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QUANTILE INEQUALITIES FOR LINEAR COMBINATIONS OF ORDER STATISTICS FROM ORDERED FAMILIES OF DISTRIBUTIONS

Abstract. The paper develops some ideas of Barlow and Proschan [5]. Inequalities and bounds for quantiles of linear combinations of order statistics and spacings are given when considered distributions are convex and starshaped ordered. A characterization of the star-ordering is also given.

1. Introduction and preliminaries. Barlow and Proschan [5] have established inequalities for moments of linear combinations of order statistics from restricted classes of distributions defined by convex and starshaped orderings. These results have been widely applied to construction of bounds and tolerance limits for life distributions (see [5]–[7]) as well as to studying robustness and stability of estimates and tests for scale parameter (see [9]–[11] and [15]). In this paper some inequalities and bounds for quantiles of linear combinations of order statistics and spacings are given. A characterization of the star-ordering of distributions is established in the last section.

Throughout the paper we identify probability distributions with their distribution functions and assume that all considered distributions are continuous and strictly increasing on their supports which are intervals. We use the term "increasing (decreasing)" for "nondecreasing (nonincreasing)".

Let random variables X and Y have the distributions F and G on the supports S_F and S_G respectively, where $F(0) = G(0) = 0$. Denote by $X_{1:n}, \ldots, X_{n:n}$ and by $Y_{1:n}, \ldots, Y_{n:n}$ order statistics of samples of size n from the distributions F and G respectively and by $F_{i:n}$ ($G_{i:n}$) the distribution of $X_{i:n}$ ($Y_{i:n}$), $i = 1, \ldots, n$. The random variables $U_{i:n} = X_{i:n} - X_{i-1:n}$

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and $V_{i:n} = Y_{i:n} - Y_{i-1:n}$, $i = 1, \ldots, n, X_{0:n} = Y_{0:n} = 0$, are called *spacings* from the distributions F and G respectively. Their respective distributions are denoted by $\widetilde{F}_{i:n}$ and $\widetilde{G}_{i:n}$. We shall consider linear combinations of order statistics

$$(1) \qquad X_{\mathbf{a}} = \sum_{i=1}^{n} a_i X_{i:n}, \quad Y_{\mathbf{a}} = \sum_{i=1}^{n} a_i Y_{i:n}$$

(analogously $X_{\mathbf{b}}, Y_{\mathbf{b}}$ etc.) and linear combinations of spacings

$$(2) \qquad U_{\mathbf{A}} = \sum_{i=1}^{n} A_i U_{i:n}, \quad V_{\mathbf{A}} = \sum_{i=1}^{n} A_i V_{i:n}$$

(analogously $U_{\mathbf{B}}, V_{\mathbf{B}}$ etc.). Their respective distributions are denoted by $F_{\mathbf{a}}, G_{\mathbf{a}}$ ($F_{\mathbf{b}}, G_{\mathbf{b}}$ etc.) and $\widetilde{F}_{\mathbf{A}}, \widetilde{G}_{\mathbf{A}}$ ($\widetilde{F}_{\mathbf{B}}, \widetilde{G}_{\mathbf{B}}$ etc.).

We say that F is *stochastically less than* G ($F \overset{\text{st}}{\leq} G$) if and only if $F(x) \geq G(x)$ for every x. We shall also use the notation $X \overset{\text{st}}{\leq} Y$ if and only if $F \overset{\text{st}}{\leq} G$. It is well known that if $X \overset{\text{st}}{\leq} Y$, then $X_{i:n} \overset{\text{st}}{\leq} Y_{i:n}$, $i = 1, \ldots, n$. The notation $X \overset{\text{st}}{=} Y$ means $F = G$.

Denote by F^{-1} the inverse of F. Thus $F^{-1}(p)$ is the p-quantile of the distribution F (and analogously for $G^{-1}, F_{i:n}^{-1}, G_{i:n}^{-1}, F_{\mathbf{a}}^{-1}, G_{\mathbf{a}}^{-1}, \widetilde{F}_{\mathbf{A}}^{-1}, \widetilde{G}_{\mathbf{A}}^{-1}$ etc.). It is well known that

$$(3) \qquad (Y_{1:n}, \ldots, Y_{n:n}) \overset{\text{st}}{=} (G^{-1}F(X_{1:n}), \ldots, G^{-1}F(X_{n:n})).$$

We say that F is *convex with respect to* G ($F <^{\text{c}} G$) if and only if $G^{-1}F$ is convex on S_F. F is *starshaped with respect to* G ($F <^* G$) if and only if $G^{-1}F$ is starshaped on S_F (i.e. $G^{-1}F(x)/x$ is increasing on S_F). It is easy to see that $F <^{\text{c}} G$ implies $F <^* G$. Since

$$(4) \qquad G^{-1}F = G_{i:n}^{-1}F_{i:n}, \quad i = 1, \ldots, n, \; n \geq 1$$

(see [7]), we see that $F <^{\text{c}} G$ implies $F_{i:n} <^{\text{c}} G_{i:n}$ and also $F <^* G$ implies $F_{i:n} <^* G_{i:n}$.

If $G(x) = 1 - e^{-x}$, $x > 0$, then $F <^{\text{c}} G$ is equivalent to F having an increasing failure rate (i.e. F is IFR), and $G <^{\text{c}} F$ is equivalent to F having a decreasing failure rate (F is DFR). Similarly $F <^* G$ is equivalent to F having an increasing failure rate average (F is IFRA) and $G <^* F$ is equivalent to F having a decreasing failure rate average (F is DFRA) (see [7]).

In the sequel we shall use results of Barlow and Proschan [5] (Theorems 3.2, 3.4, 4.2 and 4.4) concerning linear combinations of order statistics and spacings of the form (1) and (2). We formulate the lemmas for $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$ only, but they may also be stated for $U_{\mathbf{A}}$ and $V_{\mathbf{A}}$, when their coefficients

A_i, $i = 1, \ldots, n$, satisfy the relation

$$A_i = a_i + a_{i+1} + \ldots + a_n, \quad i = 1, \ldots, n. \tag{5}$$

It is easy to see that in this case $X_{\mathbf{a}} = U_{\mathbf{A}}$ and $Y_{\mathbf{a}} = V_{\mathbf{A}}$.

