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AN m-VS.-n-BULLETS SILENT DUEL WITH ARBITRARY MOTION AND ARBITRARY ACCURACY FUNCTIONS

A silent duel is considered in which Player I has m bullets, Player II has n bullets, the accuracy functions are arbitrary and the players can move as they like. Player I has greater speed than Player II and fires his bullets simultaneously. For this duel the optimal strategies are determined and the value of the game is found. The duel is solved under general assumptions on the payoff function.

1. Introduction. Consider a game which will be called the *game* $G(m,n)$. Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. Player I has m bullets which he fires simultaneously, Player II has n bullets which he can fire as he wants. These facts are known to both players. It is also known that the duel is silent: at a given moment neither player knows how many bullets his opponent has fired (obviously, for Player I this number can be only 0 or m).

At the beginning of the duel the players are at distance 1 from each other. Let $P_1(s)$ ($P_2(s)$) be the probability of succeeding (destroying the opponent) by Player I (II) by one bullet when the distance between them is $1-s$. The functions $P_1(s)$, $P_2(s)$ will be called *accuracy functions*. It is assumed that they are increasing and continuous in $[0,1]$, have continuous second derivatives in $(0,1)$ and that $P_i(s) = 0$ for $s \leq 0$, $P_i(1) \leq 1$, $i = 1,2$.

Player I gains $k > 0$ if only he succeeds, gains $-l < 0$ if only Player II succeeds, gains w if both players succeed, and gains 0 if none of them does, with $-l \leq w \leq k$. The duel is a zero-sum game.

As will be seen from the sequel, we can suppose without loss of generality

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that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

The duel is a generalization of the game considered in [10] where the case $k = l = 1$, $w = 0$, $P_1(1) = P_2(1) = 1$ was solved.

For definitions and results in the theory of games of timing see [3], [4], [11].

2. Auxiliary duel. To solve the game $G(m, n)$ presented in the previous section, it is necessary to determine the optimal strategies in the following auxiliary game $G_0(m, n)$. Consider the m-versus-n-bullets duel with accuracy functions $P_1(s)$, $P_2(s)$ in which Player I approaches Player II with constant velocity $v = 1$, all the time, even after firing his bullets. Player I fires his m bullets simultaneously and gains k if only he succeeds etc., similarly to the duel defined in the previous section.

Denote by $K_0(s; t_1, \ldots, t_n)$ the expected gain of Player I if he fires his m bullets at time s and Player II fires his n bullets at times $t_1 \le \ldots \le t_n$. Let $Q_i(s) = 1 - P_i(s)$, $i = 1, 2$. It is assumed that

(1)
$$K_0(s; t_1, \ldots, t_n) = \begin{cases} k(1 - Q_1^m(s)) & \text{if } s < t_1, \\ -l(1 - Q_2(t_1) \ldots Q_2(t_k)) \\ \quad + kQ_2(t_1) \ldots Q_2(t_k)(1 - Q_1^m(s)) & \text{if } t_k < s < t_{k+1}, \\ & \quad k = 1, \ldots, n-1, \\ -l(1 - Q_2(t_1) \ldots Q_2(t_n)) \\ \quad + kQ_2(t_1) \ldots Q_2(t_n)(1 - Q_1^m(s)) & \text{if } s > t_n. \end{cases}$$

As is easy to see $K_0(s; t_1, \ldots, t_n)$ is the expected payoff in the duel in which Player II is not allowed to fire after the salvo of Player I.

Following this idea it is easy to define $K_0(s; t_1, \ldots, t_n)$ when some t_i are equal to s.

Denote by $\bar{\xi}$ the strategy of Player I in the game $G_0(m, n)$, in which he fires m bullets at a random moment s distributed according to a density $f(s)$ in the interval $[a_1, 1]$, $0 < a_1 < 1$, and according to a probability E, $0 < E < 1$, at the point 1.

Suppose that we have defined numbers a_i, $0 < a_i < 1$, $a_i < a_{i+1}$, $i = 1, \ldots, n$, $a_{n+1} = 1$, and that

$$f(s) = f_i(s) \quad \text{when } a_i \le s < a_{i+1}\,, \tag{2}$$

where

$$f_i(s) = C_i \frac{P_2'(s)}{P_2^2(s)(\frac{k+l}{k} - Q_1^m(s))}\,, \qquad i = 1, \ldots, n\,, \tag{3}$$

C_i being constants.

