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A SILENT VERSUS PARTIALLY NOISY ONE-BULLET DUEL UNDER ARBITRARY MOTION

A duel is considered in which Player I has one silent bullet, Player II has one partially noisy bullet, the accuracy functions are the same and the players can move as they like. It is assumed that the maximal speed of Player I is greater than that of Player II.

1. Introduction. We consider the following *game* $(1,1)$. Two players, say I and II, fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. Players I and II have one bullet each and this fact is known to both players. Player II does not hear the shot of Player I, Player I hears the shot of Player II with probability p, $0 \leq p < 1$.

At the beginning of the duel the players are at distance 1 from each other. Let $P(s)$ be the probability of succeeding (destroying the opponent) by Player I (II) when the distance between them is $1-s$, $s \leq 1$. The function $P(s)$ is called the *accuracy function*. It is assumed that

(i) P is increasing and has a continuous second derivative in $[0,1]$,
(ii) $P(s) = 0$ for $s \leq 0$, $P(1) = 1$.

The duel starts at time $t = 0$ and ends when at least one player is destroyed or both bullets are shot; otherwise it continues infinitely long.

Player I gains 1 if only he achieves success, gains -1 if only Player II achieves success, and gains 0 in the remaining cases. The duel is a zero-sum game.

Suppose that Player II has fired his shot and missed and that Player I has heard that shot. In this case the best what Player I can do, if he has his

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bullet yet, is to reach Player II in pursuit and achieve success surely. This behaviour of Player I is assumed in the paper.

Without loss of generality we can suppose that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

For definitions and results in the theory of games of timing see [3]–[6], [8], [9], [15], [16].

2. Auxiliary duel. To solve the game $(1,1)$ presented in the previous section we have to determine optimal strategies in the following auxiliary *game* $(1,1)^*$. Consider a one-bullet silent versus partially noisy duel with accuracy function $P(t)$, the same for both players. It is assumed that Player I approaches Player II with constant velocity $v = 1$ all the time, even after firing his bullet. Player I gains 1 if only he achieves success etc., as in the duel defined in the previous section.

Denote by $K_0(s,t)$ the expected gain of Player I if he fires at time $s \in [0,1]$ and Player II fires at times $t \in [0,1]$. It is assumed that

$$K_0(s,t) = \begin{cases} P(s) & \text{if } s < t, \\ 0 & \text{if } s = t, \\ -P(t) + p(1-P(t)) + (1-p)(1-P(t))P(s) & \text{if } s > t. \end{cases}$$

As is easy to see, $K_0(s,t)$ is the expected payoff in the duel in which Player II is not allowed to fire after Player I has. Player I is allowed to fire after Player II has, but if he has heard the shot of his opponent he has to act just as in the duel $(1,1)$.

Denote by ξ_0^a the strategy of Player I in the game $(1,1)^*$ in which he fires at a random moment s distributed according to a density $f_1(s)$ in the interval $[a,1]$, $0 < a < 1$, and according to a probability α, $0 < \alpha < 1$, at the point 1. This distribution is chosen in such a way that if $t \in [a,1)$ then

$$\begin{aligned} K_0(\xi_0^a,t) = & \int_a^t P(s)f_1(s)\,ds \\ & + \int_t^1 (-P(t) + p(1-P(t)) + (1-p)(1-P(t))P(s))f_1(s)\,ds \\ & + (1-2P(t))\alpha = \text{const}\,. \end{aligned}$$

Here $K_0(\xi_0^a,t)$ is the expected gain of Player I if he applies the strategy ξ_0^a and Player II fires at time t.

