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TRANSITION PROBABILITY DENSITY FUNCTION OF A CERTAIN DIFFUSION PROCESS CONCENTRATED ON A HALF LINE

Abstract. We prove that under some assumptions there exists a diffusion process satisfying a one-dimensional Itô equation and living in a time-dependent half line. We give a formula on the transition probability density function of this process. This is also a probabilistic formula for a solution of a deterministic Fokker–Planck equation in a time-dependent half line.

1. Introduction. Diffusion processes are used in biology to the description of polulation dynamics. In [5] and [6] A. G. Nobile and L. M. Ricciardi consider a growth process of a population in a fluctuating environment. This process can be a solution of some one-dimensional Itô or Stratonovich equation. In some cases the transition probability density of these solutions, concentrated on a finite interval or on a half line, are given ([5]). Some sufficient conditions on the existence of a strong solution of the Itô equation

$$X(t) = X_0 + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dW(s)\,, \tag{1.1}$$

living in a bounded region in $\mathbb{R}^n$, are given in [3]. The transition probability density functions for one-dimensional diffusion processes with varying boundaries are investigated in [2]. For a one-dimensional process satisfying (1.1), the transition probability density function concentrated on a finite spatial interval is given in [4].

In the present paper we give a formula for the transition probability density function of a solution of (1.1), concentrated on a time-dependent half

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line. Using our result we obtain the transition probability density function for the growth process satisfying the equation

$$dX(t) = \left[-\varrho X\left(1 - \frac{\ln X}{\beta}\right) + \lambda X \ln X \left(1 - \frac{\ln X}{\beta}\right)\right] dt - X\left(1 - \frac{\ln X}{\beta}\right) dW(t) \tag{1.2}$$

in the diffusion interval $(\exp(\beta), \infty)$, in the case $\beta > 0$ and $\lambda = -1/(2\beta)$. This equation is investigated in [6], p. 293, but the transition probability density functions are not given there.

2. Constant boundary. To formulate our theorem concerning a diffusion process concentrated on a half line we need some definitions.

Let the coefficients $a(t,x)$ and $b(t,x)$ of (1.1) be defined for $t \geq 0$ and $x \geq 0$. If $b(t,x) \neq 0$ for $t \geq 0$ and $x > 0$, then we introduce transformations $u(t,x)$ and $v(t,x)$ by

$$u(t,x) = \int_1^x \frac{dy}{b(t,y)}, \tag{2.1}$$

$$x = \int_1^{v(t,x)} \frac{dy}{b(t,y)}. \tag{2.2}$$

Note that $v(t,\cdot)$ is the inverse to $u(t,\cdot)$ for $t \geq 0$. Then we define

$$m(t,x) = u_t(t, v(t,x)) + u_x(t, v(t,x))a(t, v(t,x)) + \frac{1}{2}u_{xx}(t, v(t,x))b^2(t, v(t,x)) \tag{2.3}$$

$$\sigma(t,x) = u_x(t, v(t,x))b(t, v(t,x)). \tag{2.4}$$

THEOREM 1. *Assume that the random variable X_0 is independent of a given Wiener process $W(t)$ and $X_0 > 0$ with probability 1. Let the coefficients $a(t,x)$ and $b(t,x)$ satisfy the following conditions*:

1) *$a(t,x)$ and $b(t,x)$ are C^2 in some open neighbourhood of the set $\{(t,x) : t \geq 0,\ x \geq 0\}$.*

2) *$b(t,0) = 0$ and $b_x(t,0) \neq 0$ for $t \geq 0$, $b(t,x) > 0$ for $t \geq 0$, $x > 0$, and*

$$\int_1^\infty \frac{ds}{b(t,s)} = \infty \quad \text{for each } t \geq 0.$$

3) *$m_x(t,x)$ and $m_t(t,x)$ are bounded for $t \geq 0$, $x \in \mathbb{R}$.*

4) *$\mathbb{E}[u(0,X_0)]^2 < \infty$.*

Then (1.1) *has a strong solution $X(t)$ satisfying the initial condition $X(t)|_{t=0} = X_0$ a.s. and such that* $\text{Prob}\{X(t) > 0, t \geq 0\} = 1$. *If $X(t)$ and*

$Y(t)$ are two such solutions of (1.1), then

$$\mathrm{Prob}\{\sup_{t\geq 0}|X(t)-Y(t)|=0\}=1\,,$$

i.e. pathwise uniqueness holds.

