theta(\mu, \sigma) = \Phi\left(- \frac{\sqrt{a}\mu}{\sigma} \right)$$

and

$$\kappa(\mu, \sigma) = \sigma^{-1} \exp\left\{ - \frac{a\mu^2}{2\sigma^2} \right\}$$

for a fixed positive constant a. The original problem corresponds to $a = 1/n$, $\kappa = \sqrt{2\pi}\varphi$. To estimate θ we use the quadratic risk $(\delta - \theta)^2$; for κ estimation the rescaled version $\sigma^2(\delta - \kappa)^2$ is more convenient.

The estimators studied here are generalized Bayes rules against prior densities with respect to reference measure $d\mu\, d\sigma/\sigma$. These densities have the form

$$\lambda(\mu, \sigma) = \sigma^{-\alpha} \exp\left\{ \frac{c\mu^2}{2\sigma^2} \right\}$$

with some real α and c, $c < 1$. As a matter of fact all admissibility results hold for more general conjugate prior distributions

$$\lambda(\mu, \sigma) = \sigma^{-\alpha} \exp\left\{ \frac{c(\mu - \mu_0)^2}{2\sigma^2} \right\}.$$

One of the reasons these priors are of interest in our problem is the form of the generalized Bayes estimators which is essentially that of δ_U and γ_U.

Proposition 2.1. *Under prior density* λ *with* $\alpha > 3 - n$ *and* $c < 1$ *the generalized Bayes estimator* γ_B *of* κ *has the form*

$$\gamma_B(X,S) = \frac{\Gamma(\varrho - 1/2)}{S\Gamma(\varrho - 1)}\sqrt{\frac{2(1-c)}{1-c+a}}\,\frac{[1-Z^2/z_0^2]^{\varrho-1}}{[1-Z^2/z_1]^{\varrho-1/2}}, \qquad |Z| < z_0\,,$$

with $Z = X/S,\ \varrho = (n+\alpha-1)/2,$

$$z_0^2 = \frac{1-c}{c} \quad \text{and} \quad z_1 = \frac{1-c+a}{c-a}\,.$$

The generalized Bayes estimator δ_B *of* θ *for* $|Z| < z_0$ *has the form*

$$\delta_B(X,S) = \frac{1}{B(\varrho,1/2)}\int_U^{z_0} (1-u^2)^{\varrho-1}\,du$$

with

$$U = \frac{\sqrt{a}Z}{\sqrt{(1-c)(1-c+a)(1-Z^2/z_1)}}\,.$$

Under definitions $\delta_B = 1$ *for* $Z \le -z_0$, $\delta_B = 0$ *for* $Z \ge z_0$ *and* $\gamma_B = 0$ *if* $|Z| \ge z_0$, *both* δ_B *and* γ_B *are pointwise limits of proper Bayes estimators.*

Proof. Let, for $m > 0$,

$$M(X,S;m,c) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_0^{\infty}\sigma^{-m-2}\exp\left\{-\frac{(X-\mu)^2+S^2-c\mu^2}{2\sigma^2}\right\}d\mu\,d\sigma\,.$$

A simple calculation shows that if $(1-c)S^2 > cX^2$ (which just means that $|Z| < z_0$) then

$$M(X,S;m,c) = \int_0^{\infty}\sigma^{-m-1}\exp\left\{-\frac{S^2-cX^2/(1-c)}{2\sigma^2}\right\}d\sigma/\sqrt{1-c}$$
$$= \left[S^2 - \frac{cX^2}{1-c}\right]^{-m/2}\frac{2^{(m-2)/2}\Gamma(m/2)}{\sqrt{1-c}}$$

and $M(X,S;m,c) = +\infty$ otherwise. Since

$$\gamma_B(X,S) = \frac{M(X,S;n+\alpha-2,c-a)}{M(X,S;n+\alpha-3,c)}\,,$$

the formula for the density estimator obtains for $|Z| < z_0$. When $|Z| \ge z_0$ the posterior risk is infinite and the generalized Bayes estimator is not defined. However, as is easy to check, any generalized Bayes estimator is a nondecreasing function of $|Z|$, which leads to defining $\gamma_B = 0$ for $|Z| \ge z_0$.