Let $K_0(\bar{\xi}; t_1, \dots, t_n)$ be the expected gain of Player I if he applies the strategy $\bar{\xi}$ and Player II fires at times $t_1, \dots, t_n$. When $a_i \le t_i < a_{i+1}$, $i = 1, \dots, n$, we obtain

$$\begin{aligned} K_0(\bar{\xi}; t_1, \dots, t_n) = & \int_{a_1}^{t_1} k(1 - Q_1^m(s)) f_1(s)\, ds \\ & + \int_{t_1}^{a_2} (-l + Q_2(t_1)(k + l - kQ_1^m(s))) f_1(s)\, ds \\ & + \int_{a_2}^{t_2} (-l + Q_2(t_1)(k + l - kQ_1^m(s))) f_2(s)\, ds \\ & + \int_{t_2}^{a_3} (-l + Q_2(t_1) Q_2(t_2)(k + l - kQ_1^m(s))) f_2(s)\, ds + \dots \\ & + \int_{a_n}^{t_n} (-l + Q_2(t_1) \dots Q_2(t_{n-1})(k + l - kQ_1^m(s))) f_n(s)\, ds \\ & + \int_{t_n}^{1} (-l + Q_2(t_1) \dots Q_2(t_n)(k + l - kQ_1^m(s))) f_n(s)\, ds \\ & + (-l + Q_2(t_1) \dots Q_2(t_n)(k + l - kQ_1^m(1)))E. \end{aligned}$$

Since

$$\int_{a_1}^{1} f(s)\, ds + E = 1, \tag{4}$$

$$\int_{a}^{b} (k + l - kQ_1^m(s)) f_i(s)\, ds = kC_i \left(\frac{1}{P_2(a)} - \frac{1}{P_2(b)} \right) \tag{5}$$

for $(a, b) \in (a_i, a_{i+1})$, we obtain

$$\begin{aligned} K_0(\bar{\xi}; t_1, \dots, t_n) = & -l + kC_1 \left(\frac{1}{P_2(a_1)} - \frac{1}{P_2(t_1)} \right) \\ & + Q_2(t_1) kC_1 \left(\frac{1}{P_2(t_1)} - \frac{1}{P_2(a_2)} \right) \\ & + Q_2(t_1) kC_2 \left(\frac{1}{P_2(a_2)} - \frac{1}{P_2(t_2)} \right) \\ & + Q_2(t_1) Q_2(t_2) kC_2 \left(\frac{1}{P_2(t_2)} - \frac{1}{P_2(a_3)} \right) + \dots \end{aligned}$$

$$
\begin{aligned}
&+Q_2(t_1)\dots Q_2(t_{n-1})kC_n\left(\frac{1}{P_2(a_n)}-\frac{1}{P_2(t_n)}\right)\\
&+Q_2(t_1)\dots Q_2(t_n)kC_n\left(\frac{1}{P_2(t_n)}-\frac{1}{P_2(1)}\right)\\
&+Q_2(t_1)\dots Q_2(t_n)(k+l-kQ_1^m(1))E\\
=&-l+kC_1\left(\frac{1}{P_2(a_1)}-1-\frac{Q_2(t_1)}{P_2(a_2)}\right)\\
&+Q_2(t_1)kC_2\left(\frac{1}{P_2(a_2)}-1-\frac{Q_2(t_2)}{P_2(a_2)}\right)+\dots\\
&+Q_2(t_1)\dots Q_2(t_{n-1})kC_n\left(\frac{1}{P_2(a_n)}-1-\frac{Q_2(t_n)}{P_2(a_n)}\right)\\
&+Q_2(t_1)\dots Q_2(t_n)(k+l-kQ_1^m(1))E\,.
\end{aligned}
$$

We obtain

$$
K_0(\overline{\xi};t_1,\dots,t_n)=-l+kC_1\frac{Q_2(a_1)}{P_2(a_1)}\,, \tag{6}
$$

and this is independent of t_i if the terms by $Q_2(t_1)$, $Q_2(t_1)Q_2(t_2),\dots$ $\dots,Q_2(t_1)\dots Q_2(t_n)$ vanish, i.e. when

$$
C_i=Q_2(a_{i+1})C_{i+1}\,,\quad i=1,\dots,n-1\,, \tag{7}
$$

$$
C_n=P_2(1)\left(\frac{k+l}{k}-Q_1^m(1)\right)E\,. \tag{8}
$$

Inserting (7) and (8) into (6) we obtain

$$
K_0(\overline{\xi};t_1,\dots,t_n)=-l+\frac{Q_2(a_1)\dots Q_2(a_n)}{P_2(a_1)}P_2(1)(k+l-kQ_1^m(1))E\,. \tag{9}
$$