A transition probability density of the process $X(t)$ exists and is given by

$$\begin{aligned}
(2.5)\qquad q(t,x,s,y) &= \frac{1}{\sqrt{2\pi(s-t)}\,b(s,y)} \\
&\times \exp\left\{-\frac{\{u(s,y)-u(t,x)\}^2}{2(s-t)}+\overline{M}(s,u(s,y))-\overline{M}(t,u(t,x))\right\} \\
&\times \mathbb{E}\exp\left\{(s-t)\int_0^1 \overline{B}(t+z(s-t),u(t,x)\right. \\
&\qquad\qquad \left.+\sqrt{s-t}\,\eta^*_{t,s}(z)+z\{u(s,y)-u(t,x)\})\,dz\right\},
\end{aligned}$$

where

$$(2.6)\qquad \overline{B}(t,x)=-\frac{1}{2}m^2(t,x)-\frac{1}{2}m'_x(t,x)-\int_0^x m'_t(t,y)\,dy\,,$$

$$(2.7)\quad \overline{M}(t,u(t,x))=\int_1^x\left(\int_1^y \frac{\partial}{\partial t}\left(\frac{1}{b(t,s)}\right)ds+\frac{a(t,y)}{b(t,y)}-\frac{1}{2}b_x(t,y)\right)\frac{dy}{b(t,y)}\,,$$

$$W^*_{t,s}=\frac{W[t+(s-t)z]-W(t)}{\sqrt{s-t}},\qquad \eta^*_{t,s}(z)=W^*_{t,s}(z)-zW^*_{t,s}(1)\,,$$

$u(t,x)$ is given by (2.1) and $m(t,x)$ is given by (2.3).

Proof. We consider the function $u(t,\cdot)$ for $t\geq 0$. By Conditions 1 and 2 we have $b(t,y)=b(t,y)-b(t,0)=yb_x(t,\xi)$ for some $\xi\in(0,y)$ and $0<b_x(t,\xi)\leq M<\infty$ for some $M\in\mathbb{R}$. Hence

$$\frac{1}{b(t,y)}=\frac{1}{yb_x(t,\xi)}\geq\frac{1}{yM}\,.$$

Thus

$$\int_x^1\frac{dy}{b(t,y)}\geq\frac{1}{M}\int_x^1\frac{dy}{y}=\frac{1}{M}\ln\frac{1}{x}\,,$$

$$\int_x^1\frac{dy}{b(t,y)}\to+\infty\quad\text{and}\quad u(t,x)\to-\infty\qquad\text{as } x\to 0^+\,.$$

By Condition 2, $u(t,x)\to\infty$ as $x\to\infty$, hence $u(t,x)$ is a one-to-one mapping from $(0,\infty)$ onto $\mathbb{R}$, for $t\geq 0$.

Now we look for a process $\xi(t)$ satisfying (1.1) with the coefficients $m(t,x)$ and $\sigma(t,x)$ given by (2.3) and (2.4) and with the initial condition $P\{\xi(t)|_{t=0} = u(0, X_0)\} = 1$.

To this end we show that $m(t,x)$ and $\sigma(t,x)$ satisfy all the assumptions of the existence and uniqueness theorem ([1], Theorem 1, p. 40). Differentiating the identity (2.2) with respect to x we obtain

$$v_x(t,x) = b(t, v(t,x)). \tag{2.8}$$

In the same way, by the identity $u(t, v(t,x)) = x$, we obtain $u_x(t, v(t,x)) \times v_x(t,x) = 1$, and consequently

$$u_x(t, v(t,x)) = \frac{1}{b(t, v(t,x))}. \tag{2.9}$$

Thus, from (2.4) and (2.9), we have

$$\sigma(t,x) = 1. \tag{2.10}$$

By (2.10) and by Conditions 3 and 4 we conclude that all assumptions of ([1], Theorem 1, p. 40) are satisfied. Thus there exists a solution $\xi(t)$ of (1.1) satisfying the conditions

(A) $\xi(t)$ is a.s. continuous and $\xi(t) = u(0, X_0)$ for $t = 0$.
(B) $\sup \mathbb{E}(\xi(t))^2 < \infty$.