Similarly

$$\delta_B(X,S) = \frac{1}{2\pi M(X,S;n+\alpha-1,c)} \int\limits_0^\infty \int\limits_{-\infty}^\infty \int\limits_0^\infty \sigma^{-n-\alpha-2}$$

$$\times \exp\{-[(X-\mu)^2 + S^2 + a(u/\sqrt{a}+\mu)^2 - c\mu^2]/(2\sigma^2)\}\, d\mu\, d\sigma\, du\,.$$

The multiple integral above equals

$$\int\limits_0^\infty \int\limits_0^\infty \exp\left\{-\frac{(1-c+a)S^2 - (c-a)X^2 + (1-c)u^2 - 2\sqrt{a}uX}{2\sigma^2(1-c+a)}\right\}$$

$$\times \frac{\sqrt{2\pi}}{\sigma^{n+\alpha+1}(1-c+a)^{1/2}}\, d\sigma\, du$$

$$= \int\limits_0^\infty \frac{(1-c+a)^\varrho 2^\varrho \sqrt{\pi}\Gamma(\varrho+1/2)}{[(1-c+a)S^2 - (c-a)X^2 + (1-c)u^2 - 2\sqrt{a}uX]^{\varrho+1/2}}\, du$$

$$= \frac{2^\varrho\Gamma(\varrho+1/2)\sqrt{\pi}}{\sqrt{1-c}\,[S^2 - cX^2/(1-c)]^\varrho} \int\limits_{\sqrt{a}X/\sqrt{(1-c+a)[(1-c)S^2-cX^2]}}^{\infty} \frac{dv}{(1+v^2)^{\varrho+1/2}}$$

$$= \frac{2^\varrho\Gamma(\varrho+1/2)\sqrt{\pi}}{\sqrt{1-c}\,[S^2 - cX^2/(1-c)]^\varrho} \int\limits_U^1 (1-u^2)^{\varrho-1}\, du\,.$$

This formula leads to the form of distribution function estimators for $|Z| < z_0$. Monotonicity in Z of generalized Bayes estimators of θ suggests the definition of δ_B outside this interval. Approximation of δ_B and γ_B by proper Bayes estimators is possible as λ is a limit of proper prior densities. ∎

The estimators δ_B and γ_B have the simplest form when $c = a$. Proposition 2.1 implies that the UMVUE δ_U can be interpreted as the generalized Bayes estimator with respect to $\lambda(\mu,\sigma) = \sigma\exp\{\mu^2/(2n\sigma^2)\}$ which corresponds to $\varrho = (n-2)/2$, i.e. to $\alpha = -1$ and $c = a = 1/n$. Notice that with $\eta = \mu/\sigma$,

$$E_{\mu,\sigma} S^{-1}\left[1 - \frac{aX^2}{(1-a)S^2}\right]_+^{n/2-2}$$

$$= \frac{e^{-a\eta^2/2}}{\sqrt{2\pi}d_{n-2}} \iint\limits_{\sqrt{a}|x|<\sqrt{1-a}s} s\left[1 - \frac{ax^2}{(1-a)s^2}\right]^{n/2-2}$$

$$\times \exp\left\{-\frac{(1-a)s^2 - ax^2 + (x-(1-a)\eta)^2}{2(1-a)}\right\} dx\, ds$$

$$= \frac{\sqrt{1-a}\,\Gamma(n/2-1)}{\sigma\sqrt{2}\,\Gamma((n-1)/2)} e^{-a\eta^2/2}\,.$$

Here and further

$$d_k = \int_0^\infty y^k e^{-y^2/2}\,dy = 2^{(k-1)/2}\Gamma\left(\frac{k+1}{2}\right).$$

Therefore

$$\gamma_U(X,S) = \frac{\sqrt{2}\Gamma((n-1)/2)}{S\sqrt{1-a}\Gamma(n/2-1)}\left[1-\frac{aX^2}{(1-a)S^2}\right]_+^{n/2-2}$$

is the unbiased estimator of κ. Thus for the generalized Bayes estimator γ_B of κ with $c = a$ and $\alpha = -1$ when $n \geq 5$,

$$\gamma_B(X,S) = \frac{(1-a)(n-4)}{n-3}\gamma_U(X,S)\,.$$

Since the risk of generalized Bayes estimators against prior density λ depends only on η we look now at a class of estimators possessing this property. Within such a class one can define the Bayes estimator as the one whose average risk with respect to some prior Λ over η is the smallest. Scale equivariant estimators δ of θ by definition are just measurable functions of Z and scale equivariant estimators of κ have the form $S^{-1}\gamma(Z)$. An explicit form of the Bayes estimator within the class of scale equivariant procedures is easily derived.

