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# CONTINUOUS DEPENDENCE OF SOLUTIONS OF SOME INVERSE PROBLEMS IN HEAT CONDUCTION

*Abstract.* We consider the problem of determination of boundary values from internal data for the heat equation. For certain types of boundary functions we prove continuous dependence upon the data. As an example we state one of our results. Let $u = u(x,t)$ and $q = q(t)$ satisfy $u_t = u_{xx} + vu_x + ru$, $0 < x < \infty$, $t > 0$, $u_x(0,t) = q(t)$, $u(x,0) = 0$, $u(1,t) = g(t)$, $|u(x,t)| \le C_1 \exp(C_2 x^\alpha + rt)$, $\alpha < 2$, $q(0) = 0$, $|q''(t)| \le Ae^{pt}$, $p > 0$. Then, for any $T > 0$, there exists $M > 0$ such that $|q(t)| \le M/(\log(K/\|g\|))^2$, $0 \le t \le T$, $\|g\| \le K < \infty$.

**1. Introduction.** In the paper a question of continuous dependence of boundary values on internal data is considered for the heat conduction equation in one spatial variable. The problem of determination of boundary values from internal data is an inverse problem, which is improperly posed in the sense that the solution does not depend continuously on the data. To obtain continuous dependence the set of permissible boundary value functions must be suitably restricted.

Inverse boundary value problems (IBVPs) of heat conduction arise often in physical applications when one or all of the boundaries of the heat conductor are inaccessible to measurements (e.g., turbine engine walls, propeller surfaces, etc.). The idea is to measure the temperature at one or more interior points and use the readings to determine the boundary temperature values and subsequently the temperature throughout the whole domain. There exists an extensive literature on the analytical and numerical aspects of the IBVPs for parabolic equations (see [2] for a bibliography). One of the first papers [8] on the subject proposed an integral equation method, which yielded a numerical algorithm with unstable behavior for small time inter-

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vals. This was due to the lack of continuous dependence of the solution on the data and not to the instability of the algorithm. Other numerical methods which were proposed later took this into the account by approximating the solutions using some stabilization techniques [1, 4, 5, 7, 9].

The question of continuous dependence upon internal measurements of the solutions of inverse problems in heat conduction were studied in [2, 4, 6] and many other papers. Some of the results showed the Hölder-continuous dependence of the solution at point $x$ in the interior of the domain upon the internal data. The Hölder exponents, however, depend on the distance of $x$ from the boundary and decrease to 0 as $x$ moves toward the boundary [2]. It is of interest to examine the question of continuous dependence of the solutions to inverse boundary value problems. Since it is well known that, in general, small errors in the data magnify errors in a high frequency component of the boundary functions, some smoothness requirements must be imposed on the boundary functions in order to recover continuous dependence on internal measurements.

In the paper we show the uniqueness and (logarithmic) continuous dependence upon the internal measurements for several IBVPs in heat conduction assuming some smoothness of the unknown boundary functions. The method of proof assumes that the solution of a corresponding direct initial-boundary problem can be expressed in terms of heat potentials. All problems considered are spatially one-dimensional.

**2. Problem description.** Given an open and simply connected set $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\Gamma$, a (direct) initial-boundary value problem for a linear parabolic equation is to determine a function $u$ satisfying

$$(2.1)\qquad \left(\frac{\partial}{\partial t} - k\nabla^2 - v\cdot\nabla - r\right)u(x,t) = F(x,t), \quad v \in \mathbb{R}^n,\ k > 0,$$

for $x \in \Omega$, $t > 0$, together with

$$(2.2)\qquad u(x,0) = \varphi(x),$$

$$(2.3)\qquad u(y,t) = g(y,t), \quad y \in \Gamma,$$

where $F$, $\varphi$ and $g$ are the source, initial, and boundary functions, respectively.

A corresponding inverse boundary value problem is to determine the boundary function $g(y,t)$, $y \in \Gamma$, from the internal data $\psi$ given by

$$(2.4)\qquad u(y,t) = \psi(y,t), \quad y \in \Gamma_1,$$

where $\Gamma_1 \subset \Omega$, $\Gamma_1 \neq \emptyset$, so that the direct problem (2.1)–(2.3) with such $g$ has the solution $u$ satisfying (2.4) on $\Gamma_1$.

