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A NOTE ON OUTERPLANARITY OF PRODUCT GRAPHS

Abstract. We prove necessary and sufficient conditions for the outerplanarity of the Cartesian product and Kronecker product of graphs. In our discussions, the class of almost bipartite graphs is defined and we show that if G is an almost bipartite graph, then it is a minor of $G \times K_2$. We conjecture that this is true for all graphs.

1. Introduction and preliminaries. A wide variety of graph products have been studied for a long time [3, 9, 12, 15, 16] and more recently [1, 5, 6, 11, 17]. Some of these products have found applications in several areas of mathematics and computer science [7, 8, 10, 14]. In this note, we deal mainly with two of these products: the *Cartesian product* ($\Box$*-product*) and *Kronecker product* ($\times$*-product*). Occasionally, we will also mention the *strong product* ($\boxtimes$*-product*). For the first two products we discuss necessary and sufficient conditions for the outerplanarity of the product graphs in terms of the factor graphs. The more challenging part of this work is the one dealing with the outerplanarity of the Kronecker product.

The paper has three sections. In Section 2 we discuss the outerplanarity of the $\Box$-product and the $\times$-product. Section 3 summarizes the results and mentions some related issues. In the remainder of this section, we present the necessary definitions, some notational conventions, and some known results that are used later. At the end of this section we will outline the situation regarding the analogous problem of the *planarity* of product graphs.

By a graph we mean a finite, simple and undirected graph. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The *Cartesian product, Kronecker product* and *strong product* of G_1 and G_2 are respectively denoted by $G_1 \Box G_2$, $G_1 \times G_2$ and $G_1 \boxtimes G_2$, and are defined as follows. The vertex set is the same for the three products: $V(G_1 \Box G_2) = V(G_1 \times G_2) = V(G_1 \boxtimes G_2) =$

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$V_1 \times V_2$. The edge sets are: $E(G_1 \boxtimes G_2) = E(G_1 \Box G_2) \cup E(G_1 \times G_2)$, where $E(G_1 \times G_2) = \{\{(x_1, x_2), (y_1, y_2)\} \mid \{x_1, y_1\} \in E_1 \text{ and } \{x_2, y_2\} \in E_2\}$, and $E(G_1 \Box G_2) = \{\{(x_1, x_2), (y_1, y_2)\} \mid \text{either } x_1 = y_1 \text{ and } \{x_2, y_2\} \in E_2 \text{ or } x_2 = y_2 \text{ and } \{x_1, y_1\} \in E_1\}$. Note that $E(G_1 \Box G_2) \cap E(G_1 \times G_2) = \emptyset$. Also observe that '$\times$' denotes Cartesian product of sets as well as Kronecker product of graphs; we use context to resolve any ambiguity.

We say that a graph G is *planar* if there is an embedding of G in the plane in which no two edges cross each other. Further, G is said to be *outerplanar* if there is a planar embedding of G in which all vertices lie on the same face. By an *elementary contraction* of a graph G, we mean a graph G' obtained from G by (i) removing an edge $\{u, v\}$ of G, (ii) identifying the vertices u and v, and (iii) discarding any multiple edges created in the process of the foregoing identification. A graph H is said to be a *minor* (or a *subcontraction*) of G if H is obtainable from a subgraph of G by a sequence of elementary contractions [2, p. 89]. Obviously, planar as well as outerplanar graphs are closed under the operation of taking minors. For a vertex subset W of a graph G, $\langle W \rangle$ will denote the subgraph of G induced by W. The complete graph on n vertices is denoted by K_n. We say that the one-vertex graph K_1 is trivial and a graph on two or more vertices is nontrivial. $K_{m,n}$ denotes the complete bipartite graph on $m + n$ vertices, where the two sets constituting a bipartition of the vertex set of $K_{m,n}$ are of cardinalities m and n respectively. A (simple) path and a (simple) cycle of length n are respectively denoted by P_n and C_n, and are defined as follows: (i) $V(P_n) = \{1, \ldots, n+1\}$, where $\{i, i+1\} \in E(P_n)$, $1 \le i \le n$, and (ii) $V(C_n) = \{1, \ldots, n\}$, where $\{1, n\}$ and $\{i, i+1\} \in E(C_n)$, $1 \le i < n$. For any undefined terms, we refer to [2].