Indeed, the Bayes scale equivariant estimator δ_Λ of θ has the form

$$\delta_\Lambda(Z) = \frac{\int \Phi(-\sqrt{a}\eta)p_{n-1}(Z,\eta)\,d\Lambda(\eta)}{\int p_{n-1}(Z,\eta)\,d\Lambda(\eta)}$$

where

$$p_k(z,\eta) = \frac{\int_0^\infty \exp\{-[(zy-\eta)^2+y^2]/2\}y^k\,dy}{\sqrt{2\pi}d_{k-1}}$$

is the noncentral t-distribution type density.

A simple calculation shows that δ_Λ coincides with the generalized Bayes estimator against the prior $\sigma^{-1}d\Lambda(\eta)\,d\sigma$ in the original θ estimation problem. The same holds for estimation of κ and rescaled quadratic loss in which case the scale equivariant Bayes estimator has the form

$$\gamma_\Lambda(Z) = \frac{\int e^{-a\eta^2/2}p_{n-2}(Z,\eta)\,d\Lambda(\eta)}{\int p_{n-3}(Z,\eta)\,d\Lambda(\eta)}\,.$$

For this reason $\alpha = 1$, not $\alpha = -1$ as for unbiased estimators, is the right choice for our prior. Indeed, in the next section we give some admissibility results concerning the Bayes scale equivariant estimators which correspond to $\alpha = 1$ and

$$\lambda(\eta) = \exp\{c\eta^2/2\}\,.$$

3. Admissibility and inadmissibility results. Our main goal is to establish the conditions under which the estimators

$$\delta_1(X,S)=\frac{1}{B(n/2,1/2)}\int_U^1[1-u^2]^{n/2-1}\,du$$

and

$$\gamma_1(X,S)=\frac{\Gamma((n-1)/2)}{S\Gamma(n/2-1)}\sqrt{\frac{2(1-c)}{1-c+a}}\frac{[1-Z^2/z_0^2]^{n/2-1}}{[1-Z^2/z_1]^{(n-1)/2}}\,,$$

$|Z|<z_0$, corresponding to $\alpha=1$ are admissible within the class of scale equivariant estimators. The following result gives a sufficient admissibility condition.

THEOREM 3.1. *The Bayes risk of δ_1 is finite if and only if $2a\ge c$. Estimator γ_1 has a finite Bayes risk if and only if $2a>c$.*

P r o o f. Since risk functions of δ_1 and γ_1 depend only on η one can put $\sigma=1$ when calculating these functions. If f is a measurable function of $t>0$ such that $f(t)=0$ for $t\le t_0$ and $f(t)\sim F(t-t_0)^{\varrho-1}$ for $t\downarrow t_0$ then

$$\begin{aligned}E_\eta S^{-\beta}f(S/|X|)&=\frac{e^{-\eta^2/2}}{\sqrt{2\pi}d_{n-2}}\int_{-\infty}^{\infty}e^{x\eta-x^2/2}\int_0^{\infty}e^{-x^2t^2/2}t^{n-2-\beta}f(t)\,dt\,dx\\&=\frac{e^{-\eta^2/2}}{\sqrt{2\pi}d_{n-2}}\int_{-\infty}^{\infty}e^{x\eta-x^2/2}Rf(x)\,dx\,.\end{aligned}$$

Laplace's method for asymptotics of integrals shows that for large x,

$$Rf(x)\sim\frac{Fe^{-x^2t_0^2/2}t_0^{n-\beta-\varrho-2}\Gamma(\varrho)}{x^{2\varrho}}$$

and as $\eta\to\infty$,

$$E_\eta S^{-\beta}f(S/|X|)\sim\frac{F\exp\{-t_0^2\eta^2/(2(1+t_0^2))\}\eta^{n-\beta-2\varrho-1}t_0^{n-\beta-\varrho-2}\Gamma(\varrho)}{(1+t_0^2)^{n-\beta-\varrho-1/2}d_{n-2}}\,.$$