An example of another inverse problem would be to determine a function $g$ such that the solution of (2.1)–(2.2) together with the (Neumann) boundary condition

$$\frac{\partial u}{\partial n_y}(y,t) = g(y,t), \qquad y \in \Gamma\,,$$

satisfies (2.4) on $\Gamma_1$. Here $\partial/\partial n_y$ denotes the normal derivative at $y \in \Gamma$.

Under certain conditions it is possible to express a solution to the direct problem stated above in terms of certain potentials, similar to the classical heat potentials, which take into account the influence of the source and the initial and boundary conditions. These potentials are defined in terms of the fundamental solution $w$ of equation (2.1) given by

$$w(x,t) = \frac{1}{(4k\pi t)^{n/2}} \exp\left[ -\frac{v\cdot x}{2k} - \left(\frac{v\cdot v}{4k} - r\right)t - \frac{|x|^2}{4kt}\right], \tag{2.5}$$

where $|x|$ is the euclidean norm of $x$. For example, assuming without loss of generality that $\varphi \equiv F \equiv 0$, $u$ can be expressed in terms of the single layer potential:

$$u(x,t) = \int_0^t \int_\Gamma w(x-\xi, t-\tau) h(\xi,\tau)\, ds_\xi\, d\tau, \qquad x \in \Omega\,, \tag{2.6}$$

where $h$ is the (unknown) potential density to be determined from the boundary condition (2.3).

For an inverse problem observe that $u$ defined by (2.6) satisfies equation (2.1) for all $x \in \Omega$, so in particular for $x = y \in \Gamma_1$ we have

$$\psi(y,t) = \int_0^t \int_\Gamma w(y-\xi, t-\tau) h(\xi,\tau)\, ds_\xi\, d\tau\,. \tag{2.7}$$

This is a Volterra integral equation of the first kind with an infinitely smooth kernel $w$. Note that if it is possible to determine the potential density $h$ uniquely from (2.7), then the values of $u$ or $\partial u/\partial u_y$ can be calculated using the properties of single layer potentials,

$$u(\zeta,t) = \int_0^t \int_\Gamma w(\zeta-\xi, t-\tau) h(\xi,\tau)\, ds_\xi\, d\tau, \qquad \zeta \in \Gamma\,,$$

$$\frac{\partial u}{\partial n_\zeta}(\zeta,t) = -\frac{1}{2}h(\zeta,t) + \int_0^t \int_\Gamma \frac{\partial w}{\partial n_\zeta}(\zeta-\xi, t-\tau) h(\xi,\tau)\, ds_\xi\, d\tau\,,$$

where the kernels of the above integrals have weak singularities at $\zeta = \xi$, $\tau = t$.

Note that the uniqueness of the density $h$ for the integral equation (2.7) does not imply the uniqueness of the solution of the inverse boundary value problem. The uniqueness of the IBVP solution will follow if it is known that a solution to the corresponding direct boundary value problem given by (2.6) is unique.

In the rest of the paper we assume that given an inverse boundary value problem there exists a potential representation of $u$ for the corresponding direct problem. We shall focus on the determination of the potential density functions from Volterra integral equations similar to equation (2.7). Moreover, we restrict our attention to initial-boundary problems depending on a single spatial variable. This will have an effect of reducing the Volterra equations to the convolution equations of the form

$$\psi(y,t) = \int_0^t H(y, t-\tau)h(\tau)\,d\tau\,, \tag{2.8}$$

where $y \in \Gamma_1$, $\xi \in \Gamma$, $t > 0$ and $H(y, 0) = 0$, for $y \neq 0$. Equations of this form cannot be reduced to the equation of the second kind by the standard procedure of differentiation. It is easy to see that the solution $h$ does not exist unless $\psi(y, 0) = 0$. However, if the solution $h$ exists, a suitable restriction placed on the set of admissible solutions results in uniqueness and continuous dependence of $h$ upon $\psi$. The next sections of this paper serve to support and illustrate this assertion.

**3. Uniqueness and continuous dependence of the solution of a Volterra convolution equation of the first kind.** Assuming that the functions $\psi$ and $H$ are of exponential type we can take the Laplace transform $\mathcal{L}$ of both sides of equation (2.8) and obtain

$$\mathcal{L}H\mathcal{L}h = \mathcal{L}\psi\,.$$

It is clear that the solution $\mathcal{L}h$ of the above equation exists and it is unique if $\mathcal{L}H \neq 0$. The existence and uniqueness of $h$ follows provided that $\mathcal{L}\psi/\mathcal{L}H$ is invertible.