If $\xi_1(t)$ and $\xi_2(t)$ are two solutions of (1.1) satisfying (A) and (B), then

$$\text{Prob}\{\sup_{0\le t\le T} |\xi_1(t) - \xi_2(t)| = 0\} = 1,$$

i.e. pathwise uniqueness holds.

By Condition 1 and by (2.1), the coefficients $m(t,x)$ and $\sigma(t,x)$ are continuous in both arguments. Hence by ([1], Theorem 2, p. 68) the process $\xi(t)$ is a diffusion with diffusion coefficient $\sigma^2(t,x)$ and displacement coefficient $m(t,x)$.

Now we show that all conditions of ([1], Theorem 2, p. 96) are satisfied. By (2.1), (2.3), (2.10) and by Condition 3 we conclude that $m_x(t,x)$ and $\sigma_x(t,x)$ exist and are bounded. The derivatives σ''_{xx}, σ''_{xt}, σ'_t and σ''_{tt} exist. By (2.10) we can calculate the following functions (cf. the statement of ([1], Theorem 2, p. 96)):

$$x = \int_0^{h(t,x)} \frac{dy}{\sigma(t,y)} = h(t,x) \tag{2.11}$$

and

$$\overline{m}(t,x) = \int_0^{h(t,x)} \frac{\sigma_t(t,y)}{\sigma^2(t,y)}\,dy + \frac{m(t,h(t,x))}{\sigma(t,h(t,x))} - \frac{1}{2}\sigma_x(t,h(t,x)) = m(t,x) \tag{2.12}$$

so $\overline{B}(t,x)$ is given by (2.6).

Taking into account Condition 3 we conclude that there exists some constant $K > 0$ such that

$$\left| -\frac{1}{2}m'_x(t,x) - \int_0^x m'_t(t,y)\,dy \right| \le K(1+|x|)\,.$$

Hence

$$\limsup_{|x|\to\infty} \frac{1}{1+x^2} \sup_{0\le t\le T} \overline{B}(t,x) = \limsup_{|x|\to\infty} \frac{1}{1+x^2} \sup_{0\le t\le T} \left(-\frac{1}{2}m^2(t,x) \right) \le 0\,.$$

Thus all conditions of ([1], Theorem 2, p. 96) are satisfied.

Hence the transition probability of the process $\xi(t)$:

$$P(t,x,s,y) = \mathrm{Prob}\{\xi(s) \le y \mid \xi(t) = x\}, \qquad t < s\,,$$

has a density given by

$$\begin{aligned} (2.13) \qquad p(t,x,s,y) &= \frac{1}{\sqrt{2\pi(s-t)}\sigma(s,y)} \\ &\times \exp\left\{ -\frac{(\varphi(s,y)-\varphi(t,x))^2}{2(s-t)} + \overline{M}(s,\varphi(s,y)) - \overline{M}(t,\varphi(t,x)) \right\} \\ &\times \mathbb{E}\exp\left\{ (s-t) \int_0^1 \overline{B}(t+z(s-t), \varphi(t,x) \right. \\ &\qquad \left. + \sqrt{s-t}\,\eta^*_{t,s}(z) + z[\varphi(s,y)-\varphi(t,x)])\,dz \right\}, \end{aligned}$$

where $h(t,x)$, $\overline{m}(t,x)$ and $\overline{B}(t,x)$ are given by (2.11), (2.12) and (2.6) respectively, and moreover

$$\overline{M}(t,x) = \int_0^x \overline{m}(t,y)\,dy, \qquad \varphi(t,x) = \int_0^x \frac{dy}{\sigma(t,y)}\,,$$

$$W^*_{t,s} = \frac{W(t+(s-t)z) - W(t)}{\sqrt{s-t}}, \qquad \eta^*_{t,s}(z) = W^*_{t,s}(z) - zW^*_{t,s}(1)\,.$$