LEMMA 1. *Let $F <^{c} G$. If the a_i satisfy* (5) *and $0 \le A_i \le 1$, $i = 1, \ldots, n$, then $F(X_{\mathbf{a}}) \overset{\text{st}}{\le} G(Y_{\mathbf{a}})$, i.e. $G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) \ge G^{-1}F(x)$ for every x.*

LEMMA 2. *Let $F <^{c} G$. If the a_i, $i = 1, \ldots, n$, satisfy* (5) *and $A_i \ge 1$, $i = 1, \ldots, k$, for some k $(1 \le k \le n)$ while $A_i \le 0$, $i = k+1, \ldots, n$, then $F(X_{\mathbf{a}}) \overset{\text{st}}{\ge} G(Y_{\mathbf{a}})$, i.e. $G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) \le G^{-1}F(x)$ for every x.*

LEMMA 3. *Let $F <^{*} G$. If the a_i, $i = 1, \ldots, n$, satisfy* (5) *and if there exists k $(1 \le k \le n)$ such that $0 \le A_1 \le \ldots \le A_k \le 1$ and when $k < n$, $A_{k+1} = \ldots = A_n = 0$, then $F(X_{\mathbf{a}}) \overset{\text{st}}{\le} G(Y_{\mathbf{a}})$, i.e. $G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) \ge G^{-1}F(x)$ for every x.*

LEMMA 4. *Let $F <^{*} G$. If $a_i \ge 0$, $i = 1, \ldots, k-1$, $a_k \ge 1$ and $a_i = 0$, $i = k+1, \ldots, n$ $(1 \le k \le n)$, then $F(X_{\mathbf{a}}) \overset{\text{st}}{\ge} G(Y_{\mathbf{a}})$, i.e. $G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) \le G^{-1}F(x)$ for every x.*

The next well known lemma (Barlow and Proschan [5]) gives important properties of monotone failure rate distributions.

LEMMA 5. *If F is IFR (DFR), then $(n - i + 1)U_{i:n}$ is*

(a) *stochastically decreasing (increasing) in $i = 1, \ldots, n$ for fixed n,*
(b) *stochastically increasing (decreasing) in $n \ge 1$ for fixed i; moreover,*
(c) *$U_{n-1:n}$ is stochastically decreasing (increasing) in $n > 1$ for fixed i.*

From Lemmas 1 and 2 we obtain the following results.

LEMMA 6. *Let $F <^{c} G$ and the a_i, $i = 1, \ldots, n$, satisfy* (5).

(a) *If $A_i \ge 0$, $i = 1, \ldots, n$, and $A_k > 0$ for some k, then*

$$\overline{A} G^{-1} F(x/\overline{A}) \le G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) = \widetilde{G}_{\mathbf{A}}^{-1} \widetilde{F}_{\mathbf{A}}(x), \quad x > 0, \tag{6}$$

where $\overline{A} = \max_{1 \le i \le n} A_i$.

(b) *If $A_i > 0$ for $i = 1, \ldots, k$, and $A_i = 0$ for $i = k+1, \ldots, n$ $(1 \le k \le n)$, then*

$$\overline{A} G^{-1} F(x/\overline{A}) \le G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(x) = \widetilde{G}_{\mathbf{A}}^{-1} \widetilde{F}_{\mathbf{A}}(x) \le \underline{A} G^{-1} F(x/\underline{A}), \quad x > 0, \tag{7}$$

where $\underline{A} = \min_{1 \le i \le n} A_i$.

Proof. (a) Let $B_i = A_i/\overline{A}$, $i = 1, \ldots, n$. It is obvious that $\widetilde{F}_{\mathbf{B}}(x) = \widetilde{F}_{\mathbf{A}}(\overline{A}x)$ and also $\widetilde{G}_{\mathbf{B}}(x) = \widetilde{G}_{\mathbf{A}}(\overline{A}x)$ and hence $\widetilde{G}_{\mathbf{B}}^{-1} \widetilde{F}_{\mathbf{B}}(x) = \widetilde{G}_{\mathbf{A}}^{-1} \widetilde{F}_{\mathbf{A}}(\overline{A}x)/\overline{A}$ $= G_{\mathbf{a}}^{-1} F_{\mathbf{a}}(\overline{A}x)/\overline{A}$. Since the assumptions of Lemma 1 are satisfied for B_i

and $b_i = a_i/\overline{A}$, we have $\widetilde{G}_{\mathbf{B}}^{-1}\widetilde{F}_{\mathbf{B}}(x) = G_{\mathbf{b}}^{-1}F_{\mathbf{b}}(x) \geq G^{-1}F(x)$, $x > 0$, which is equivalent to (6).

(b) Let now $C_i = A_i/\underline{A}$, $c_i = a_i/\underline{A}$, $i = 1,\ldots,n$. We have also $G_{\mathbf{c}}^{-1}F_{\mathbf{c}}(x) = \widetilde{G}_{\mathbf{C}}^{-1}\widetilde{F}_{\mathbf{C}}(x) = \widetilde{G}_{\mathbf{A}}^{-1}\widetilde{F}_{\mathbf{A}}(\underline{A}x) = G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(\underline{A}x)$. The assumptions of Lemma 2 are satisfied for C_i and c_i ($C_i \geq 1$, $i = 1,\ldots,k$), hence $G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) = \widetilde{G}_{\mathbf{A}}^{-1}\widetilde{F}_{\mathbf{A}}(x) \leq \underline{A}G^{-1}F(x/\underline{A})$, $x > 0$. Combining this with (6) we obtain (7).

LEMMA 7. *Let $F <^{\mathrm{c}} G$. If $a_i \geq 0$, $i = 1,\ldots,k-1$, $a_k > 0$ and $a_i = 0$, $i = k+1,\ldots,n$ $(1 \leq k \leq n)$, then*

$$a^*G^{-1}F(x/a^*) \leq G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) \leq a_kG^{-1}F(x/a_k), \quad x > 0,$$

where $a^* = \sum_{i=1}^k a_i$.

Proof. The result follows directly from Lemma 6(b). We have $\underline{A} = a_k$ and $\overline{A} = a^*$.

Notice that the function $G_{\mathbf{a}}^{-1}F_{\mathbf{a}}$ (or $\widetilde{G}_{\mathbf{A}}^{-1}\widetilde{F}_{\mathbf{A}}$) lies between two convex functions and if $\underline{A} = \overline{A}$, i.e. $U_{\mathbf{A}} = cX_{k:n}$ and $V_{\mathbf{A}} = cY_{k:n}$ for some $c > 0$ ($X_{\mathbf{a}} = a_kX_{k:n}$, $Y_{\mathbf{a}} = a_kY_{k:n}$ respectively), then $\widetilde{G}_{\mathbf{A}}^{-1}\widetilde{F}_{\mathbf{A}}(x) = G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) = a_kG^{-1}F(x/a_k)$.