It is well known that the three graph products are commutative and associative, up to isomorphism. With respect to connectivity, $G_1 \Box G_2$ is connected if and only if both G_1 and G_2 are connected [15]. For G_1 and G_2 nontrivial, $G_1 \times G_2$ is connected if and only if both G_1 and G_2 are connected and either G_1 or G_2 is non-bipartite; moreover, if G_1 and G_2 are both bipartite, then $G_1 \times G_2$ has exactly two connected components [16]. It is easy to see that $G_1 \boxtimes G_2$ is connected if and only if both G_1 and G_2 are connected. The following theorem characterizes outerplanar graphs using graph minors [2, p. 89]; it will be useful in the sequel.

THEOREM 1.1. *A graph is outerplanar if and only if neither K_4 nor $K_{2,3}$ is a minor of G.* ∎

The next theorem says that while dealing with the outerplanarity of the $\Box$-product and $\boxtimes$-product of graphs, it suffices to consider only those factor graphs which are themselves outerplanar. A similar result for the $\times$-product will be proved later.

THEOREM 1.2. *Let G_1 and G_2 be nontrivial, connected graphs. If one of G_1 and G_2 is non-outerplanar, then so is each of $G_1 \square G_2$ and $G_1 \boxtimes G_2$.* ■

We now offer several remarks regarding the planarity of product graphs. For the $\square$-product and $\times$-product the planarity issue is completely characterized in [3] and [4]. The following theorem gives a complete characterization of the planarity of the $\boxtimes$-product. The proofs are left out.

THEOREM 1.3. *Let G_1 and G_2 be nontrivial, connected (planar) graphs. Then $G_1 \boxtimes G_2$ is planar if and only if one of the following holds:*

1. *One graph is a tree and the other is K_2.*
2. *Both graphs are P_2.* ■

2. Main results. First, we note that *for all graphs G_1 and G_2, each containing at least one edge, $G_1 \boxtimes G_2$ is non-outerplanar.* Next, we dispose of the easy case of $\square$-product. The following lemma provides a basis for the characterization of the outerplanarity of $\square$-product. The proof is routine and omitted.

LEMMA 2.1. *For $n \geq 3$, $C_n \square K_2$ is a non-outerplanar graph.* ■

The characterization of the outerplanarity of the $\square$-product of two graphs is given in the next theorem. By Theorem 1.2, there is no loss of generality in assuming that the factor graphs are themselves outerplanar.

THEOREM 2.2. *The Cartesian product of two nontrivial, connected (outerplanar) graphs is outerplanar if and only if one graph is a path and the other is K_2.*

Proof. The $\square$-product of a path and K_2 is obviously outerplanar. For the converse, let G_1 and G_2 be nontrivial, connected graphs, and assume that the condition of the lemma is not satisfied. If one graph is K_2, then the other cannot be a path, and hence must contain a cycle C_n, $n \geq 3$, or $K_{1,3}$ as a subgraph. By Lemma 2.1, $C_n \square K_2$ is non-outerplanar. Since $K_{2,3}$ is a minor of $K_{1,3} \square K_2$, by Theorem 1.1, $K_{1,3} \square K_2$ is non-outerplanar. Alternatively, if none of G_1 and G_2 is K_2, then each must contain P_2 so the product has $K_{2,3}$ as a minor. ■

To discuss the necessary and sufficient conditions for the outerplanarity of the $\times$-product we introduce the concept of an *almost bipartite graph.* Let $G = (V, E)$ be a connected graph. For $x, y \in V$, an *(x, y)-path* is simply a path between the vertices x and y in G. Let C_m be a cycle of G. We say that C_m is a *minimal cycle* of G if no proper vertex subset of C_m induces a smaller cycle in G. A *minimal odd cycle* is a minimal cycle which is of odd length. Note that if a graph contains an odd cycle, then it necessarily contains a minimal odd cycle. We say that G is an *almost bipartite graph* (or an *a-b graph*) if it contains a unique minimal odd cycle. Figure 1 shows an example of an a-b graph. In the following lemma, we state a useful property of such graphs.