Now we define the process

$$X(t) = v(t,\xi(t))\,.$$

Using Itô's formula ([1], Theorem 4, p. 24) it is easy to prove that the process $X(t)$ satisfies the equation (1.1) with coefficients $a(t,x)$ and $b(t,x)$. Thus $X(t)$ is a strong solution of (1.1), and moreover

$$X(0) = v(0,u(0,X_0)) = X_0\,.$$

If $Y(t)$ is a solution of (1.1) and $Y(0) = X_0$, then

$$\operatorname{Prob}\{\sup_{0\le t\le T}|X(t)-Y(t)|=0\}$$
$$=\operatorname{Prob}\{\sup_{0\le t\le T}|u(t,X(t))-u(t,Y(t))|=0\}=1,$$

i.e. pathwise uniqueness holds.

Furthermore,

$$\operatorname{Prob}\{X(t)>0, t\ge 0\}=1.$$

The process $X(t)$ is a diffusion with diffusion coefficient $b^2(t,x)$ and drift coefficient $a(t,x)$ ([1], p. 66). By the definition of the process $X(t)$, and by Lemma 1 of [4], the transition probability

$$Q(t,x,s,y)=\operatorname{Prob}\{X(s)\le y \mid X(t)=x\},\quad t<s,$$

has a density given by

$$q(t,x,s,y)=p(t,u(t,x),s,u(s,y))u_y(s,y) \tag{2.14}$$

where $p(t,x,s,y)$ is given by (2.13).

Now we want to express the density (2.14) in terms of the coefficients of (1.1). To this end we make some calculations. The functions $\varphi(t,x)$ and $\overline{M}(t,x)$ are as follows:

$$\varphi(t,x)=\int_0^x \frac{dy}{\sigma(t,y)}=\int_0^x dy=x,$$
$$\overline{M}(t,x)=\int_0^x \overline{m}(t,y)\,dy=\int_0^x m(t,y)\,dy.$$

After some standard calculations we find that

$$m(t,x)=\int_1^{v(t,x)}\frac{\partial}{\partial t}\left(\frac{1}{b(t,s)}\right)ds+\frac{a(t,v(t,x))}{b(t,v(t,x))}-\frac{1}{2}b_x(t,v(t,x)). \tag{2.15}$$

Next we calculate

$$\overline{M}(t,u(t,x))=\int_0^{u(t,x)}\left(\int_1^{v(t,z)}\frac{\partial}{\partial t}\left(\frac{1}{b(t,s)}\right)ds\right.$$
$$\left.+\frac{a(t,v(t,z))}{b(t,v(t,z))}-\frac{1}{2}b_x(t,v(t,z))\right)dz.$$

We substitute in this integral $v(t,z)=y$ (for fixed t). From the identity $v(t,u(t,z))=z$, we obtain $z=u(t,y)$. By (2.1) we have $u_x(t,x)=1/b(t,x)$, hence $dz=dy/b(t,y)$, and consequently we obtain the formula (2.7) for $\overline{M}(t,u(t,x))$.

From (2.6) we find that $\overline{B}(t,x) = W(t, v(t,x))$, where

$$\begin{aligned} W(t,z) = & -\frac{1}{2}\bigg(\int_1^z \frac{\partial}{\partial t}\bigg(\frac{1}{b(t,s)}\bigg)\, ds + \frac{a(t,z)}{b(t,z)} - \frac{1}{2} b_x(t,z)\bigg)^2 \\ & - \frac{1}{2}\bigg(a_x(t,z) - \frac{b_t(t,z) + a(t,z) b_x(t,z)}{b(t,z)} - \frac{1}{2} b_{xx}(t,z) b(t,z)\bigg) \\ & - \int_1^z \bigg\{ \int_1^y \frac{\partial^2}{\partial t^2}\bigg(\frac{1}{b(t,s)}\bigg)\, ds + \bigg(\frac{b_t(t,y)}{b(t,y)} - a_x(t,y) + \frac{a(t,y) b_x(t,y)}{b(t,y)} \\ & + \frac{1}{2} b_{xx}(t,y) b(t,y)\bigg) \int_1^y \frac{\partial}{\partial t}\bigg(\frac{1}{b(t,s)}\bigg)\, ds \\ & + \frac{a_t(t,y)}{b(t,y)} - \frac{a(t,y) b_t(t,y)}{b^2(t,y)} - \frac{1}{2} b_{tx}(t,y) \bigg\} \frac{dy}{b(t,y)}, \end{aligned}$$

which leads to the formula (2.5) for the transition probability density of the process X.