Under some additional assumptions on $G^{-1}F$ and using a result of Birge and Teboulle [12] we obtain another upper bound on $G_{\mathbf{a}}^{-1}F_{\mathbf{a}}$.

LEMMA 8. *Let $F <^{\mathrm{c}} G$ with $G^{-1}F$ differentiable, let*

$$\eta(x) = \frac{d}{dx}G^{-1}F(x)$$

and

$$0 < \alpha \leq \eta(x) \leq \beta \quad \text{for some } \alpha \text{ and } \beta \text{ and every } x. \tag{8}$$

If $a_i \geq 0$, $i = 1,\ldots,n$, and $a_k > 0$ for some k, then

$$G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) \leq a^*G^{-1}F\left(\frac{\beta x}{\alpha a^*}\right), \quad x > 0, \tag{9}$$

where $a^* = \sum_{i=1}^n a_i$.

Proof. Birge and Teboulle [12] (Theorem 2.1) have proved that if $\phi : S \to \mathbb{R}$ is a convex differentiable increasing function on the interval S, Z is a random variable taking values in S and the expectations $\mathrm{E}[\phi(Z)]$, $\mathrm{E}[\phi'(Z)] > 0$, $\mathrm{E}[Z\phi'(Z)]$ exist and are finite then

$$\mathrm{E}[\phi(Z)] \leq \phi\left(\frac{\mathrm{E}[Z\phi'(Z)]}{\mathrm{E}[\phi'(Z)]}\right). \tag{10}$$

Under our assumptions it follows from (10) that

$$\frac{1}{a^*}\sum_{i=1}^{n} a_i G^{-1}F(X_{i:n}) \le G^{-1}F\left(\frac{(1/a^*)\sum_{i=1}^{n} a_i X_{i:n}\eta(X_{i:n})}{(1/a^*)\sum_{i=1}^{n} a_i\eta(X_{i:n})}\right)$$
$$\le G^{-1}F\left(\frac{\beta}{\alpha a^*}\sum_{i=1}^{n} a_i X_{i:n}\right).$$

Since $G^{-1}F(X_{i:n}) \overset{\text{st}}{=} Y_{i:n}$, we have

$$G\left(\frac{1}{a^*}Y_{\mathbf{a}}\right) \overset{\text{st}}{\le} F\left(\frac{\beta}{\alpha a^*}X_{\mathbf{a}}\right),$$

which is equivalent to (9).

Immediately from Lemmas 3 and 4 we obtain the following results.

LEMMA 9. *Let $F <^{c} G$. If a_i, $i = 1,\ldots,n$, satisfy* (5) *and* $0 \le A_1 \le \ldots \le A_k$, $A_k > 0$, *and* $A_{k+1} = \ldots = A_n = 0$ $(1 \le k \le n)$, *then*

$$G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) \ge a_k G^{-1}F(x/a_k), \quad x > 0.$$

LEMMA 10. *Let $F <^{*} G$. If $a_i \ge 0$, $i = 1,\ldots,k-1$, $a_k > 0$ and $a_{k+1} = \ldots = a_n = 0$ $(1 \le k \le n)$, then*

$$G_{\mathbf{a}}^{-1}F_{\mathbf{a}}(x) \le a_k G^{-1}F(x/a_k), \quad x > 0.$$

2. Inequalities for quantiles of order statistics and spacings. The following theorem is an analogue of Theorem 3.6 of Barlow and Proschan [5].

THEOREM 1. *If $F <^{*} G$, then for every $p \in (0,1)$, $F_{i:n}^{-1}(p)/G_{i:n}^{-1}(p)$ is*

(a) *decreasing in i for fixed n,*
(b) *increasing in $n \ge i$ for fixed i; moreover,*
(c) *$F_{n-i:n}^{-1}(p)/G_{n-i:n}^{-1}(p)$ is decreasing in $n > i$ for fixed i.*

Proof. (a) From (4) it follows that

$$\frac{G_{i:n}^{-1}F_{i:n}(x)}{x} = \frac{G_{i+1:n}^{-1}F_{i+1:n}(x)}{x}, \quad i = 1,\ldots,n-1,\ x > 0, \tag{11}$$

which is equivalent to

$$\frac{G_{i:n}^{-1}(p)}{F_{i:n}^{-1}(p)} = \frac{G_{i+1:n}^{-1}F_{i+1:n}F_{i:n}^{-1}(p)}{F_{i:n}^{-1}(p)}, \quad p \in (0,1). \tag{12}$$

Since $G_{i+1:n}^{-1}F_{i+1:n}(x)/x = G^{-1}F(x)/x$ is increasing in $x > 0$ and $F_{i:n}^{-1}(p) \le F_{i+1:n}^{-1}(p)$, $p \in (0,1)$, we obtain from (12)

$$\frac{G_{i:n}^{-1}(p)}{F_{i:n}^{-1}(p)} \le \frac{G_{i+1:n}^{-1}(p)}{F_{i+1:n}^{-1}(p)}, \quad i = 1,\ldots,n-1,\ p \in (0,1),$$

which completes the proof.

(b) The proof is similar to that of (a) by noticing that $F^{-1}_{i:n+1}(p) \le F^{-1}_{i:n}(p)$, $i = 1,\dots,n$, $p \in (0,1)$.

(c) The proof is similar to that of (a) by noticing that $F_{n-i:n}(p) \le F^{-1}_{n+1-i:n+1}(p)$, $i = 0,1,\dots,n-1$, $p \in (0,1)$.

The next theorem concerns inequalities for quantiles of linear combinations of spacings.

THEOREM 2. *Let $F <^{c} G$. If $A_i > 0$ for $i = 1,\dots,k$, and $A_i = 0$ for $i = k+1,\dots,n$ $(1 \le k \le n)$, then*

$$\frac{\underline{A}}{\overline{A}}\frac{F^{-1}_{k:n}(p)}{G^{-1}_{k:n}(p)} \le \frac{\widetilde{F}^{-1}_{\mathbf{A}}(p)}{\widetilde{G}^{-1}_{\mathbf{A}}(p)} \le \frac{\overline{A}}{\underline{A}}\frac{F^{-1}_{k:n}(p)}{G^{-1}_{k:n}(p)}, \quad p \in (0,1), \tag{13}$$

where $\underline{A} = \min_{1\le i\le k} A_i$ and $\overline{A} = \max_{1\le i\le k} A_i$.