Applying this formula with $\beta=1$ and $\beta=2$ to γ_1 for which $t_0^2=c/(1-c)$ so that $t_0^2/(1+t_0^2)=c$, one obtains

$$R(\gamma,\eta)\sim\frac{C_1e^{-c\eta^2/2}}{\eta^{4\varrho+1-n}}-\frac{C_2e^{-(a+c)\eta^2/2}}{\eta^{2\varrho+2-n}}+e^{-a\eta^2}$$

with positive constants C_1,C_2. Therefore

$$\int R(\gamma,\eta)e^{c\eta^2/2}\,d\eta<\infty$$

if and only if $2a>c$ and $\varrho>n/4$ (which holds automatically). The same conclusion holds for estimator δ whose risk function is also a symmetric function of η (see Appendix in [13]) except that now

$$R(\delta,\eta) \sim \frac{C_3 e^{-c\eta^2/2}}{\eta^{4\varrho+3-n}} + \frac{e^{-a\eta^2}}{2\pi\eta^2}\,,$$

and the Bayes risk integral converges when $2a \geq c$. ■

This theorem shows that estimators δ of the normal distribution function and γ of the normal density are scale equivariant admissible when $c < 2/n$. In particular, δ_1 and γ_1 are admissible when $c = a$.

Next comes an inadmissibility result in density estimation.

THEOREM 3.2. *Estimator γ_U is inadmissible.*

Proof. We show that a shrinkage estimator $b\gamma_u$ for some b, $0 < b < 1$, improves on γ_U. A simple calculation shows that such an estimator improves on γ_U if

$$b \geq 2\sup_{\eta} \kappa^2/E_\eta\gamma_U^2 - 1\,.$$

Because of the information inequality (see Theorem 4.3.1 in [14]) for parametric function $\kappa = \exp\{-\frac{\alpha\xi^2}{2\sigma^2}\}$ with $\alpha = an$,

$$E_\eta\gamma_U^2 - \kappa^2 = E_\eta(\gamma_U - \kappa)^2 \geq \frac{\kappa^2}{2n}(2a^2n\eta^2 + (a\eta^2-1)^2)\,.$$

Therefore if $an \geq 1$ then

$$\sup_{\eta} \frac{\kappa^2}{E_\eta\gamma_U^2} \leq \frac{2n}{2n+1} < 1$$

so that one can take $b = (2n-1)/(2n+1)$ to improve on γ_U, and if $an < 1$ then

$$\sup_{\eta} \frac{\kappa^2}{E_\eta\gamma_U^2} \leq \frac{2n}{2n + an(2-an)} < 1$$

and the choice $b = (2-a(2-an))/(2+a(2-an))$ leads to a better estimator. Notice that this result does not imply that γ_1 is better than γ_U (which in fact is not true). ■

The risk functions of δ_U and δ_1 are given in Figures 1–3 for $n = 2, 3, 4$. Similar graphs for γ_U, γ_B with $\alpha = -1$ and γ_1 are shown in Figures 4–6 for $n = 5, 6, 7$. Although neither γ_1 nor δ_1 dominates the corresponding unbiased estimator they can be seriously recommended since they are admissible and have reasonable risk functions.

To calculate the risk of density estimators the following formulas were used. One has, for $p > (n-4)/2$,

$$E\frac{1}{S}\left[1-\frac{aX^2}{(1-a)S^2}\right]_+^p =$$

$$\frac{2d_{2p+1}(1-a)^{1/2}a^{(n-2)/2}e^{-\eta^2/2}}{d_{n-2}d_{2p-n+3}}\int_0^1 \frac{e^{(1-a)x\eta^2/2}x^{(n-3)/2}(1-x)^{p-(n-2)/2}}{[1-(1-a)x]^{p+1}}\,dx$$

and for $p > (n-5)/4$,

$$E\frac{1}{S^2}\left[1-\frac{aX^2}{(1-a)S^2}\right]_+^{2p} =$$

$$\frac{2d_{4p+1}(1-a)^{1/2}a^{(n-3)/2}e^{-\eta^2/2}}{d_{n-2}d_{4p-n+4}}\int_0^1 \frac{e^{(1-a)x\eta^2/2}x^{(n-4)/2}(1-x)^{2p-(n-3)/2}}{[1-(1-a)x]^{2p+1}}\,dx.$$