This completes the proof of Theorem 1.

In this way we have obtained the probabilistic formula (2.5) for a solution of the following deterministic Fokker–Planck equation:

$$\frac{\partial q(t,x,s,y)}{\partial s} = -\frac{\partial}{\partial y}[a(s,y) q(t,x,s,y)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[b^2(s,y) q(t,x,s,y)], \qquad t < s,$$

with the initial condition $q(t,x,t,y) = \delta(x-y)$, $x \geq 0$.

This solution has the following properties: $q(t,x,s,y) \geq 0$ for $s \geq t$ and $y \geq 0$, $q(t,x,s,y) = 0$ for $s \geq t$ and $y < 0$, and moreover $\int_0^\infty q(t,x,s,y)\, dy = 1$ for each $s \geq 0$.

3. Time-dependent half line. Let $G = \{(t,x) : 0 \leq t \leq T,\ x \geq \gamma(t)\}$, where γ is defined for $t \in [0,T]$. Define $f(t,x) = x - \gamma(t)$. So $f(t,\cdot)$ is a one-to-one mapping from $(\gamma(0), \infty)$ onto $(0,\infty)$, for $t \in [0,T]$. Let $g(t,\cdot)$ denote the inverse of $f(t,\cdot)$, i.e. $g(t,x) = x + \gamma(t)$.

Assume that the random variable $X_{1,0}$ is independent of a Wiener process $W(t)$ and $\mathrm{Prob}\{X_{1,0} > \gamma(0)\} = 1$.

Now we consider (1.1) with the coefficients $a_1(t,x)$ and $b_1(t,x)$ defined in G, with the initial condition $X_1(t)|_{t=0} = X_{1,0}$ a.s. We define

(3.1) $\qquad a(t,x) = a_1(t, g(t,x)),$

(3.2) $\qquad b(t,x) = b_1(t, g(t,x)) \quad$ for $t \in [0,T]$ and $x > 0$.

If $a(t,x)$ and $b(t,x)$ given by (3.1) and (3.2) satisfy all the conditions of Theorem 1, then the following corollary is true:

COROLLARY 1. *Equation* (1.1) *with coefficients* $a_1(t,x)$ *and* $b_1(t,x)$ *has a strong solution* $X_1(t)$ *satisfying the initial condition* $\text{Prob}\{X_1(t)|_{t=0} = X_{1,0}\} = 1$. *Moreover*, $\text{Prob}\{X_1(t) > \gamma(t),\ 0 \le t \le T\} = 1$.

If X_1 *and* Y_1 *are two such solutions of* (1.1) *then*

$$\text{Prob}\{\sup_{0 \le t \le T} |X_1(t) - Y_1 t)| = 0\} = 1\,,$$

i.e. pathwise uniqueness holds.

A transition probability density of the process $X_1(t)$ *exists and is given by*

$$q_1(t,x,s,y) = q(t, f(t,x), s, f(s,y)) \tag{3.3}$$

where $q(t,x,s,y)$ *is given by* (2.5) *for the coefficients* $a(t,x)$ *and* $b(t,x)$ *defined by* (3.1) *and* (3.2). *The function* q_1 *is a solution of the Fokker–Planck equation and satisfies the condition*

$$\int_{\gamma(t)}^{\infty} q_1(t,x,s,y)\,dy = 1 \qquad \textit{for } s \ge t\,.$$