Proof. From Lemma 6 and (4) it follows that

$$\frac{F^{-1}_{k:n}G_{k:n}(x/\underline{A})}{x/\underline{A}} \le \frac{\widetilde{F}^{-1}_{\mathbf{A}}\widetilde{G}_{\mathbf{A}}(x)}{x} \le \frac{F^{-1}_{k:n}G_{k:n}(x/\overline{A})}{x/\overline{A}}, \quad x > 0,$$

which is equivalent to

$$\frac{F^{-1}_{k:n}G_{k:n}(\widetilde{G}^{-1}_{\mathbf{A}}(p)/\underline{A})}{\widetilde{G}^{-1}_{\mathbf{A}}(p)/\underline{A}} \le \frac{\widetilde{F}^{-1}_{\mathbf{A}}(p)}{\widetilde{G}^{-1}_{\mathbf{A}}(p)} \le \frac{F^{-1}_{k:n}G_{k:n}(\widetilde{G}^{-1}_{\mathbf{A}}(p)/\overline{A})}{\widetilde{G}^{-1}_{\mathbf{A}}(p)/\overline{A}}, \quad p \in (0,1). \tag{14}$$

It is obvious that $\underline{A}Y_{k:n} \le V_{\mathbf{A}} \le \overline{A}Y_{k:n}$, hence

$$\underline{A}G^{-1}_{k:n}(p) \le \widetilde{G}^{-1}_{\mathbf{A}}(p) \le \overline{A}G^{-1}_{k:n}(p), \quad p \in (0,1). \tag{15}$$

Since $F^{-1}_{k:n}G_{k:n}$ is increasing, from (14) and (15) it follows that

$$\begin{aligned} &\frac{\underline{A}F^{-1}_{k:n}G_{k:n}(\widetilde{G}^{-1}_{\mathbf{A}}(p)/\overline{A})}{\overline{A}\widetilde{G}^{-1}_{\mathbf{A}}(p)} \\ &\qquad \le \frac{\widetilde{F}^{-1}_{\mathbf{A}}(p)}{\widetilde{G}^{-1}_{\mathbf{A}}(p)} \le \frac{\overline{A}F^{-1}_{k:n}G_{k:n}(\widetilde{G}^{-1}_{\mathbf{A}}(p)/\underline{A})}{\overline{A}\widetilde{G}^{-1}_{\mathbf{A}}(p)/\underline{A}}, \quad p \in (0,1). \end{aligned} \tag{16}$$

Now from the assumption $F <^{c} G$ and (4) we find that $F^{-1}_{k:n}G_{k:n}(x)/x$ is decreasing and hence from (15) and (16) we obtain (13).

In the same way using Lemma 9 one can obtain the following result.

THEOREM 3. *Let $F <^{*} G$. If $0 \le A_1 \le \ldots \le A_k$, $A_k > 0$, $A_{k+1} = \ldots = A_n = 0$ $(1 \le k \le n)$, then*

$$\frac{\widetilde{F}^{-1}_{\mathbf{A}}(p)}{\widetilde{G}^{-1}_{\mathbf{A}}(p)} \le \frac{A_k}{\underline{A}}\frac{F^{-1}_{k:n}(p)}{G^{-1}_{k:n}(p)}, \quad p \in (0,1),$$

where $\underline{A} = \min\{A_i : A_i > 0\}$.

The next three theorems give inequalities for nonnegative linear combinations of order statistics.

THEOREM 4. *Let $F <^{c} G$. If r and s $(1 \le r \le s \le n)$ are such that $a_1 = \ldots = a_{r-1} = 0$, $a_r > 0$, $a_i \ge 0$ for $i = r+1, \ldots, s-1$, $a_s > 0$ and $a_{s+1} = \ldots = a_n = 0$, then*

$$\frac{a_s}{a^*}\frac{F_{s:n}^{-1}(p)}{G_{s:n}^{-1}(p)} \le \frac{F_{\mathbf{a}}^{-1}(p)}{G_{\mathbf{a}}^{-1}(p)} \le \frac{F_{r:n}^{-1}(p)}{G_{r:n}^{-1}(p)}, \quad p \in (0,1), \tag{17}$$

where $a^* = \sum_{i=r}^{s} a_i$.

Proof. From Lemma 7 and (4) it follows that

$$\frac{F_{s:n}^{-1}G_{s:n}(x/a_s)}{x/a_s} \le \frac{F_{\mathbf{a}}^{-1}G_{\mathbf{a}}(x)}{x} \le \frac{F_{r:n}^{-1}G_{r:n}(x/a^*)}{x/a^*}, \quad x > 0,$$

which is equivalent to

$$\frac{F_{s:n}^{-1}G_{s:n}(G_{\mathbf{a}}^{-1}(p)/a_s)}{G_{\mathbf{a}}^{-1}(p)/a_s} \le \frac{F_{\mathbf{a}}^{-1}(p)}{G_{\mathbf{a}}^{-1}(p)} \le \frac{F_{r:n}^{-1}G_{r:n}(G_{\mathbf{a}}^{-1}(p)/a^*)}{G_{\mathbf{a}}^{-1}(p)/a^*}, \quad p \in (0,1). \tag{18}$$

Since $a^*Y_{r:n} \le Y_{\mathbf{a}} \le a^*Y_{s:n}$, we have

$$a^*G_{r:n}^{-1}(p) \le G_{\mathbf{a}}^{-1}(p) \le a^*G_{s:n}^{-1}(p), \quad p \in (0,1). \tag{19}$$

The function $F_{s:n}^{-1}G_{s:n}(x)$ is increasing and the functions $F_{s:n}^{-1}G_{s:n}(x)/x$ and $F_{r:n}^{-1}G_{r:n}(x)/x$ are decreasing, hence from (18) and (19) we obtain (17).

THEOREM 5. *Let the assumptions of Theorem 4 be satisfied and in addition $G^{-1}F$ be differentiable. If $0 < \alpha \le \frac{d}{dx}G^{-1}F(x) \le \beta$ for some α and β and every x, then*

$$\max\left(\frac{\alpha}{\beta}, \frac{a_s}{a^*}\right)\frac{F_{s:n}^{-1}(p)}{G_{s:n}^{-1}(p)} \le \frac{F_{\mathbf{a}}^{-1}(p)}{G_{\mathbf{a}}^{-1}(p)} \le \frac{F_{r:n}^{-1}(p)}{G_{r:n}^{-1}(p)}, \quad p \in (0,1). \tag{20}$$

Proof. Using Lemma 8 in the same way as in the proof of Theorem 4 we obtain the inequality

$$\frac{\alpha}{\beta}\frac{F_{s:n}^{-1}(p)}{G_{s:n}^{-1}(p)} \le \frac{F_{\mathbf{a}}^{-1}(p)}{G_{\mathbf{a}}^{-1}(p)}, \quad p \in (0,1).$$

Combining this with (17) we have (20).