Inadmissibility of δ_U and γ_B with $\alpha = -1$ remains an open problem. In the particular case $n = 2, a = 1/2$, Kolmogorov's estimator δ_U is defined by

$$\delta_U(X_1, X_2) = \begin{cases} 1, & X_1 + X_2 < -|X_1 - X_2|\,, \\ \frac{1}{2}, & |X_1 + X_2| \le |X_1 - X_2|\,, \\ 0, & X_1 + X_2 > |X_1 - X_2|\,, \end{cases}$$

and is admissible. Indeed, if there is a better estimator δ, i.e. if

$$E_\eta(\delta(Z)-\theta)^2 \le E_\eta(\delta_U(Z)-\theta)^2 = \frac{\theta(1-\theta)}{2}$$

then

$$\lim_{\eta\to+\infty}\frac{2}{\eta^2}E_\eta\delta^2 \le \lim_{\eta\to+\infty}\frac{2}{\eta^2}\theta = -1\,.$$

One can assume that $\delta(-Z) = 1-\delta(Z)$ so that its risk is a symmetric function of η. A Tauberian theorem for functions of exponential type (see [3, Sec. 4.12]) shows that for a nonincreasing function g,

$$\limsup_{\eta\to+\infty}\frac{2}{\eta^2}\log\int_0^\infty \exp\left\{x\eta - \frac{x^2}{2} - g(x)\right\}dx = \frac{1}{1+\liminf_{x\to+\infty}2g(x)/x^2}\,.$$

In our application one can take

$$g(x) = -\log\int_0^\infty e^{-s^2/2}s^{n-2}\delta^2(-x/s)\,ds$$

so that

$$\liminf_{x\to-\infty}\frac{2}{x^2}\log\int_0^\infty e^{-s^2/2}s^{n-2}\delta^2(x/s)\,ds \le -1\,.$$

Now the classical Tauberian theorem implies that

$$\delta(Z) = 0 \quad \text{if } Z \le -z_0 < 0$$

with $z_0 \le 1$. (This argument concerning the structure of the potential improvements on δ_B and γ_B is valid for any n.)

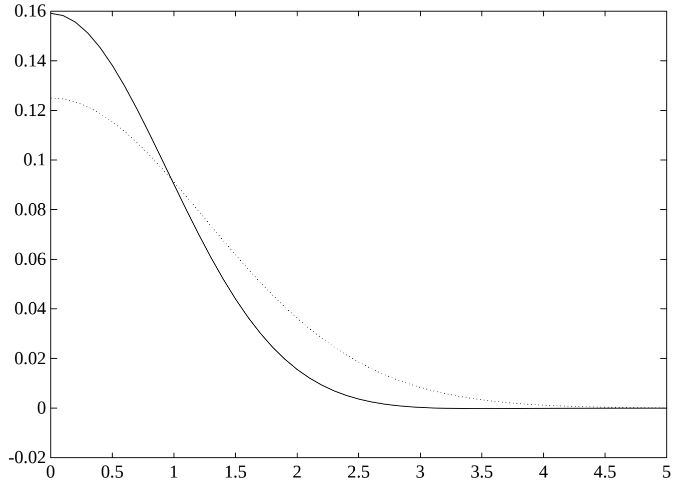

Fig. 1. Risk functions of estimators δ_U (dotted line), and δ_1 (solid line) for $n = 2$

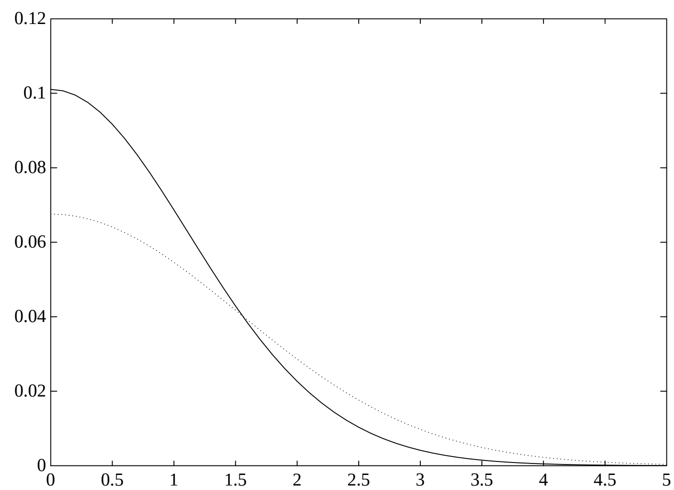

Fig. 2. Risk functions of estimators δ_U (dotted line), and δ_1 (solid line) for $n = 3$

