ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,4 (1993), pp. 591–595

A. KRZYWICKI and T. NADZIEJA (Wrocław)

# A NOTE ON THE POISSON–BOLTZMANN EQUATION

In this note we complete our earlier results [4] concerning the integro-differential equation

$$-\Delta u = \sigma\mu(u)\exp u \tag{1}$$

considered in a bounded domain $\Omega \subset \mathbb{R}^3$, where $\mu(u) = (\int_\Omega \exp u)^{-1}$ and $\sigma$ is a positive parameter. One of the possible physical interpretations of (1) is to look at $\exp(-u)$ as the density of a gas in thermodynamical equilibrium, consisting of gravitationally interacting particles, and filling up $\Omega$. In this case $\sigma$ should be identified with $M_0/(kT)$, where $k$ is the Boltzmann constant, $T$ is the absolute temperature and $M_0$ is the total mass of the gas.

One of the possible boundary conditions imposed upon $u$ may be

$$u_{|\partial\Omega} = 0\,; \tag{2}$$

however, only in case of radial symmetry (with $\Omega$ being a ball) (2) is physically acceptable. Assuming that $\Omega$ is the annulus $\Omega = \{x : a < |x| < 1\}$, $0 < a < 1$, physically reasonable conditions are

$$u(a) = 0, \quad u'(1) = -\sigma. \tag{3}$$

The last condition means that the gravitational force acting at the exterior boundary of $\Omega$ is proportional to $\sigma$.

Although, in general, (2) has no direct physical interpretation, the problem (1), (2) is interesting from the mathematical point of view, due to the fact that the existence of solution of (1), (2) depends on the geometry of $\Omega$.

It was shown in [4], by using the Pokhozhaev identity, that in the case of star-shaped $\Omega$, the problem (1), (2) has no solution for sufficiently large $\sigma$. However, if $\Omega$ is an annulus, radially symmetric solutions of (1), (2) exist for all positive $\sigma$.

---

1991 *Mathematics Subject Classification*: Primary 35J60.
*Key words and phrases*: Poisson–Boltzmann equation, radially symmetric solutions.
The preparation of this paper was supported by the KBN grant 649/2/91.

Recently similar results, obtained by using variational methods, have been presented in [1], together with another physical interpretation of the problem (1), (2).

In [1], [4] the problem of uniqueness has not been considered. In this note it will be shown, by applying the contraction principle, that the uniqueness of radially symmetric solutions in spherical shells holds.

We will consider here radially symmetric solutions, defined on $[a, 1]$, of an equation slightly more general than (1), namely

$$-(r^2u')' = \sigma\mu(u)r^2 f(u) \tag{4}$$

with $f$ being a continuous positive function on $\mathbb{R}$, and

$$\mu(u) = \Big(\int_a^1 s^2 f(u(s))\,ds\Big)^{-1},$$

together with the boundary conditions (3).

Other types of boundary conditions may be treated similarly, so we restrict ourselves to (3) only.

First, note that integrating (4) over $[a, 1]$ we get $u'(a) = 0$, therefore

$$u'(r) = -\sigma\mu(u)r^{-2}\int_a^r s^2 f(u(s))\,ds\,. \tag{5}$$

Hence for any $f > 0$, $-\sigma r^{-2} \le u'(r) \le 0$, and consequently

$$-A \le u(r) \le 0\,, \tag{6}$$

where $A = \sigma(1/a - 1)$.

THEOREM 1. *If $f$ is Lipschitz continuous and positive on $\mathbb{R}$, then for any positive $\sigma$ the problem* (4), (3) *has a unique solution.*

Proof. Let $X$ denote the Banach space of continuous functions defined on $[a, 1]$ with the norm

$$\|v\|_N = \sup_{r\in[a,1]} r^{1-N}|v(r)|\,, \qquad v \in X\,,$$

where $N \ge 1$ will be chosen later.

Integrating (5), we get the integral equation equivalent to (4), (3)

$$u = \mathcal{T}_f(u)$$

with the nonlinear operator $\mathcal{T}_f$ defined by

$$(\mathcal{T}_f(u))(r) = -\sigma\mu(u)\int_a^r h_r(s)f(u(s))\,ds\,,$$

where $h_r(s) = s^2(1/s - 1/r)$.

Assume for a moment that

$$m \leq f \leq M \tag{7}$$

for some positive constants $m$, $M$. We will show that the operator $\mathcal{T}_f : X \to X$ is a contraction if $N$ is sufficiently large.

Let $v, w \in X$, and consider the difference

$$(\mathcal{T}_f(v) - \mathcal{T}_f(w))(r) = -\sigma \int_a^r h_r(s)(\mu(v)f(v(s)) - \mu(w)f(w(s)))\, ds.$$

We can write

$$\mu(v)f(v) - \mu(w)f(w) = \mu(v)(f(v) - f(w)) + f(w)(\mu(v) - \mu(w)).$$

Due to the assumptions imposed upon $f$ we have

$$\Big| \int_a^r h_r(s)\mu(v)(f(v(s)) - f(w(s)))\, ds\Big| \leq C\|v - w\|_N \int_a^r s^{N-1}\, ds$$
$$\leq CN^{-1}\|v - w\|_N r^N .$$

For notational convenience here, as well as in the sequel, the constants depending only on $\sigma$, $a$, $m$, $M$, and the Lipschitz constant of $f$ are denoted by the same letter $C$.