From Lemma 10 one can easily obtain the following result.

THEOREM 6. *Let $F <^{*} G$. If $a_i \ge 0$, $i = 1, \ldots, k-1$, $a_k > 0$ and $a_i = 0$ for $i = k+1, \ldots, n$ $(1 \le k \le n)$, then*

$$\frac{F_{\mathbf{a}}^{-1}(p)}{G_{\mathbf{a}}^{-1}(p)} \ge \frac{F_{k:n}^{-1}(p)}{G_{k:n}^{-1}(p)}, \quad p \in (0,1). \tag{21}$$

3. Bounds on quantiles of linear combinations of spacings and order statistics. In many situations the distribution G is known, e.g. exponential. Generally we assume that F is unknown. However, we can have some information about F: moments, bounds on the failure rate function, even we can know $F^{-1}(p)$ for some p, e.g. $F^{-1}(p) = G^{-1}(p)$. Such a situation is possible if we replace elements having exponentially distributed life time by elements which have IFR (or IFRA) life distribution with the same *mission time* with probability p. At the worst, having some additional information about F we may use bounds for $F^{-1}(p)$ derived from results of Barlow and Marshall [3] and Barlow and Proschan [4], [7]. By the assumption that G and $F^{-1}(p)$ are known for some p, using theorems of the preceding section we give bounds on p-quantiles of linear combinations of order statistics and spacings from the distribution F. We start from bounds on p-quantiles of ith order statistics. The result is an analogue of the formula (3.7) of [5].

THEOREM 7. *If $F <^* G$, then*

$$F^{-1}(p)\frac{G_{i:n}^{-1}(p)}{G_{i:i}^{-1}(p)} \le F_{i:n}^{-1}(p) \le F^{-1}(p)\frac{G_{i:n}^{-1}(p)}{G_{1:n-i+1}^{-1}(p)}, \quad p \in (0,1),\ i = 1, \ldots, n. \tag{22}$$

P r o o f. It is obvious that

$$F_{1:n}^{-1}(p) \le F^{-1}(p) \le F_{n:n}^{-1}(p), \quad p \in (0,1). \tag{23}$$

Applying Theorem 1 we have

$$\frac{F_{i:i}^{-1}(p)}{G_{i:i}^{-1}(p)} \le \frac{F_{i:n}^{-1}(p)}{G_{i:n}^{-1}(p)} \le \frac{F_{i:n-i+1}^{-1}(p)}{G_{i:n-i+1}^{-1}(p)}, \quad p \in (0,1),\ i = 1, \ldots, n, \tag{24}$$

where the first inequality follows from Theorem 1(b) and the second one from Theorem 1(c). Combining (23) with (24) we obtain immediately (22).

From Theorem 2 and (22) the following result follows.

COROLLARY 1. *Let $F <^{c} G$. If $A_i > 0$, $i = 1, \ldots, k$, and $A_{k+1} = \ldots = A_n = 0$ $(1 \le k \le n)$, then*

$$F^{-1}(p)\frac{\underline{A}\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\overline{A}G_{k:k}^{-1}(p)} \le \widetilde{F}_{\mathbf{A}}^{-1}(p) \le F^{-1}(p)\frac{\overline{A}\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\underline{A}G_{1:n-k+1}^{-1}(p)}, \quad p \in (0,1), \tag{25}$$

where $\underline{A} = \min_{1\le i\le k} A_i$ and $\overline{A} = \max_{1\le i\le k} A_i$.

Analogously, from (22) and Theorems 3–6 we obtain the respective corollaries.

COROLLARY 2. *Let $F <^* G$. If $0 \leq A_1 \leq \ldots \leq A_k$, $A_k > 0$ and $A_{k+1} = \ldots = A_n = 0$ $(1 \leq k \leq n)$, then*

$$\widetilde{F}_{\mathbf{A}}^{-1}(p) \leq \frac{A_k}{\underline{A}} F^{-1}(p) \frac{\widetilde{G}_{\mathbf{A}}^{-1}(p)}{G_{1:n-k+1}^{-1}(p)}, \quad p \in (0,1),$$

where $\underline{A} = \min\{A_i : A_i > 0\}$.

COROLLARY 3. *Let $F <^c G$. If r and s $(1 \leq r \leq s \leq n)$ are such that $a_1 = \ldots = a_{r-1} = 0$, $a_r > 0$, $a_i \geq 0$ for $i = r+1, \ldots, s-1$, $a_s > 0$ and $a_{s+1} = \ldots = a_n = 0$, then*

$$\frac{a_s}{a^*} F^{-1}(p) \frac{G_{\mathbf{a}}^{-1}(p)}{G_{s:s}^{-1}(p)} \leq F_{\mathbf{a}}^{-1}(p) \leq F^{-1}(p) \frac{G_{\mathbf{a}}^{-1}(p)}{G_{1:n-r+1}^{-1}(p)}, \quad p \in (0,1),$$

where $a^* = \sum_{i=r}^{s} a_i$.

COROLLARY 4. *Let the assumptions of Corollary 3 be satisfied and in addition $G^{-1}F$ be differentiable. If $0 < \alpha \leq \frac{d}{dx} G^{-1}F(x) \leq \beta$ for some α and β and every x, then*

$$F^{-1}(p) \frac{G_{\mathbf{a}}^{-1}(p)}{G_{s:s}^{-1}(p)} \max\left(\frac{\alpha}{\beta}, \frac{a_s}{a^*}\right) \leq F_{\mathbf{a}}^{-1}(p) \leq F^{-1}(p) \frac{G_{\mathbf{a}}^{-1}(p)}{G_{1:n-r+1}^{-1}(p)}, \tag{26}$$

$p \in (0,1)$, where $a^* = \sum_{i=r}^{s} a_i$.