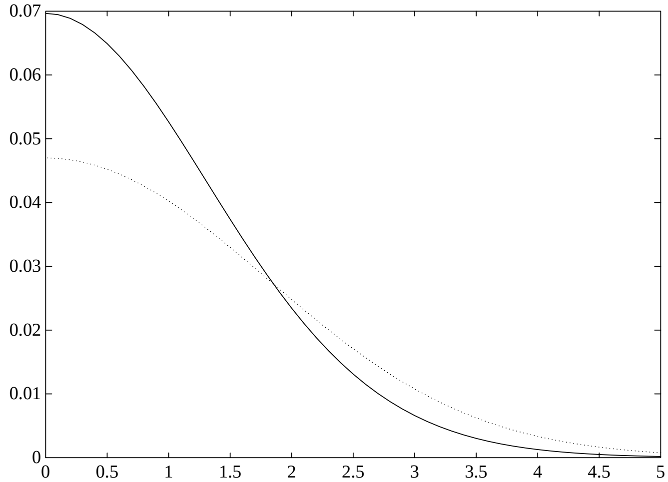

Fig. 3. Risk functions of estimators δ_U (dotted line), and δ_1 (solid line) for $n = 4$

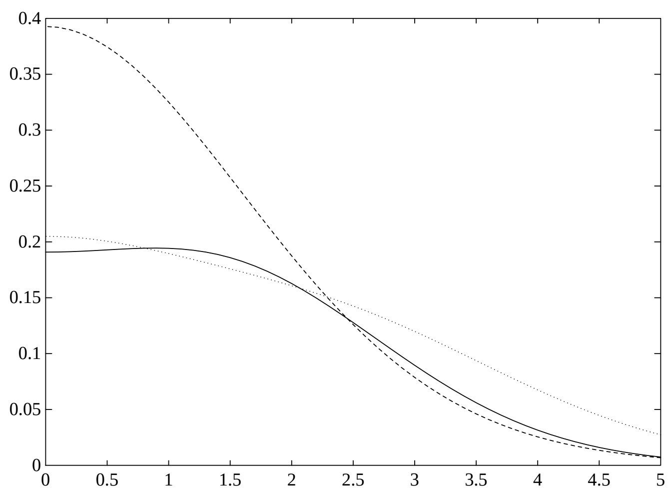

Fig. 4. Risk functions of density estimators γ_U (dotted line), γ_B (dashed line) and γ_1 (solid line) for $n = 5$

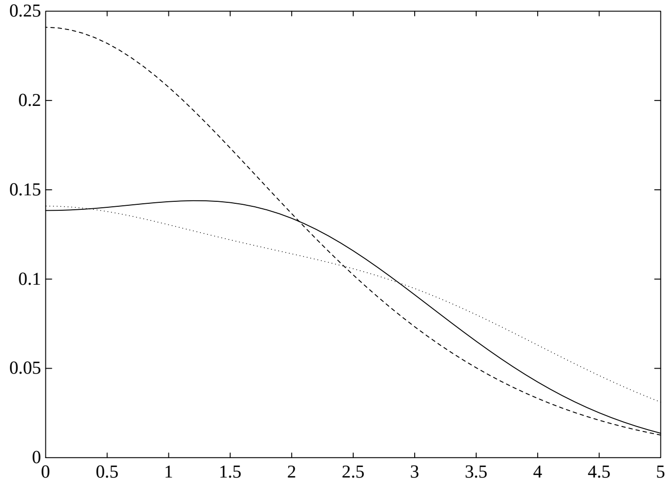

Fig. 5. Risk functions of density estimators γ_U (dotted line), γ_B (dashed line) and γ_1 (solid line) for $n = 6$