Let $\overline{\eta}$ be the strategy of Player II in the game $G_0(m,n)$ in which he chooses at random stochastically independent moments $t_1,\dots,t_n$ for his shots according to the densities $g_1(t_1),\dots,g_2(t_n)$, where

(10)
$$
g(t_j)\,dt_j=\begin{cases} D_j\dfrac{d(\frac{k+l}{k}-Q_1^m(t_j))}{P_2(t_j)(\frac{k+l}{k}-Q_1^m(t_j))^2} & \text{if } a_j<t_j<a_{j+1},\ j=1,\dots,n,\\ 0 & \text{otherwise,}\end{cases}
$$

$a_{n+1}=0$, and D_j are normalizing constants.

For $s\in(a_j,a_{j+1})$, $j=1,\dots,n$, we obtain

$$
K_0(s;\overline{\eta})=\int\limits_{a_1}^{a_2}\dots\int\limits_{a_{j-1}}^{a_j}\Big\{\int\limits_{a_j}^{s}(-l+Q_2(t_1)\dots Q_2(t_j)
$$
$$
\times(k+l-kQ_1^m(s)))g_j(t_j)\,dt_j
$$

$$+ \int_{s}^{a_{j+1}} (-l + Q_2(t_1)\dots Q_2(t_{j-1})(k+l-kQ_1^m(s)))g_j(t_j)\,dt_j \Big\} \times g_1(t_1)\dots g_{j-1}(t_{j-1})\,dt_1\dots dt_{j-1}$$

$$= \int_{a_1}^{a_2}\dots\int_{a_{j-1}}^{a_j} \Bigg(-l + Q_2(t_1)\dots Q_2(t_{j-1})(k+l-kQ_1^m(s)) \times\Bigg(1 - D_j\Bigg(\frac{1}{\frac{k+l}{k} - Q_1^m(a_j)} - \frac{1}{\frac{k+l}{k} - Q_1^m(s)}\Bigg)\Bigg)\Bigg) \times g_1(t_1)\dots g_{j-1}(t_{j-1})\,dt_1\dots dt_{j-1}\,.$$

Let

$$D_i = \frac{k+l}{k} - Q_1^m(a_i)\,. \tag{11}$$

We obtain

$$\int_{a_i}^{a_{i+1}} Q_2(t_i)g_i(t_i)\,dt_i = 1 - D_i\Bigg(\frac{1}{\frac{k+l}{k} - Q_1^m(a_i)} - \frac{1}{\frac{k+l}{k} - Q_1^m(a_{i+1})}\Bigg)$$

and

$$K_0(s;\overline{\eta}) = -l + kD_j\sum_{i=1}^{j-1}\Bigg(1 - D_i\Bigg(\frac{1}{\frac{k+l}{k} - Q_1^m(a_i)} - \frac{1}{\frac{k+l}{k} - Q_1^m(a_{i+1})}\Bigg)\Bigg) = k(1 - Q_1^m(a_1))\,. \tag{12}$$

Then for D_i given by (11) the function $K(s;\overline{\eta})$ is independent of s when $a_1 \le s \le 1$.

From the definition of the strategy $\overline{\xi}$ we obtain

$$\sum_{i=1}^{n} C_i \int_{a_i}^{a_{i+1}} \frac{P_2'(s)\,ds}{P_2^2(s)(\frac{k+l}{k} - Q_1^m(s))} + E = 1\,. \tag{13}$$

Integrating by parts and taking into account that

$$\int_{a_i}^{a_{i+1}} \frac{d(\frac{k+l}{k} - Q_1^m(s))}{P_2(s)(\frac{k+l}{k} - Q_1^m(s))^2} = \frac{1}{D_i} = \frac{1}{\frac{k+l}{k} - Q_1^m(a_i)}$$

we come to the equation

$$\sum_{i=1}^{n} C_i\Bigg(\frac{Q_2(a_i)}{P_2(a_i)(\frac{k+l}{k} - Q_1^m(a_i))} - \frac{1}{P_2(a_{i+1})(\frac{k+l}{k} - Q_1^m(a_{i+1}))}\Bigg) + E = 1\,, \tag{14}$$

with $a_{n+1} = 1$. Taking into account (7) and comparing neighbouring terms in (14) we obtain

$$C_n\left(\frac{Q_2(a_1)\dots Q_2(a_n)}{P_2(a_1)(\frac{k+l}{k}-Q_1^m(a_1))}-\frac{1}{P_2(1)(\frac{k+l}{k}-Q_1^m(1))}\right)=1-E\,.$$