We also have the inequality

$$|\mu(v) - \mu(w)| \leq CN^{-1}\|v - w\|_N ,$$

which gives the estimate

$$\Big| \int_a^r h_r f(w)(\mu(v) - \mu(w))\Big| \leq CN^{-1}\|v - w\|_N \int_a^r h_r$$
$$\leq CN^{-2}\|v - w\|_N \|h_1\|_N r^N .$$

Therefore, whenever $N \geq 1$,

$$|\mathcal{T}_f(v) - \mathcal{T}_f(w)| \leq Cr^N N^{-1}\|v - w\|_N ,$$

so we have

$$\|\mathcal{T}_f(v) - \mathcal{T}_f(w)\|_N \leq CN^{-1}\|v - w\|_N,$$

and, for $N$ sufficiently large, $\mathcal{T}_f$ becomes a contraction in the norm $\|\cdot\|_N$.

We have proved our theorem under the additional assumption (7). Now, let $f$ be an arbitrary Lipschitz continuous and positive function. We define a new function $g$ such that $g(x) = f(x)$ for $-A \leq x \leq 0$, $g(x) = f(-A)$ for $x \leq -A$ and $g(x) = f(0)$ for $x \geq 0$. For the function $g$ the corresponding operator $\mathcal{T}_g$ has a unique fixed point which is also a fixed point of $\mathcal{T}_f$ (cf. (6)). Moreover, it follows from (6) that $\mathcal{T}_f$ has no other fixed points.

Remark 1. Theorem 1 is valid for arbitrary dimensions with obvious modifications in the proof.

Remark 2. In the two-dimensional case with $f(u) = \exp u$ our problem is integrable, and the existence and uniqueness may be proved using the methods of [5].

Remark 3. The methods used in this note may be applied to obtain the existence and uniqueness of a radially symmetric solution of the problem considered in [5], [6].

The following version of the well known Pokhozhaev identity has been proved in [3]: If $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $u$ is a solution of the problem

$$-\Delta u = g(x,u), \quad u_{|\partial\Omega} = 0,$$

then

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle x, \nu\rangle = \int_{\Omega} (2\langle \nabla_x G, x\rangle + 2nG - (n-2)ug)\, dx\,,$$

where

$$G(x,u) = \int_0^u g(x,s)\, ds$$

and $\nu$ is the exterior unit normal vector.

Using this identity we will prove

THEOREM 2. *The problem*

$$-\Delta u = \sigma\mu \exp u, \quad u_{|\partial\Omega} = 0, \quad \sigma > 0\,, \tag{8}$$

*where $\Omega$ is a bounded simply connected domain in the plane with $C^2$ boundary, has no solution for large $\sigma$.*

This result generalizes and improves Theorem 3 of [4].

Proof. Using a conformal mapping $T$ we can map the unit disk $B$ onto $\Omega$. If $u$ is a solution of (8), then the function $v = u \circ T$ satisfies

$$-\Delta v = \frac{\sigma \mathcal{J} \exp v}{\int_B \mathcal{J} \exp v}, \quad v_{|\partial B} = 0, \tag{9}$$

where $\mathcal{J}$ denotes the Jacobian of $T$.

Applying the Pokhozhaev identity to (9) we get

$$\int_{\partial B} \left(\frac{\partial v}{\partial \nu}\right)^2 = \int_B \left(2\frac{\sigma}{\int_B \mathcal{J}\exp v}(\exp v - 1)\langle \nabla \mathcal{J}, x\rangle + 4\frac{\sigma \mathcal{J}}{\int_B \mathcal{J}\exp v}(\exp v - 1)\right). \tag{10}$$

Because the boundary of $\Omega$ is $C^2$ we have $\nabla \mathcal{J} \in C^1(\overline{\Omega})$ (cf. [7]). Therefore the right hand side of (10) can be estimated by a linear function of $\sigma$.

Since $\int_{\partial B} \partial v/\partial \nu = -\sigma$, we have $\sigma^2 \leq 2\pi \int_{\partial B} (\partial v/\partial \nu)^2$, which implies that a solution of (9) cannot exist for sufficiently large $\sigma$.

## References


[1] E. Caglioti, P.-L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description*, Comm. Math. Phys. 143 (1992), 501–525.

[2] A. Friedman and K. Tintarev, *Boundary asymptotics for solutions of the Poisson–Boltzmann equation*, J. Differential Equations 69 (1987), 15–38.

[3] J. L. Kazdan and F. W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. 28 (1975), 567–597.

[4] A. Krzywicki and T. Nadzieja, *Some results concerning the Poisson–Boltzmann equation*, Zastos. Mat. 21 (1991), 265–272.

[5] K. Nagasaki and T. Suzuki, *Radial and nonradial solutions for the nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ on annuli in $\mathbb{R}^2$*, J. Differential Equations 87 (1990), 144–168.

[6] I. Rubinstein, *Counterion condensation as an exact limiting property of solutions of the Poisson–Boltzmann equation*, SIAM J. Appl. Math. 46 (1986), 1024–1038.

[7] V. Smirnoff, *Über die Ränderzuordnung bei konformer Abbildung*, Math. Ann. 107 (1933), 313–323.



ANDRZEJ KRZYWICKI
TADEUSZ NADZIEJA
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND


*Received on 27.10.1992*