In order to show continuous dependence of $h$ upon $\psi$ we use the following lemma, which appeared in [3] with more restrictive hypotheses.

LEMMA 1. *Let $h \in C^2[0, \infty)$, $h(0) = 0$ and $|h''(t)| < A\exp(pt)$, $A > 0$, $p > 0$. If $h$ is the solution of the integral equation*

$$g(t) = \int_0^t H(t-\tau)h(\tau)\,d\tau, \quad 0 < t < \infty\,,$$

*where the kernel $H$ satisfies $|\mathcal{L}H(s)|^{-1} \leq C(\exp(B|s+D|^q))$, $0 < q < 1$, $B, C, D > 0$, then given $T > 0$ there exists $M > 0$ such that*

$$|h(t)| \leq M(\log(K/\|g\|))^{-1/q}, \quad 0 \leq t \leq T\,,$$

*where $\|g\|^2 = \int_0^\infty (g(t))^2\,dt < K < \infty$.*

Proof. Setting $s = \sigma + i\eta$, $\sigma > p$ we have

$$|\mathcal{L}g(s)| = \left|\int_0^\infty g(t)e^{-ts}\,dt\right| \le \|g\|(2\sigma)^{-1/2}$$

by the Cauchy–Schwarz inequality. Now, if $\sigma > p$ then

$$h(t) = \frac{1}{2\pi i}\int_{\sigma-iN}^{\sigma+iN} \mathcal{L}h(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{\sigma+iN}^{\sigma+i\infty} \mathcal{L}h(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma-iN} \mathcal{L}h(s)e^{st}ds\,.$$

The lemma follows after estimating each of the above integrals. For the first we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma-iN}^{\sigma+iN} \mathcal{L}h(s)e^{st}\,ds\right| &\le \frac{1}{2\pi}\int_{-N}^{N}\left|\frac{\mathcal{L}g(\sigma+i\eta)}{\mathcal{L}H(\sigma+i\eta)}\right|e^{\sigma t}\,d\eta \\ &\le \frac{e^{\sigma t}}{2\pi}\frac{C\|g\|}{(2\sigma)^{1/2}}\int_{-N}^{N}\exp(B|\sigma+D+i\eta|^q)\,d\eta \\ &\le \frac{e^{\sigma t}}{\pi}\frac{B\|g\|}{(2\sigma)^{1/2}}N\exp[B((\sigma+D)^2+N^2)^{q/2}] \\ &\le C_1e^{\sigma t}\|g\|\exp(B_1N^q)\,, \quad \text{for some } B_1 > B,\ C_1 > C\,.\end{aligned}$$

For the second integral we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma+iN}^{\sigma+i\infty} \mathcal{L}h(s)e^{st}\,ds\right| &= \frac{1}{2\pi}\left|\int_{\sigma+iN}^{\sigma+i\infty} s^{-2}[h'(0)+\mathcal{L}h''(s)]e^{st}\,ds\right| \\ &\le \frac{1}{\pi}\int_N^\infty |s|^{-2}\{|h'(0)|+|\mathcal{L}h''|\}|e^{ts}|\,d\eta \\ &\le \frac{e^{\sigma t}}{\pi}\left[|h'(0)|+\frac{A}{\sigma-p}\right]\int_N^\infty(\sigma^2+\eta^2)^{-1}\,d\eta \le C_2e^{\sigma t}N^{-1}\,.\end{aligned}$$

The third integral can be bound in the similar way. Hence,

$$|h(t)| \le C_3\{\|g\|\exp(B_1N^q)+1/N\}, \quad 0 \le t \le T\,,$$

where $C_3$ depends on $\sigma$, $p$, $h'(0)$, $A$, $B$, and $q$. Choosing $N^q = \log(K/\|g\|)^{1/2}/B_1$ we obtain

$$|h(t)| \le C_4\{\|g\|^{1/2}+(\log(K/\|g\|))^{-1/q}\}, \quad 0 \le t \le T\,. \blacksquare$$