Then by (8) we get

$$C_n=\frac{P_2(a_1)(\frac{k+l}{k}-Q_1^m(a_1))}{Q_2(a_1)\dots Q_2(a_n)}\,. \tag{15}$$

On the other hand, from (10) and (11) we get

$$\left(\frac{k+l}{k}-Q_1^m(a_i)\right)\int\limits_{a_i}^{a_{i+1}}\frac{d(\frac{k+l}{k}-Q_1^m(t_i))}{P_2(t_i)(\frac{k+l}{k}-Q_1^m(t_i))^2}=1\,, \tag{16}$$

$$i=1,\dots,n,\ a_{n+1}=1\,.$$

Suppose that the system of equations (16) has a solution $(a_1,\dots,a_n)$ with $0<a_i<1$, $i=1,\dots,n$. Then the constant C_n is determined from (15), and from (7) and (8) we determine the remaining C_i and E. Moreover, from (4) it follows that $0<E<1$. Then the system of equations (7), (8), (11), (14), (16) has a solution $(a_1,\dots,a_n;C_1,\dots,C_n;D_1,\dots,D_n;E)$ with $0<a_i<1$, $C_i>0$, $D_i>0$, $0<E<1$ provided (16) has a solution $(a_1,\dots,a_n)$ with $0<a_i<1$, $i=1,\dots,n$. Moreover, from (8), (9), (12) and (15) we obtain

$$\begin{aligned}K_0(\bar{\xi};t_1,\dots,t_n)&=-l+k\frac{Q_2(a_1)\dots Q_2(a_n)}{P_2(a_1)}P_2(1)(k+l-kQ_1^m(1))E\\&=k(1-Q_1^m(a_1))=K_0(s;\bar{\eta})\end{aligned} \tag{17}$$

if $a_1\le s\le 1$, $a_i\le t_i\le a_{i+1}$, $i=1,\dots,n-1$, $a_n\le t_n<1$.

Consider the functions

$$\varphi_i(t)=\left(\frac{k+l}{k}-Q_1^m(t)\right)\int\limits_{t}^{a_{i+1}}\frac{d(\frac{k+l}{k}-Q_1^m(t))}{P_2(t)(\frac{k+l}{k}-Q_1^m(t))^2}$$

for $0\le t\le a_{i+1}$, $i=1,\dots,n$, $a_{n+1}=1$. We obtain

$$\begin{aligned}\frac{d\varphi_i(t)}{dt}&=\frac{d(\frac{k+l}{k}-Q_1^m(t))}{dt}\left[\int\limits_{t}^{a_{i+1}}\frac{d(\frac{k+l}{k}-Q_1^m(t))}{P_2(t)(\frac{k+l}{k}-Q_1^m(t))^2}\right.\\&\qquad\left.-\frac{1}{P_2(t)(\frac{k+l}{k}-Q_1^m(t))}\right]\\&=\frac{d(\frac{k+l}{k}-Q_1^m(t))}{dt}\left[-\frac{1}{P_2(a_{i+1})(\frac{k+l}{k}-Q_1^m(a_{i+1}))}\right.\\&\qquad\left.-\int\limits_{t}^{a_{i+1}}\frac{P_2'(t)\,dt}{P_2^2(t)(\frac{k+l}{k}-Q_1^m(t))}\right]<0\end{aligned} \tag{18}$$

for $0 \le t \le a_{i+1}$. From the above it follows that there exists at most one solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$.

Notice that the system (16) has a solution $(a_1, \ldots, a_n)$ with $0 < a_1 < 1$ for any $k > 0$, $l > 0$, m, n when $P_1(s) = P_2(s)$.