COROLLARY 5. *Let $F <^* G$. If $a_i \geq 0$, $i = 1, \ldots, k-1$, $a_k > 0$ and $a_i = 0$ for $i = k+1, \ldots, n$ $(1 \leq k \leq n)$, then*

$$F_{\mathbf{a}}^{-1}(p) \geq F^{-1}(p) \frac{G_{\mathbf{a}}^{-1}(p)}{G_{1:n-k+1}^{-1}(p)}, \quad p \in (0,1).$$

Remark 1. If $F \overset{\text{st}}{\leq} G$ or $G \overset{\text{st}}{\leq} F$ and some assumptions on the supports S_F and S_G are satisfied, then Corollary 1 may be modified using the results of Bartoszewicz [8] and Oja [16]. If $F <^c G$, $F \overset{\text{st}}{\leq} G$ and $S_F = [0, t_1]$, $S_G = [0, t_2]$, $0 < t_1 \leq t_2 \leq \infty$, then $F <^{\text{disp}} G$, i.e. $F^{-1}(\delta) - F^{-1}(\gamma) \leq G^{-1}(\delta) - G^{-1}(\gamma)$ whenever $0 < \gamma \leq \delta < 1$, which is equivalent to $G^{-1}F(x) - x$ being increasing (see [17]). Since (3) holds and $G^{-1}F(X_{i:n}) - G^{-1}F(X_{i-1;n}) \geq X_{i:n} - X_{i-1:n} = U_{i:n}$, $i = 1, \ldots, n$, then we have $(V_{1:n}, \ldots, V_{n:n}) \overset{\text{st}}{\geq} (U_{1:n}, \ldots, U_{n:n})$ and hence $\widetilde{F}_{\mathbf{A}} \overset{\text{st}}{\leq} \widetilde{G}_{\mathbf{A}}$ for $A_i \geq 0$, $i = 1, \ldots, n$ (for the definition and properties of the stochastic ordering in $\mathbb{R}^n$ see [14]). Therefore from (25) we obtain

$$F^{-1}(p) \frac{\underline{A} \widetilde{G}_{\mathbf{A}}^{-1}(p)}{\overline{A} G_{k:k}^{-1}(p)} \leq \widetilde{F}_{\mathbf{A}}^{-1}(p) \leq \min\left(F^{-1}(p) \frac{\overline{A} \widetilde{G}_{\mathbf{A}}^{-1}(p)}{\underline{A} G_{1:n-k+1}^{-1}(p)}, \widetilde{G}_{\mathbf{A}}^{-1}(p)\right),$$

$p \in (0,1)$. Analogously, if $F <^{c} G$, $G \overset{st}{\leq} F$ and $S_F = [t, \infty)$, $t \geq 0$, $F(0) = 0$, $S_G = [0, \infty)$, then $G <^{\text{disp}} F$ and hence $\widetilde{G}_{\mathbf{A}} \overset{st}{\leq} \widetilde{F}_{\mathbf{A}}$. Therefore from (25) we have

$$\max\left(F^{-1}(p)\frac{\underline{A}\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\overline{A}G_{k:k}^{-1}(p)}, \widetilde{G}_{\mathbf{A}}^{-1}(p)\right) \leq \widetilde{F}_{\mathbf{A}}^{-1}(p) \leq F^{-1}(p)\frac{\overline{A}\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\underline{A}G_{1:n-k+1}^{-1}(p)},$$

$p \in (0,1)$.

It is obvious that the appropriate modification of Corollary 3 is also possible.

Remark 2. If $F <^{c} G$ and in addition $F \overset{st}{\leq} G$ and $S_F = [0, t_1]$, $S_G = [0, t_2]$, $0 < t_1 \leq t_2 \leq \infty$, we have $F <^{\text{disp}} G$ and hence $\frac{d}{dx} G^{-1}F(x) \geq 1$, provided that $G^{-1}F$ is differentiable. This also implies $F_{\mathbf{a}} \overset{st}{\leq} G_{\mathbf{a}}$ for $a_i > 0$, $i = 1, \ldots, n$. If moreover $\frac{d}{dx} G^{-1}F(x) \leq \beta$, then the inequality (26) in Corollary 4 may be modified as follows:

$$F^{-1}(p)\frac{G_{\mathbf{a}}^{-1}(p)}{G_{s:s}^{-1}(p)} \max\left(\frac{1}{\beta}, \frac{a_s}{a^*}\right)$$

$$\leq F_{\mathbf{a}}^{-1}(p) \leq \min\left(F^{-1}(p)\frac{G_{\mathbf{a}}^{-1}(p)}{G_{1:n-r+1}^{-1}(p)}, G_{\mathbf{a}}^{-1}(p)\right),$$

$p \in (0,1)$.

Remark 3. If F and G are absolutely continuous with densities f and g respectively, then

$$\frac{d}{dx} G^{-1}F(x) = \frac{f(x)}{gG^{-1}F(x)}$$

is called the *generalized failure rate function* (see [1], p. 242). If $G(x) = 1 - e^{-x}$, $x > 0$, we have the common *failure rate function*

$$r(x) = \frac{f(x)}{1 - F(x)}.$$

So the condition (8) means the boundedness of the failure rate function.

4. Inequalities when one distribution is exponential. If $G(x) = 1 - e^{-x}$, $x > 0$, we can obtain inequalities and bounds on quantiles of linear combinations of spacings using a characteristic property of the exponential distribution.

THEOREM 8. *Let* $G(x) = 1 - e^{-x}$, $x > 0$, *and* $F <^{c} G$ *(i.e. F is IFR). If there exist r and s* $(1 \leq r \leq s \leq n)$ *such that* $A_i = 0$ *for* $i = 1, \ldots, r-1$,

$A_i \geq 0$ for $i = r+1, \ldots, s-1$, $A_s > 0$ and $A_i = 0$ for $i = s+1, \ldots, n$, then

$$\frac{\widetilde{F}_{\mathbf{A}}^{-1}(p)}{\widetilde{G}_{\mathbf{A}}^{-1}(p)} \leq -(s-r+1) \max\left(1, \frac{\overline{A}}{A^*}\right) \frac{F_{1:s-r+1}^{-1}(p)}{\log(1-p)}, \quad p \in 0,1), \tag{27}$$

where $\overline{A} = \max_{r \leq i \leq s} A_i$ and $A^* = \sum_{i=r}^{s} A_i/(n-i+1)$.