**4. Continuous dependence of the heat flux upon the internal measurements for a semi-infinite slab.** Consider the following inverse problem: find a piecewise continuous function $q$ satisfying $q(t) = u_x(0,t)$ for $0 < t < T$, with some $T > 0$, given that $u$ satisfies

$$u_t = u_{xx} + vu_x + ru, \quad 0 < x < \infty,\ t > 0,\ v, r > 0, \tag{4.1}$$

$$u(x,0) = 0,\ u(1,t) = g(t),\ |u(x,t)| \le C_1 \exp(C_2 x^\alpha + rt),\ \alpha < 2. \tag{4.2}$$

It can be shown [2] that a solution of the corresponding (direct) initial-boundary problem with the boundary condition $q(t) = u_x(0,t)$ can be expressed as a single layer potential,

$$u(x,t) = -2 \int_0^t w(x, t-\tau) q(\tau)\, d\tau, \quad x > 0,\ t > 0, \tag{4.3}$$

where $w$ is given by formula (2.5). For $x = 1$ we have

$$g(t) = \int_0^t H(t-\tau) q(\tau)\, d\tau,$$

$$H(t) = -2\ w(1,t) = \frac{-1}{\sqrt{\pi t}} \exp\left[ -\frac{v}{2} + (r - v^2/4)t - 1/(4t) \right].$$

After application of the Laplace transform $\mathcal{L}$ one gets

$$\mathcal{L}q(s) = \mathcal{L}g(s)/\mathcal{L}H(s),$$

where

$$\mathcal{L}H(s) = -\exp(-v/2) \frac{\exp(-(s-A)^{1/2})}{(s-A)^{1/2}}, \quad A = r - v^2/4.$$

Since $\mathcal{L}H(s) \ne 0$, when $\mathrm{Re}(s - A) > 0$, the uniqueness of the solution $q$ of the inverse problem follows if $u$ given by (4.3) is the unique solution of the direct problem. This is assured under the assumption $|u(x,t)| \le C_1 \exp(C_2 x^\alpha + rt)$, $\alpha < 2$. Thus we have

THEOREM 1. *There exists at most one solution of the problem* (4.1)–(4.2).

Also, since $1/\mathcal{L}H(s) = \mathrm{const}\sqrt{s-A}\exp(\sqrt{s-A})$, it follows that

$$|\mathcal{L}H(s)|^{-1} < B\exp(C\sqrt{|s|}), \quad \text{for some } B, C > 0.$$

Thus a simple application of Lemma 1 yields

THEOREM 2. *Let* $q(0) = 0$, $|q''(t)| < A\exp(pt)$, $\|g\| \le K < \infty$, $T > 0$. *If* $q$ *is the solution of the problem* (4.1)–(4.2), *then there exists* $M > 0$ *such that*

$$|q(t)| \le \frac{M}{(\log(K/\|g\|))^2}, \quad 0 \le t \le T.$$

**5. Continuous dependence of boundary temperature upon internal data for the finite layer.** Consider the problem of determination of piecewise continuous boundary functions $u_0(t) = u(0,t)$ and $u_1(t) = u(1,t)$ so that $u$ is bounded, continuous in $x$ on [0,1] and satisfies

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \tag{5.1}$$

$$u(0,t) = 0, \quad u(A,t) = a(t), \quad u(B,t) = b(t), \quad 0 < A < B < 1. \tag{5.2}$$

There exists an integral representation of the unique solution $u$ to the direct initial-boundary problem with $u_0$, $u_1$ as the boundary data given by

$$u(x,t) = \int_0^t \Big[K(x,0,t-\tau)h_0(\tau) - \int_0^t K(x,1,t-\tau)h_1(\tau)\Big]\,d\tau,$$

where

$$K(x,y,t-\tau) = \frac{1}{4\sqrt{\pi}}\frac{x-y}{(t-\tau)^{3/2}}\exp\frac{-(x-y)^2}{4(t-\tau)}.$$

By substituting $A$ and $B$ in the above equation, a system of two Volterra equations of the first kind for the density functions $h_0$ and $h_1$ is obtained. Note that a necessary condition for the existence of the solution $\{h_0, h_1\}$ to the system is that the internal data $\{u_A, u_B\}$ be smooth. Once the densities are obtained the boundary functions can be recovered from

$$u_0(t) = \frac{1}{2}h_0(t) - \int_0^t K(0,1,t-\tau)h_1(\tau)\,d\tau, \tag{5.4}$$

$$u_1(t) = \int_0^t K(1,0,t-\tau)h_0(\tau)\,d\tau + \frac{1}{2}h_1(t) \tag{5.5}$$

(the jump conditions).