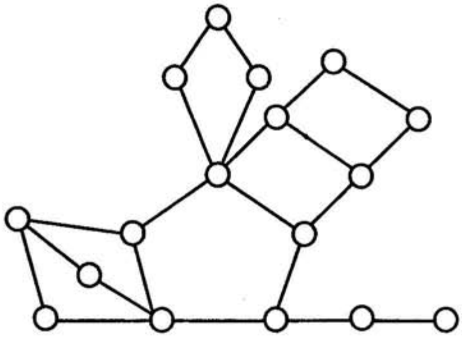

Fig. 1. An almost bipartite graph

LEMMA 2.3. *Let G be an almost bipartite graph with C_{2k+1} as its unique minimal odd cycle, and let $W \subseteq V(G)$. Then $\langle W \rangle$ is a bipartite graph in its own right if and only if $C_{2k+1} \not\subseteq W$.* ∎

Let G and C_{2k+1} be as above. For $i \in C_{2k+1}$, let

$$A_i = \{x \in V(G) \mid x \neq i \text{ and for all } j \in C_{2k+1}, \; i \text{ appears on every } (x,j)\text{-path}\}.$$

Note that the set A_i may be empty for some i, and that if $A_i \neq \emptyset$, then the induced subgraph $\langle A_i \rangle$ is bipartite in its own right. Next, let v be a vertex of G such that (i) $v \notin A_i$ for any i, (ii) $v \notin C_{2k+1}$, and (iii) for some distinct $i, j \in C_{2k+1}$, there are (v,i)- and (v,j)-paths, none of which contains any other vertex of C_{2k+1}. We claim that $\{i,j\} \in E(C_{2k+1})$. Assume otherwise. Let w be a vertex which is common to a (v,i)-path and a (v,j)-path such that there exist a (w,i)-path and a (w,j)-path which are vertex-disjoint (except for w, of course). By Lemma 2.3, every cycle of G which does not include all vertices of C_{2k+1} is even. Consequently, every cycle consisting of (i) a (w,i)-path, (ii) an (i,j)-path along C_{2k+1}, and (iii) a (j,w)-path must be even, since (by our assumption) it does not include all of C_{2k+1}. However, this condition cannot always be satisfied as there are two paths between the vertices i and j in the cycle C_{2k+1}, one of which is of even length while the other is of odd length. This contradiction shows that $\{i,j\} \in E(G)$ as claimed. Based on the foregoing argument, for every edge $e = \{i,j\}$ of C_{2k+1}, let

$$B_e = \{x \in V(G) \setminus C_{2k+1} \mid \text{for every } m \in C_{2k+1} \setminus \{i,j\}, \text{ there is an } (x,m)\text{-path in which } i \text{ appears but } j \text{ does not, and an } (x,m)\text{-path in which } j \text{ appears but } i \text{ does not}\}.$$

Note again that B_e may be empty for some e and that if B_e is nonempty, then the induced subgraph $\langle B_e\rangle$ is bipartite in its own right. It is clear that the A_i's and B_e's are all mutually disjoint. It is also easy to see that each vertex of G is in exactly one of the following sets: (i) C_{2k+1}, (ii) A_i for some i, and (iii) B_e for some e. We now prove an interesting result which, we conjecture, holds for arbitrary graphs, but were only able to prove for almost bipartite graphs. Luckily, this special case suffices for our goals.

LEMMA 2.4. *If G is an almost bipartite graph, then G is a minor of $G \times K_2$.*

Proof. Let G be an *a-b* graph with C_{2k+1} as its unique minimal odd cycle. For $i \in C_{2k+1}$ and $e \in E(C_{2k+1})$, let A_i and B_e be the vertex subsets of G as defined in the discussion preceding the statement of this lemma. Obviously, the corresponding induced subgraphs $\langle A_i\rangle$ and $\langle B_e\rangle$ are bipartite, and hence the graph $G \times K_2$ will contain two disjoint copies of each of $\langle A_i\rangle$ and $\langle B_e\rangle$ (see also [13]). Let u and v be the two (adjacent) vertices of K_2 so that the vertex set of $G \times K_2$ is simply $V(G) \times \{u, v\}$. Note that corresponding to the (unique, minimal) odd cycle C_{2k+1} of G, the graph $G \times K_2$ contains the even cycle C_{4k+2}, and that $\{i, j\}$ is an edge of C_{2k+1} if and only if $\{(i,u),(j,v)\}$ and $\{(i,v),(j,u)\}$ are (antipodal) edges of C_{4k+2}.