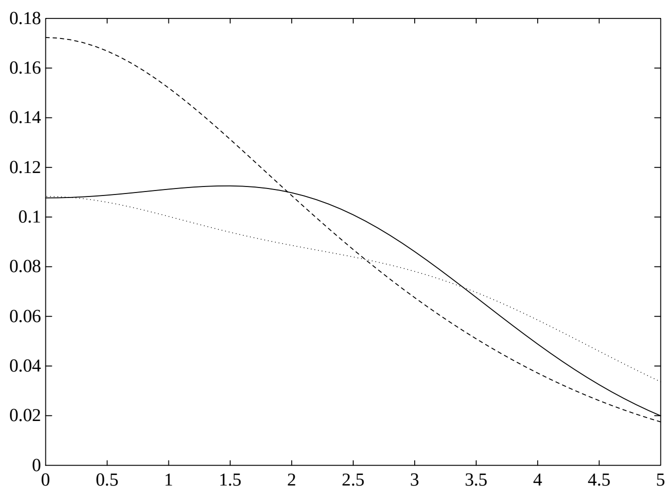

Fig. 6. Risk functions of density estimators γ_U (dotted line), γ_B (dashed line) and γ_1 (solid line) for $n = 7$

The comparison of risk functions at $\eta = 0$ shows that $\delta = 1/2$ for $|Z| \le z_0$ and that to have equal risk at this parametric value, δ must coincide with δ_U. Of course this result is based on the fact that Kolmogorov's estimator on the set $|Z| \le z_0$ coincides with a Bayes estimator with respect to the point mass prior at $\eta = 0$. It is conceivable that a similar phenomenon might hold for other values of n and also for γ_B.

References

[1] D. E. Barton, *Unbiased estimation of a set of probabilities*, Biometrika 48 (1961), 227–229.

[2] A. P. Basu, *Estimates of reliability for some distributions useful in life testing*, Technometrics 6 (1964), 215–219.

[3] N. Bingham, C. Goldie and J. Teugels, *Regular Variation*, Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 1987.

[4] G. G. Brown and H. C. Rutemiller, *The efficiencies of maximum likelihood and minimum variance unbiased estimators of fraction defective in the normal case*, Technometrics 15 (1973), 849–855.

[5] L. D. Brown, *Fundamentals of Statistical Exponential Families with Applications in Statistical Decision Theory*, Institute of Mathematical Statistics, Lecture Notes–Monograph Series, Volume 9, Hayward, CA, 1968.

[6] J. Folks, D. A. Pierce and C. Stewart, *Estimating the fraction of acceptable product*, Technometrics 7 (1965), 43–50.

[7] I. Gertsbakh and A. Winterbottom, *Point and interval estimation of normal tail probabilities*, Comm. Statist. Theory Methods 4 (1991), 1497–1514.

[8] W. C. Geunther, *A note on the minimum variance unbiased estimate of the fraction of a normal distribution below a specification limit*, Amer. Statist. 25 (1971), 18–20.

[9] L. B. Klebanov, *Unbiased parametric distribution estimation*, Mat. Zametki 25 (1979), 743–750 (in Russian).

[10] A. N. Kolmogorov, *Unbiased estimates*, Izv. Akad. Nauk SSSR Ser. Mat. 14 (1950), 303–326 (in Russian).

[11] E. L. Lehmann, *Theory of Point Estimation*, Wiley, New York, 1983.

[12] G. J. Lieberman and G. J. Resnikoff, *Sampling plans for inspections by variables*, J. Amer. Statist. Assoc. 50 (1955), 457–516.

[13] A. L. Rukhin, *Estimating normal tail probabilities*, Naval Res. Logist. Quart. 33 (1986), 91–99.

[14] S. Zacks, *The Theory of Statistical Inference*, Wiley, New York, 1971.

[15] S. Zacks and R. C. Milton, *Mean square errors of the best unbiased and maximum likehood estimators of tail probabilities in normal distributions*, J. Amer. Statist. Assoc. 66 (1971), 590–593.

ANDREW L. RUKHIN
DEPARTMENT OF MATHEMATICS AND STATISTICS
UMBC
BALTIMORE, MARYLAND 21228
U.S.A.

Received on 4.3.1993