To show continuous dependence of $u_i$'s upon $a$ and $b$ apply the Laplace transform to (5.3) (with $x = A$ and $x = B$) to obtain

$$\mathcal{L}a(s) = \mathcal{L}K(A,0,s)\mathcal{L}h_0(s) + \mathcal{L}K(A,1,s)\mathcal{L}h_1(s), \tag{5.6}$$

$$\mathcal{L}b(s) = \mathcal{L}K(B,0,s)\mathcal{L}h_0(s) + \mathcal{L}K(B,1,s)\mathcal{L}h_1(s). \tag{5.7}$$

The solution of this system of equations exists if the system's determinant

$$\det(s) = \mathcal{L}K(A,0,s)\mathcal{L}K(B,1,s) - \mathcal{L}K(B,0,s)\mathcal{L}K(A,1,s)$$

is not identically zero. Since for any $c$ and $d$

$$\mathcal{L}K(c,d,s) = \begin{cases} \frac{1}{2}\exp(-\sqrt{s}(c-d)), & c > d, \\ -\frac{1}{2}\exp(-\sqrt{s}(d-c)), & c < d, \end{cases}$$

it follows that

$$\det(s) = -\frac{1}{4}[\exp(-\sqrt{s}(1+A-B)) - \exp(-\sqrt{s}(1+B-A))] \neq 0,$$

unless $A = B$. Moreover,

$$|1/\det(s)| = C\exp(B|s|^{1/2}), \quad \text{for some } C, B > 0.$$

Combining this with an argument similar to Lemma 1 we obtain

THEOREM 3. *Let* $u_i(0) = 0$, $|u_i''(t)| < A\exp(pt)$ $(i = 0, 1)$, $\|g\| = \|a\| + \|b\| \le K < \infty$, $T > 0$. *If* $u_0$, $u_1$ *are the solutions of* (5.1)–(5.2), *then there exists* $M > 0$ *such that*

$$|u_i(t)| \le \frac{M}{(\log(K/\|g\|))^2}, \quad 0 \le t \le T.$$

Proof. Since $u_i$ are calculated from (5.4)–(5.5) it can be shown that $|u_i(t)| \le C\max_i(|h_i(t)|)$, $t > 0$, for some $C > 0$. Thus it suffices to show that

$$|h_i(t)| \le \frac{\text{const}}{(\log(K/\|g\|))^2}, \quad t < T.$$

Solving equations (5.6)–(5.7) for $\mathcal{L}h_i(s)$ we obtain

$$|\mathcal{L}h_i(s)| \le C\exp(B\sqrt{|s|})(\|a\| + \|b\|).$$

The theorem follows from an argument similar to that in Lemma 1 applied to the functions $h_i$. Note here that the conditions $h_i(0) = 0$, $|h_i''(t)| < A\exp(pt)$ $(i = 0, 1)$, needed in the argument, follow from the hypotheses of the theorem via the equations (5.4)–(5.5). (The fact that these integral equations are of the second kind greatly simplifies this check.)

**6. An inverse boundary value problem in the unit ball.** Let $\Omega \subset \mathbb{R}^3$ be the unit ball; given $g : [0, \infty) \to \mathbb{R}$ and a fixed point $z \in \Omega$, find a piecewise continuous function $q$ such that a bounded, radially symmetric function $u$ solves the direct initial-boundary value problem

$$\left(\frac{\partial}{\partial t} - k\nabla^2 - r\right)u(x,t) = 0, \quad x \in \Omega,\ t > 0,\ k > 0, \tag{6.1}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial n}(\xi, t) = q(t), \quad \xi \in \partial\Omega, \tag{6.2}$$

together with

$$u(z,t) = g(t), \quad g(0) = 0, \quad z \in \Omega \quad \text{(internal data)}. \tag{6.3}$$

An application of Lemma 1 yields:

THEOREM 4. *Let* $u$ *be a bounded, radially symmetric function satisfying equations* (6.1)–(6.3) *with* $q(0) = 0$, $|q''(t)| \le Be^{pt}$, $\|g\| \le K < \infty$. *Then, given* $T > 0$, *there exists* $M > 0$ *such that*

$$|q(t)| \le \frac{M}{(\log(K/\|g\|))^2}, \quad 0 \le t \le T.$$

Proof. The solution $u$ can be expressed as a potential of the single layer,

$$u(x,t) = \int_0^t \int_\Gamma w(x-\xi, t-\tau) h(\xi,\tau)\, ds_\xi \, d\tau\,,$$

where $w$ is given by

$$w(x,t) = \frac{1}{(4k\pi t)^{3/2}} \exp\left[rt - \frac{|x|^2}{4kt}\right].$$

Note that as a consequence of the radial symmetry of $u$ the potential density $h$ does not depend on $\xi$. To see this assume that $x, y \in \Omega$ with $|x| = |y|$ and $F$ is a rotation about the center of a unit sphere such that $F(x) = y$. Given $\zeta \in \Gamma$, define $\xi$ by $F(\xi) = \zeta$. Then

$$\begin{aligned}
&\int_0^t \int_\Gamma w(y-\zeta, t-\tau) h(\zeta,\tau)\, ds_\zeta \, d\tau \\
&\qquad = \int_0^t \int_\Gamma w(y-F(\xi), t-\tau) h(F(\xi),\tau)\, ds_\zeta \, d\tau \\
&\qquad = \int_0^t \int_\Gamma w(x-\xi, t-\tau) h(F(\xi),\tau)\, ds_\zeta \, d\tau \\
&\qquad = \int_0^t \int_\Gamma w(x-\xi, t-\tau) h(\xi,\tau)\, ds_\xi \, d\tau\,.
\end{aligned}$$

The second equality holds since $w(x-\xi, t-\tau) = w(y-\zeta, t-\tau)$ and the last one follows from $u(x,t) = u(y,t)$. Thus,

$$\int_0^t \int_\Gamma w(x-\xi, t-\tau)[h(F(\xi),\tau) - h(\xi,\tau)]\, ds_\xi \, d\tau = 0, \qquad t > 0,\ x \in \Omega\,,$$

which implies $h(F(\xi),\tau) - h(\xi,\tau) \equiv 0$.

For $x = z$ we can write

$$g(t) = \int_0^t H_z(t-\tau) h(\tau)\, d\tau\,,$$

where

$$H_z(t) = \int_\Gamma w(z-\xi, t)\, ds_\xi\,.$$

Since $q$ is given by

$$q(t) = -\frac{1}{2} h(t) + \int_0^t \int_\Gamma \frac{\partial w}{\partial n}(\zeta - \xi, t-\tau)\, ds_\xi \, h(\tau)\, d\tau \quad \text{for any } \zeta \in \partial\Omega\,,$$

the hypotheses of the theorem imply that $h(0) = 0$, $|h''(t)| \leq Be^{pt}$. Hence it suffices to show that $|\mathcal{L}H_z(s)|^{-1} \leq \exp(C|s|^\alpha)$, $\alpha > 0$. After some computations we obtain

$$H_z(t) = \int_\Gamma \frac{1}{(4k\pi t)^{3/2}} \exp\left[rt - \frac{|z-\xi|^2}{4kt}\right] ds_\xi$$
$$= \frac{\exp(rt)}{(4k\pi t)^{1/2}} \frac{1}{|z|}\left[\exp\frac{(1-|z|)^2}{-4kt} - \exp\frac{(1+|z|)^2}{-4kt}\right],$$

where $|z|^2 = z_1^2 + z_2^2 + z_3^2$ . In the case $z = 0$ this reduces to

$$H_0(t) = \frac{1}{(4k\pi t)^{3/2}} 4\pi \exp\left[rt - \frac{1}{4kt}\right].$$

The Laplace transforms of $H_z$ and $H_0$ are:

$$\mathcal{L}H_z(s) = A[\exp(-B\sqrt{s-r}) - \exp(-C\sqrt{s-r})]/\sqrt{s-r},$$
$$\mathcal{L}H_0(s) = A\exp(-B\sqrt{s-r}),$$

for some generic, positive constants $A, B$ and $C$. This implies that

$$|\mathcal{L}H_z(s)|^{-1} \leq \exp(D|s|^{1/2})$$

for all $z \in \Omega$, including $z = 0$, and Lemma 1 can be applied to prove the theorem.

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