Proof. Considering the combinations $U_{\mathbf{A}}/\overline{A}$ and $V_{\mathbf{A}}/\overline{A}$ we have from Lemma 6(a)

$$\widetilde{G}_{\mathbf{A}}^{-1} \widetilde{F}_{\mathbf{A}}(x) \geq \overline{A} G^{-1} F(x/\overline{A}), \quad x > 0,$$

or equivalently

$$\frac{\widetilde{F}_{\mathbf{A}}^{-1}(p)}{\widetilde{G}_{\mathbf{A}}^{-1}(p)} \leq \frac{F^{-1} G(\widetilde{G}_{\mathbf{A}}^{-1}(p)/\overline{A})}{\widetilde{G}_{\mathbf{A}}^{-1}(p)/\overline{A}}, \quad p \in (0,1). \tag{28}$$

Let now $C_i = A_i/[(n-i+1)A^*]$. The linear combinations

$$U_{\mathbf{C}} = \sum_{i=r}^{s} C_i(n-i+1) U_{i:n} \quad \text{and} \quad V_{\mathbf{C}} = \sum_{i=r}^{s} C_i(n-i+1) V_{i:n}$$

are convex and hence

$$\min_{r \leq i \leq s} (n-i+1) V_{i:n} \leq V_{\mathbf{C}} \leq \max_{r \leq i \leq s} (n-i+1) V_{i:n} \tag{29}$$

(and analogously for $U_{\mathbf{C}}$). It is well known that the normalized spacings $(n-i+1)V_{i:n}$ from the exponential distribution G are independent with the same distribution G. Therefore from (29) we have

$$Y_{1:s-r+1} \overset{\text{st}}{\leq} V_{\mathbf{C}} \overset{\text{st}}{\leq} Y_{s-r+1:s-r+1},$$

or equivalently

$$G_{1:s-r+1}^{-1}(p) \leq \widetilde{G}_{\mathbf{A}}^{-1}(p)/A^* \leq G_{s-r+1:s-r+1}^{-1}(p), \quad p \in (0,1), \tag{30}$$

since $\widetilde{G}_{\mathbf{C}}^{-1}(p) = \widetilde{G}_{\mathbf{A}}^{-1}(p)/A^*$. Since $F^{-1}G(x) = F_{1:s-r+1}^{-1} G_{1:s-r+1}(x)$ is increasing in x and $F_{1:s-r+1}^{-1} G_{1:s-r+1}(x)/x$ is decreasing in x, from (28) and (30) we obtain

$$\frac{\widetilde{F}_{\mathbf{A}}^{-1}(p)}{\widetilde{G}_{\mathbf{A}}^{-1}(p)} \leq \frac{F_{1:s-r+1}^{-1} G_{1:s-r+1}(A^* G_{1:s-r+1}^{-1}(p)/\overline{A})}{A^* G_{1:s-r+1}^{-1}(p)/\overline{A}}$$

$$\leq \begin{cases} \dfrac{\overline{A} F_{1:s-r+1}^{-1}(p)}{A^* G_{1:s-r+1}^{-1}(p)} & \text{if } A^* \leq \overline{A}, \\[2ex] \dfrac{F_{1:s-r+1}^{-1}(p)}{G_{1:s-r+1}^{-1}(p)} & \text{if } A^* > \overline{A}, \end{cases}$$

which is equivalent to (27).

Remark 4. Let $0 = A_1 = \ldots = A_{r-1} < A_r \le \ldots \le A_s$ and $A_i = 0$ for $i = s+1, \ldots, n$. Then (27) also holds if F is an IFRA distribution ($F <^* G$). This follows from Lemma 9.

Similarly to the proof of Theorem 8, with slight modifications, one can obtain from Lemma 6(b) the following result.

THEOREM 9. *Let* $G(x) = 1 - e^{-x}$, $x > 0$, *and* $F <^c G$ *(i.e. F is IFR). If* $A_i > 0$ *for* $i = 1, \ldots, k$, *and* $A_{k+1} = \ldots = A_n = 0$ $(1 \le k \le n)$, *then for* $p \in (0,1)$,

$$-\min\left(1, \frac{\underline{A}}{A^*}\right) \frac{F_{k:k}^{-1}(p)}{\log(1 - p^{1/k})} \le \frac{\widetilde{F}_{\mathbf{A}}^{-1}(p)}{\widetilde{G}_{\mathbf{A}}^{-1}(p)} \le -k \max\left(1, \frac{\overline{A}}{A^*}\right) \frac{F_{1:k}^{-1}(p)}{\log(1-p)},$$

where $\underline{A} = \min_{1 \le i \le k} A_i$, $\overline{A} = \max_{1 \le i \le k} A_i$ *and* $A^* = \sum_{i=1}^k A_i/(n-i+1)$.

From Theorem 9 and (25), in the same way as in the preceding section, one can obtain bounds on $\widetilde{F}_{\mathbf{A}}^{-1}(p)$ if F is an IFR distribution and $F^{-1}(p)$ is known.

COROLLARY 6. *Under the assumptions of Theorem* 9,

$$\text{(31)} \quad -F^{-1}(p) \frac{\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\log(1 - p^{1/k})} \max[\underline{A}/\overline{A}, \min(1, \underline{A}/A^*)] \le \widetilde{F}_{\mathbf{A}}^{-1}(p)$$
$$\le -F^{-1}(p) \frac{\widetilde{G}_{\mathbf{A}}^{-1}(p)}{\log(1-p)} \min[(n-k+1)\overline{A}/\underline{A}, k\max(1, \overline{A}/A^*)], \quad p \in (0,1).$$

EXAMPLE 1. Let F be an IFR distribution and $G(x) = 1 - e^{-x}, x > 0$. Consider the *total time on test statistic*

$$U_{\mathbf{A}} = \sum_{i=1}^{r} (n-i+1)(X_{i:n} - X_{i-1:n}) = \sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}$$

in the life test of the II type censoring (without replacement, until the rth failure). It is well known that if the X_i have the exponential distribution, then $2U_{\mathbf{A}}$ has the chi-square distribution with $2r$ degrees of freedom. Thus write $\widetilde{G}_{\mathbf{A}}^{-1}(p) = \chi^2_{2r}(p)/p$. Note that $\underline{A} = n - r + 1$, $\overline{A} = n$ and $A^* = r$. Therefore from (31) we obtain

$$\text{(32)} \quad -\frac{F^{-1}(p)\chi^2_{2r}(p)}{2\log(1-p^{1/r})} \max\left[\frac{n-r+1}{n}, \max\left(1, \frac{n-r+1}{r}\right)\right]$$
$$\le \widetilde{F}_{\mathbf{A}}^{-1}(p) \le -\frac{nF^{-1}(p)\chi^2_{2r}(p)}{2\log(1-p)}, \quad p \in (0,1).$$