We now outline the construction of a subgraph of $G \times K_2$ of which G will be a minor. First include the even cycle C_{4k+2} whose vertices are labeled (i,u) or (j,v) as stated above. Next, for a nonempty vertex subset A_i of G (where $i \in C_{2k+1}$), let $v_1, \dots, v_m$ be the vertices of A_i such that $\{i, v_p\} \in E(G)$, $1 \le p \le m$. "Prepare and attach" one copy of $\langle A_i\rangle$ to C_{4k+2} as follows: if i is odd (resp. even), then introduce an edge between the vertex (i,u) (resp. (i,v)) of C_{4k+2} and each of $v_1, \dots, v_m$ of $\langle A_i\rangle$. (Note that in the graph $G\times K_2$, there is a copy of $\langle A_i\rangle$ attached to the "diametrically opposite" vertex of C_{4k+2}, but we do not include that in our subgraph.) Similarly, for a nonempty vertex subset B_e of G, where $e = \{i, j\} \in E(C_{2k+1})$, let $v_1, \dots, v_m$ and $w_1, \dots, w_n$ be the vertices of B_e such that $\{i, v_p\}, \{j, w_q\} \in E(G)$, $1 \le p \le m, 1 \le q \le n$. Prepare a copy of $\langle B_e\rangle$ and attach it to C_{4k+2} as follows: (i) if $i = 1$ and $j = 2$, then introduce an edge between the vertex $(1,u)$ of C_{4k+2} and each of the vertices $v_1, \dots, v_m$ of $\langle B_e\rangle$, and an edge between the vertex $(2,v)$ and each of $w_1, \dots, w_n$, (ii) if $i = 1$ and $j = 2k+1$, then do a similar attachment of $\langle B_e\rangle$ to the (adjacent) vertices $(2k+1,u)$ and $(1,v)$ of C_{4k+2}, and (iii) if $i, j \neq 1$, then assume that $j = i+1$ and for odd (resp. even) i, do an analogous attachment of a copy of $\langle B_e\rangle$ to the adjacent vertices (i,u) and (j,v) (resp. (i,v) and (j,u)) of the even cycle C_{4k+2}. (Note again that in the graph $G\times K_2$, there is an identical copy of $\langle B_e\rangle$ attached to the antipodal edge of C_{4k+2}, but we do not include that in our subgraph.) We perform the foregoing operations for all nonempty vertex subsets A_i and B_e of G. It is clear that the graph H thus obtained is (isomorphic to) a subgraph of $G \times K_2$. Finally, we contract the following $2k+1$ edges of (the cycle C_{4k+2} of) H: $\{(1,v),(2,u)\}$, $\{(2,u),(3,v)\}$, ... , $\{(2k+1,v),(1,u)\}$, whence the vertices $(1,u)$ and $(1,v)$ get identified. The resulting graph is isomorphic to G. ■

LEMMA 2.5. *If a connected graph G contains at least two distinct, minimal odd cycles, then $G \times K_2$ is a non-outerplanar graph.*

Proof. It suffices to show that if G is a connected graph in which the number of distinct, minimal odd cycles is exactly two, then $G \times K_2$ is non-outerplanar. There are three cases: the two (minimal odd) cycles (i) are vertex-disjoint, (ii) share exactly one vertex and (iii) share one or more edges. In each case, $K_{2,3}$ is a minor of $G \times K_2$. ■

The following lemma shows that while dealing with the outerplanarity of the $\times$-product of graphs, it suffices to consider only those factor graphs which are themselves outerplanar.

LEMMA 2.6. *If G_1 and G_2 are nontrivial, connected graphs, one of which is non-outerplanar, then the graph $G_1 \times G_2$ is non-outerplanar.*

Proof. It suffices to show that if G is non-outerplanar, then so is $G \times K_2$. So assume that G is a connected, non-outerplanar graph. If G is bipartite, then $G \times K_2$ contains exactly two disjoint copies of G, in which case we are done. On the other hand, if G is non-bipartite, then there are two cases: (i) G is an *a-b* graph, i.e., it contains exactly one minimal odd cycle, and (ii) G contains two or more minimal odd cycles. In the former case, the claim follows from Lemma 2.4 and Theorem 1.1 while in the latter case, it follows from Lemma 2.5. ■

LEMMA 2.7. *If G is a connected, outerplanar, almost bipartite graph, then $G \times K_2$ is an outerplanar graph.*

