ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,4 (1993), pp. 571–573

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A NOTE ON OPTIMALITY CONDITIONS FOR DUAL PROBLEMS TO THE TRAVELING SALESMAN PROBLEM

Abstract. Different duals for the traveling salesman problem are considered. A simple example shows that it is in general difficult to check the optimality.

1. Introduction. The traveling salesman problem (TSP) is a well-known combinatorial optimization problem. The interested reader is referred to the excellent bibliography [3]. The problem can be described as follows:

$$\sum_{i,j} c_{ij}x_{ij} \to \min \quad \text{subject to} \tag{1}$$

$$\sum_{i} x_{ij} = 1 \quad \text{for all } j\,, \qquad \sum_{j} x_{ij} = 1 \quad \text{for all } i\,, \tag{2}$$

$$x_{ij} \in \{0,1\} \quad \text{for all } i,j \tag{3}$$

and

$$\sum_{i\in S, j\in \overline{S}} x_{ij} \ge 1 \quad \text{for all } S \subset N,\ 1 \le |S| \le n-1\,, \tag{4}$$

where $N := \{1,\ldots,n\}$.

The problem stated by (1)–(3) is called the assignment problem and the inequalities (4) subtour elimination constraints. In connection with the dual

1991 *Mathematics Subject Classification*: Primary 05C45; Secondary 90C27.
Key words and phrases: traveling salesman problem, duality.

problems we substitute (3) by

$$(5) \qquad 0 \le x_{ij} \quad \text{for all } i, j.$$

DEFINITION. Let $G(x)$ be a graph with n vertices such that the arc (i, j) exists iff $x_{ij} > 0$.

2. The dual problems. First we consider the duality concept of Balas & Christofides (cf. [1]). For the constraints of type (4) we introduce Lagrangean multipliers w_k, $k \in K := \{1, \ldots, 2^n - 2\}$. The Lagrange function takes the form

$$L(x, w) := \sum_{i,j} c_{ij} x_{ij} + \sum_{k \in K} w_k \Big(1 - \sum_{i \in S_k, j \in \overline{S}_k} x_{ij}\Big).$$

The dual function becomes

$$\varphi(w) = \min\{L(x, w) : x \text{ satisfies (2) and (5)}\}$$

and we can define the dual problem:

$$\varphi(w) \to \max, \quad w \ge 0.$$

Let $w^* \ge 0$ with

$$\varphi(w^*) \ge \varphi(w) \quad \text{for all } w \ge 0.$$

Now we consider the duality approach of Held & Karp (cf. [2]) in the continuous form. They introduce Lagrangean multipliers for the constraints (2). We obtain:

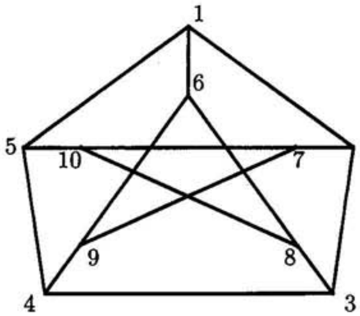
$$\psi(u, v) := \min\Big\{\sum_{i,j} (c_{ij} - u_i - v_j) x_{ij} : x \text{ satisfies (4) and (5)}\Big\} + \sum_i u_i + \sum_j v_j.$$

Let $\psi(u^*, v^*) \ge \psi(u, v)$ for all u, v.

Then from the classical Lagrangean theory we obtain the following statement:

COROLLARY. *The optimal objective function value of the problem* (1), (2), (4) *and* (5) *is equal to* $\varphi(w^*) = \psi(u^*, v^*)$.

Now we inspect the Peterson graph (cf. [3]):



We set $x'_{ij} := 1/3$ if the arc (i, j) exists, else $x'_{ij} := 0$. Obviously, the vector x' satisfies (2), (4) and (5). But the graph $G(x')$ does not contain a tour.

EXAMPLE. Let C' be a distance matrix with: if $x'_{ij} = 0$ then $c'_{ij} > 0$ else $c'_{ij} = 0$.

From the Corollary we obtain $0 = \varphi(0) = \psi(0, 0)$ as the optimal objective function value for this example. The weighted optimal assignment problem solutions and the weighted optimal 1-trees do not contain a tour.

In the section "Find a tour and improving the bound" in [1] the authors suggest that the optimal assignment problem solutions contain a tour. Our example contradicts the effort to construct a tour with the constraints of type (4), in general.

References

[1] E. Balas and N. Christofides, *A restricted Lagrangean approach to the traveling salesman problem*, Math. Programming 21 (1981), 19–46.

[2] M. Held and R. M. Karp, *The traveling-salesman problem and minimum spanning trees*, Oper. Res. 18 (1970), 1138–1162.

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Received on 2.9.1992