Barlow and Proschan [4] (Theorem 4.6) give bounds on $F^{-1}(p)$ if F is an IFR distribution with the expected value μ:

$$(33)\qquad \begin{aligned} -\mu\log(1-p) \le F^{-1}(p) &\le -\frac{\mu\log(1-p)}{p} \quad \text{if } p \le 1-e^{-1}, \\ \mu \le F^{-1}(p) &\le -\frac{\mu\log(1-p)}{p} \quad \text{if } p > 1-e^{-1}. \end{aligned}$$

Thus if $F^{-1}(p)$ is not known but μ is known one obtains from (32) and (33) the bounds on $\widetilde{F}_{\mathbf{A}}^{-1}(p)$:

$$(34)\qquad \frac{\mu\log(1-p)\chi^2_{2r}(p)}{2\log(1-p^{1/r})}\max\left[\frac{n-r+1}{n}, \max\left(1, \frac{n-r+1}{r}\right)\right] \le \widetilde{F}_{\mathbf{A}}^{-1}(p) \le \frac{n\mu\chi^2_{2r}(p)}{2p} \quad \text{if } p \le 1-e^{-1},$$

$$(35)\qquad -\frac{\mu\chi^2_{2r}(p)}{2\log(1-p^{1/r})}\max\left[\frac{n-r+1}{n}, \max\left(1, \frac{n-r+1}{r}\right)\right] \le \widetilde{F}_{\mathbf{A}}^{-1}(p) \le \frac{n\mu\chi^2_{2r}(p)}{2p} \quad \text{if } p > 1-e^{-1}.$$

Generally, if F is IFR, then $\widetilde{F}_{\mathbf{A}}$ need not be IFR. But if $r = n$, i.e.

$$U_{\mathbf{A}} = \sum_{i=1}^{n}(n-i+1)(X_{i:n} - X_{i-1:n}) = \sum_{i=1}^{n} X_{i:n} = \sum_{i=1}^{n} X_i,$$

then $\widetilde{F}_{\mathbf{A}}$ is also IFR, as a convolution of IFR distributions (see [4]). In this case one can use the bounds (33) for $\widetilde{F}_{\mathbf{A}}^{-1}(p)$ with the expected value $\mathrm{E}(U_{\mathbf{A}}) = n\mu$. It is easy to see that these bounds are more exact than (34) and (35), where $r = n$.

EXAMPLE 2. Let F be an IFRA distribution and $G(x) = 1 - e^{-x}$, $x > 0$. Consider the (r,s)-range

$$U_{\mathbf{A}} = X_{s:n} - X_{r:n}, \quad 1 \le r < s \le n.$$

It is easy to notice that $U_{\mathbf{A}} = \sum_{i=1}^{n} A_i U_{i:n}$, where $A_1 = \ldots = A_r = 0$, $A_{r+1} = \ldots = A_s = 1$, $A_{s+1} = \ldots = A_n = 0$. Therefore from Lemma 3 we have $\widetilde{G}_{\mathbf{A}}^{-1}\widetilde{F}_{\mathbf{A}}(x) \ge G^{-1}F(x)$, $x > 0$, or equivalently

$$(36)\qquad \widetilde{F}_{\mathbf{A}}^{-1}(p) \le F^{-1}G(\widetilde{G}_{\mathbf{A}}^{-1}(p)), \quad p \in (0,1).$$

It is well known that $\widetilde{G}_{\mathbf{A}} \overset{\mathrm{st}}{=} G_{s-r:s-r}$. Thus from (36) we have

$$\widetilde{F}_{\mathbf{A}}^{-1}(p) \le F_{s-r:s-r}^{-1} G_{s-r:s-r}(G_{s-r:s-r}^{-1}(p)) = F_{s-r:s-r}^{-1}(p), \quad p \in (0,1).$$

Now from (22) we obtain the upper bound for the p-quantile of the (r,s)-

range from the IFRA distribution F:

$$\widetilde{F}_{\mathbf{A}}^{-1}(p) \le F^{-1}(p) \frac{\log(1-p^{1/(s-r)})}{\log(1-p)} .$$

5. Characterization of the star-ordering of distributions. Langberg *et al.* [13] have characterized the IFRA class of distributions via monotonicity of $\mathrm{E}(X_{i:n})/\sum_{k=1}^{i}(n-k+1)^{-1}$. Now we prove the following analogue of their result.

THEOREM 10. *Let F and G be continuous life distributions, $F(0) = G(0) = 0$. Then $F <^* G$ if and only if for some $p \in (0,1)$ and infinitely many n, $F_{i:n}^{-1}(p)/G_{i:n}^{-1}(p)$ is decreasing in $i \le n$.*

Proof. The necessity part is Theorem 1(a), so we only need to prove sufficiency. Let $t \in (0,1)$ and $p \in (0,1)$ be fixed. Notice that

$$\frac{F_{[nt]:n}^{-1}(p)}{G_{[nt]:n}^{-1}(p)} = \frac{F^{-1}B_{[nt]:n}^{-1}(p)}{G^{-1}B_{[nt]:n}^{-1}(p)},$$

where $B_{[nt]:n}^{-1}(p)$ is the p-quantile of the $[nt]$th order statistic $R_{[nt]:n}$ from the uniform distribution $R(0,1)$, i.e.

$$B_{[nt]:n}(\xi) = \mathrm{P}\{R_{[nt]:n} \le \xi\} .$$

It is easy to see that

$$\lim_{n\to\infty} \mathrm{P}\{R_{[nt]:n} \le \xi\} = \begin{cases} 0 & \text{if } \xi < t, \\ 1 & \text{if } \xi \ge t, \end{cases}$$

which means that $R_{[nt]:n} \to t$ a.s. as $n \to \infty$. Therefore $B_{[nt]:n}^{-1}(p) \to t$ as $n \to \infty$. Since F^{-1} and G^{-1} are continuous, we have

$$\frac{F_{[nt]:n}^{-1}(p)}{G_{[nt]:n}^{-1}(p)} \to \frac{F^{-1}(p)}{G^{-1}(p)} \quad \text{for every } t \in (0,1)\,, \text{ as } n \to \infty\,.$$

From the assumption it follows that $F_{[nt]:n}^{-1}(p)/G_{[nt]:n}^{-1}(p)$ is decreasing in t for infinitely many n, so $F^{-1}(p)/G^{-1}(p)$ is also decreasing in $t \in (0,1)$, which means that $G^{-1}F(x)/x$ is increasing in $x > 0$. Thus $F <^* G$.

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