EXAMPLE 1. Let $m = n = 1$, $P_1(t) = t$, $P_2(t) = t^\alpha$, $\alpha > 0$. We obtain

$$\left(\frac{l}{k} + P_1(a_1)\right) \int\limits_{a_1}^{1} \frac{P_1'(t)\,dt}{P_2(t)(\frac{l}{k} + P_1(t))^2} = \left(\frac{l}{k} + a_1\right) \int\limits_{a_1}^{1} \frac{dt}{t^\alpha(t + \frac{l}{k})^2}$$

$$= \left(\frac{l}{k} + a_1\right) \int\limits_{k/(k+l)}^{k/(ka_1+l)} \left(\frac{x}{1 - \frac{l}{k}x}\right)^\alpha dx$$

$$\le \left(\frac{l}{k} + a_1\right) \int\limits_{k/(k+l)}^{k/(ka_1+l)} \frac{dx}{(1 - \frac{l}{k}x)^\alpha} \xrightarrow[\alpha \to 0]{} (1 - a_1)\frac{k^2}{k + l} < k$$

for each $0 < a_1 < 1$. Then for these $P_1(s)$, $P_2(s)$, m, n the equation (16) has no solution a_1 with $0 < a_1 < 1$ when α is small.

LEMMA. *If there exists a solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$, then for these a_i the strategy $\overline{\xi}$ is maximin and the strategy $\overline{\eta}$ is minimax in the game $G_0(m, n)$. The value of the game is $v_{mn}^0 = k(1 - Q_1^m(a_1))$.*

The proof is similar to one given in [10] and is omitted.

3. Main result. Let us return to the duel $G(m, n)$ defined at the beginning of the paper. Assume that there exists a solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$. For a given natural N, let constants c_i be defined as follows:

$$c_0 = a_1\,, \quad \int\limits_{c_i - 1}^{c_i} f(s)\,ds = \frac{1}{N}\,, \quad i = 1, \ldots, N_0\,, \quad c_{N_0+1} = 1\,, \tag{19}$$

where N_0 is defined from the inequalities

$$1 - \frac{1}{N} \le \int\limits_{c_0}^{c_{N_0}} f(s)\,ds + E < 1\,. \tag{20}$$

Define the strategy ξ^ε of Player I in the game $G(m, n)$ as follows: If there exists a solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$ (case 1), Player I moves back and forth with maximal speed in the following manner: at first between 0 and c_1, then between 0 and $c_2, \ldots$, and finally between 0 and C_{N_0+1}. At the ith step, $i = 1, \ldots, N_0 + 1$, he can fire his salvo at random only if he is between the points c_{i-1} and c_i and goes forward, and he fires it with probability density $f(s)$. If he has fired at the ith step, he reaches the point c_i, escapes to 0 and never approaches Player II. If Player I has not fired his salvo between the points a_1 and 1 and survives, he fires it when he is at 1, as soon as possible.

If no solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$, exists (case 2), Player I, following ξ^ε, does not approach Player II.

The strategy η^0 of Player II is defined in case 1 as follows: If Player I reaches the point t the first time and his velocity is $v(\tau)$, τ being the time, $a_i < t < a_{i+1}$, then Player II fires at random his ith bullet with probability density $v(\tau)g_i(t(\tau))$. Otherwise he does not fire.

In case 2, when the system (16) has no solution $(a_1, \ldots, a_n)$ with $0 < a_i < 1$, $i = 1, \ldots, n$, the strategy η^0 is defined similarly, but if $a'_i < t < a'_{i+1}$ the firing has probability density $v(\tau)g_i^0(t(\tau))$ for

$$(21)\quad g_i^0(t)\,dt = \begin{cases} D_i^0 \dfrac{d(\frac{k+l}{k} - Q_1^m(t))}{P_2(t)(\frac{k+l}{k} - Q_1^m(t))^2} & \text{if } a'_i < t < a'_{i+1},\ i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$(22)\qquad D_i^0 = MD'_i \stackrel{\text{def}}{=} M\left(\frac{k+l}{k} - Q_1^m(a'_i)\right).$$

The constants a'_i and the constant $M \ge 1$ are chosen in such a way that

$$(23)\qquad \int_{a'_i}^{a'_{i+1}} g_i^0(t)\,dt = 1, \quad i = 1, \ldots, n,\ a'_0 = 0,\ a'_{n+1} = 1.$$

For any strategy ξ of Player I it is assumed that the function $v(\tau)$ is piecewise continuous.