4. Example. We can use Corollary 1 to calculate the transition probability density function of the diffusion process $X_1(t)$ satisfying the equation (1.2), in the case $\lambda = -1/(2\beta)$, $\beta > 0$, on the interval (e^β, ∞), with the initial condition $\text{Prob}\{X_1(0) = x_0\} = 1$, where $x_0 > e^\beta$. In this case $\gamma(t) = e^\beta$ for $t \ge 0$. Hence

$$\begin{aligned} a(x) = &-\varrho(x+e^\beta)\left(1 - \frac{\ln(x+e^\beta)}{\beta}\right) \\ &+ \lambda(x+e^\beta)\ln(x+e^\beta)\left(1 - \frac{\ln(x+e^\beta)}{\beta}\right), \\ b(x) = &\,(x+e^\beta)\left(\frac{\ln(x+e^\beta)}{\beta} - 1\right). \end{aligned}$$

Now we can show that $a(x)$ and $b(x)$ satisfy all the assumptions of Theorem 1 in the interval $(0,\infty)$:

$$b(0) = 0, \quad b'(0) = 1/\beta > 0, \quad b(x) > 0 \quad \text{for } x > 0,$$
$$\int_1^\infty \frac{dx}{b(x)} \ge \beta \int_1^\infty \frac{dx}{(x+e^\beta)\ln(x+e^\beta)} = \infty\,.$$

From (2.15) we have

$$m(x) = \frac{a(v(x))}{b(v(x))} - \frac{1}{2}b'(v(x))\,. \tag{4.1}$$

Consequently, taking into account (2.8), we obtain

$$m'(x) = \lambda + \frac{1}{2\beta} - \left(\lambda + \frac{1}{2\beta}\right)\frac{\ln(x + e^{\beta})}{\beta}.$$

So $m'(x)$ is bounded if and only if $\lambda + 1/(2\beta) = 0$. Then $m'(x) \equiv 0$ and all the assumptions of Theorem 1 are satisfied. From (2.2) and (4.1) we have

$$m(x) \equiv m(0) = \varrho + \frac{1}{2} - \frac{1}{2\beta}.$$

Define

$$\mu = \varrho + \frac{1}{2} - \frac{1}{2\beta}.$$

From (2.6) and (2.12) we have, respectively,

$$\overline{B}(x) = -\frac{1}{2}\mu^2, \tag{4.2}$$

$$\overline{M}(x) = \mu x. \tag{4.3}$$

By Corollary 1 we conclude that the probability

$$\mathrm{Prob}\{X_1(t) \le y \mid X_1(0) = x_0\}$$

has a density function given by

$$q_1(0, x_0, s, y) = q(0, x_0 - e^{\beta}, s, y - e^{\beta})$$

where q is given by (2.5). Taking into account (4.2) and (4.3) we have

$$\begin{aligned} q_1(0, x_0, s, y) = \frac{1}{\sqrt{2\pi s}\, b(y - e^{\beta})} \exp\bigg\{ &- \frac{[u(y - e^{\beta}) - u(x_0 - e^{\beta})]^2}{2s} \\ &+ \mu[u(y - e^{\beta}) - u(x_0 - e^{\beta})]\bigg\} \exp(-\tfrac{1}{2}\mu^2 s). \end{aligned}$$

By (2.1),

$$\begin{aligned} u(y - e^{\beta}) - u(x_0 - e^{\beta}) &= \int_{z_1}^{z_2} \frac{dz}{b(z)} = \beta \int_{z_1}^{z_2} \frac{dz}{(z + e^{\beta})[\ln(z + e^{\beta}) - \beta]} \\ &= \beta \ln\left(\frac{\ln y - \beta}{\ln x_0 - \beta}\right), \end{aligned}$$

where $z_1 = x_0 - e^{\beta}$, $z_2 = y - e^{\beta}$. So

$$q_1(0, x_0, s, y) = \frac{\beta \exp(-\frac{1}{2}\mu^2 s)}{\sqrt{2\pi s}\, y(\ln x_0 - \beta)} (q(y))^{w(y)}$$

where

$$q(y) = \frac{\ln y - \beta}{\ln x_0 - \beta}, \qquad w(y) = \mu\beta - 1 - \frac{\beta^2}{2\varrho} \ln q(y).$$

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