Proof. First, observe that if G is an odd cycle, say C_{2m+1}, then $G \times K_2$ is C_{4m+2}, which is trivially outerplanar. So consider the general case when (a connected, outerplanar *a-b* graph) G contains exactly one minimal odd cycle, say C_{2k+1}, as a proper subgraph. Our construction will be somewhat similar to that in the proof of Lemma 2.4. Note that every vertex of G is a member of one of the following sets: (i) C_{2k+1}, (ii) A_i, where $i \in C_{2k+1}$, and (iii) B_e, where $e \in E(C_{2k+1})$. (Definitions of the sets A_i and B_e appear just before the statement of Lemma 2.4.) Let u and v be the two vertices of K_2 so that the vertex set of $G \times K_2$ is $V(G) \times \{u,v\}$. We outline an outerplanar embedding of $G \times K_2$ based on an outerplanar embedding of G. First embed the even cycle C_{4k+2} as an outerplanar graph corresponding to the cycle C_{2k+1} of G. Next, for a vertex i of C_{2k+1}, if the set A_i is nonempty, then prepare two copies of (the induced subgraph) $\langle A_i \rangle$, and "attach and embed" the first copy to C_{4k+2} through the vertex (i,u) and the second copy through the "diametrically opposite" vertex (i,v) in exactly the same manner as $\langle A_i \rangle$ is connected to i in G. Similarly, for an edge $e = \{i,j\}$ of C_{2k+1}, if B_e is nonempty, then prepare two copies of $\langle B_e \rangle$, and attach and embed the first copy to C_{4k+2} through the edge $\{(i,u),(j,v)\}$ and the second copy through (the antipodal edge) $\{(i,v),(j,u)\}$—again in exactly the same manner as $\langle B_e \rangle$ is connected to $e = \{i,j\}$ in G. We perform the foregoing operations for all nonempty sets A_i and B_e. Our embedding of $G \times K_2$ closely "mimics" an outerplanar embedding of G. It follows that $G \times K_2$ is outerplanar. ∎

The following two lemmas are needed in our characterization of outerplanar $\times$-product graphs. The proofs are routine and omitted.

LEMMA 2.8. *For all $m, n \geq 1$, $P_m \times P_n$ is outerplanar if and only if either $m \leq 3$ or $n \leq 3$.* ∎

LEMMA 2.9. *For $n \geq 3$, the graph $C_n \times P_2$ is non-outerplanar.* ∎

The characterization for the outerplanarity of the $\times$-product of graphs is as follows.

THEOREM 2.10. *Let G_1 and G_2 be nontrivial, connected (outerplanar) graphs.*

(1) *If G_1 and G_2 are paths of lengths m and n respectively, then $G_1 \times G_2$ is outerplanar if and only if either $m \leq 3$ or $n \leq 3$.*

(2) *If G_1 and G_2 are both bipartite and G_1 is not a path, then $G_1 \times G_2$ is outerplanar if and only if $G_2 \cong K_2$.*

(3) *If G_1 is non-bipartite, then $G_1 \times G_2$ is outerplanar if and only if G_1 is an a-b graph (i.e., contains exactly one minimal odd cycle) and $G_2 \cong K_2$.*

Proof. (1) follows from Lemma 2.8 while (3) follows from Lemmas 2.5, 2.7 and 2.9. For (2), let G_1 and G_2 be (nontrivial, connected and) bipartite graphs, where G_1 is different from a path. First observe that if $G_2 \cong K_2$, then $G_1 \times G_2$ consists of simply two disjoint copies of G_1, and hence outerplanarity of $G_1 \times G_2$ follows from that of G_1. For the converse, assume that $G_2 \ncong K_2$. Then P_2 must be a subgraph of G_2, and since G_1 is not a path, it must contain either $K_{1,3}$ or an even cycle as a subgraph. The graph $K_{1,3} \times P_2$ is non-outerplanar as it contains $K_{2,3}$. Further, by Lemma 2.9, $C_{2n} \times P_2$ is non-outerplanar. It follows that $G_1 \times G_2$ is non-outerplanar and (2) is established. ∎

3. Concluding remarks. In this paper, we have discussed necessary and sufficient conditions for the outerplanarity of product graphs. While dealing with the outerplanarity of the $\times$-product, we have introduced an interesting class of graphs called almost bipartite graphs which are connected graphs containing a unique minimal odd cycle. We have shown that if G is an almost bipartite graph, then it is a minor of $G \times K_2$. (For bipartite graphs, the analogous statement is trivially true.) We *conjecture* that every graph G is a minor of the graph $G \times K_2$ and note that an analogous "conjecture" for the other two products is trivially true.

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