We obtain

THEOREM. *The strategy ξ^ε is ε-maximin and the strategy η^0 is minimax in the game $G(m,n)$. The value of the game is $v_{mn} = k(1 - Q_1^m(a_1))$ if there exists a solution $(a_1, \ldots, a_n)$ of (16) with $0 < a_i < 1$, $i = 1, \ldots, n$, and $v_{mn} = 0$ otherwise.*

The long proof of the Theorem is similar to one in [10] and is omitted.

4. Examples. When $m = 1$, $P_1(s) = P_2(s) \stackrel{\text{def}}{=} P(s)$, from (16) we obtain

$$(24)\quad \left(\frac{l}{k} + P(a_i)\right)\left(\frac{k^2}{l^2}\log\frac{\frac{l}{k} + P(a_i)}{P(a_i)} - \frac{k^2}{l^2}\log\frac{\frac{l}{k} + P(a_{i+1})}{P(a_{i+1})} + \frac{\frac{k}{l}}{\frac{l}{k} + P(a_{i+1})}\right) = 1 + \frac{k}{l},$$
$$i = 1, \ldots, n,\ P(a_{n+1}) = 1.$$

This system has a solution $(a_1, \ldots, a_n)$ for any n.

EXAMPLE 2. Let $m = n = 1$, $k = l = 1$, $|w| \leq 1$, $P_1(s) = P_2(s) = P(s)$, $P(1) = 1$. From (24) we obtain

$$v_{11} = P(a_1) \cong 0.177655.$$

Then from (15), (11) and (8) we get

$$C_1 \cong 0.254414, \quad D_1 \cong 1.177655, \quad E \cong 0.127207.$$

EXAMPLE 3. Let $m = 1$, $n = 2$, $k = l = 1$, $|w| \leq 1$, $P_1(s) = P_2(s) = P(s)$, $P(1) = 1$. In this case we have

$$v_{12} = P(a_1) \cong 0.056044, \quad P(a_2) \cong 0.177655,$$
$$C_1 \cong 0.062698, \quad D_1 \cong 1.056044,$$
$$C_2 \cong 0.076243, \quad D_2 \cong 1.177655,$$
$$E \cong 0.038122.$$

EXAMPLE 4. Let $m = n = 1$, $k = 2$, $l = 1$, $-1 \leq w \leq 2$, $P_1(s) = P_2(s) = P(s)$, $P(1) = 1$. We obtain

$$P(a_1) \cong 0.271598, \quad v_{11} \cong 0.543196,$$
$$C_1 \cong 0.431556, \quad D_1 \cong 0.771598, \quad E \cong 0.287704.$$

Duels under arbitrary motion, as far as the author knows, have never been considered before, except in the papers of the author (see [9], [10]).

For other results in the theory of games of timing see [1], [2], [4–8].

References

[1] A. Cegielski, *Game of timing with uncertain number of shots*, Math. Japon. 31 (1986), 503–532.

[2] M. Fox and G. Kimeldorf, *Noisy duels*, SIAM J. Appl. Math. 17 (1969), 353-361.

[3] S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. 2, Addison-Wesley, Reading, Mass., 1959.

[4] G. Kimeldorf, *Duels: an overview*, in: *Mathematics of Conflict*, North-Holland, 1983, 55–71.

[5] T. Radzik, *Games of timing with resources of mixed type*, J. Optim. Theory Appl. 58 (1988), 473–500.

[6] R. Restrepo, *Tactical problems involving several actions*, in: *Contributions to the Theory of Games*, Vol. III, Ann. of Math. Stud. 39, Princeton Univ. Press, 1957, 313–335.

[7] A. Styszyński, *An n-silent-vs.-noisy duel with arbitrary accuracy functions*, Zastos. Mat. 14 (1974), 205–225.

[8] S. Trybuła, *A silent n versus 1 bullet duel with retreat after the shots*, Optimization 21 (1990), 609–627.

[9] —, *A noisy duel under arbitrary moving. I–VI*, Zastos. Mat. 20 (1990), 491–495, 497–515, 517–530; Zastos. Mat. 21 (1991), 43–61, 63–81, 83–98.

[10] S. Trybuła, *Solution of a silent duel with arbitrary moving and arbitrary accuracy functions*, Optimization, submitted.

[11] N. N. Vorob'ev, *Foundations of the Theory of Games. Uncoalition Games*, Nauka, Moscow 1984 (in Russian).

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