We have

$$(1)\quad \frac{\partial K_0(\xi_0^a,t)}{\partial t} = (2P(t) - p(1-P(t)) - (1-p)(1-P(t))P(t))f_1(t) - P'(t)\int_t^1 (1+p+(1-p)P(s))f_1(s)\,ds - 2P'(t)\alpha = 0\,,$$

$$(2)\quad \frac{\partial^2 K_0(\xi_0^a,t)}{\partial t^2} = (2P(t) - p(1-P(t)) - (1-p)(1-P(t))P(t))f_1'(t) + P'(t)(3+2p-(1-p)(1-3P(t)))f_1(t) - P''(t)\int_t^1 (1+p+(1-p)P(s))f_1(s)\,ds - 2P''(t)\alpha = 0\,.$$

Eliminating the integral from (1) and (2) we obtain the equation

$$\frac{f_1'(t)}{f_1(t)} - \frac{P''(t)}{P'(t)} + \frac{3+2p-(1-p)(1-3P(t))}{(1-p)P^2(t)+(1+2p)P(t)-p}P'(t) = 0$$

whose solution is

$$(3)\qquad f_1(t) = \frac{CP'(t)}{(P(t)-P_1)^E(P(t)-P_2)^F}\,,$$

where

$$(4)\qquad P_1 = \frac{1}{2(1-p)}(-1-2p+\sqrt{1+8p}) = \frac{2p}{1+2p+\sqrt{1+8p}}\,,$$

$$P_2 = \frac{1}{2(1-p)})(-1-2p-\sqrt{1+8p})\,,$$

$$E = \frac{3}{2} + \frac{\frac{1}{2}}{\sqrt{1+8p}}\,,\qquad F = \frac{3}{2} - \frac{\frac{1}{2}}{\sqrt{1+8p}}\,.$$

From (4) it follows that $0 < P_1 < 1$.

Obviously, the constant C in (3) satisfies the equation

$$(5)\qquad C\int_a^1 \frac{P'(t)\,dt}{(P(t)-P_1)^E(P(t)-P_2)^F} + \alpha = 1\,.$$

Let η_0^a be the strategy of Player II in the game $(1,1)^*$ in which Player II chooses at random the moment t at the shot according to a density $f_2(t)$ in $[a,1]$ to obtain

$$(6)\quad K_0(s,\eta_0^a) = \int_a^s (-P(t)+p(1-P(t))+(1-p)(1-P(t))P(s))f_2(t)\,dt + \int_s^1 P(s)f_2(t)\,dt = \text{const}$$

if $s \in [a,1]$, where $K_0(s,\eta_0^a)$ is the expected gain of Player I if Player II applies the strategy η_0^a and Player I fires at s.

Acting in the same way as before, we obtain

$$f_2(t) = \frac{DP'(t)}{(P(t)-P_1)^F(P(t)-P_2)^E}, \tag{7}$$

$$D\int_a^1 \frac{DP'(t)\,dt}{(P(t)-P_1)^F(P(t)-P_2)^E} = 1. \tag{8}$$

From (1) and (3) it follows that

$$\frac{\partial K_0(\xi_0^a,t)}{\partial t}\frac{1}{P'(t)}\bigg|_{t=1} = \frac{C(1-p)}{(1-P_1)^{E-1}(1-P_2)^{F-1}} - 2\alpha = 0. \tag{9}$$

From (6) and (7) it follows that

$$\frac{\partial K_0(s,\eta_0^a)}{\partial s}\frac{1}{P'(s)}\bigg|_{s=1} = -\frac{D(1-p)}{(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}} + 1 = 0. \tag{10}$$

From (5), (8), (9) and (10) we determine the unknown parameters C, D, α, a, $0<\alpha<1$, $P_1<a<1$. We prove that such a solution exists and is unique for any $P(t)$ satisfying our conditions.

Inserting the constant D from (10) to the equation (8) we obtain

$$(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}\int_a^1 \frac{P'(s)\,ds}{(P(s)-P_1)^F(P(s)-P_2)^E} = 1-p. \tag{11}$$

Denote by $\varphi(a)$ the left side of the above equation. We have

$$\begin{aligned}\varphi'(a) = \Big\{&[(F-1)(P(a)-P_1)^{F-2}(P(a)-P_2)^{E-1}\\ &+(E-1)(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-2}]\\ &\times\int_a^1 \frac{P'(s)\,ds}{(P(s)-P_1)^F(P(s)-P_2)^E}\\ &-\frac{1}{(P(a)-P_1)(P(a)-P_2)}\Big\}P'(a)\\ =\ &\frac{(1-p)P(a)+p-1}{(P(a)-P_1)(P(a)-P_2)}P'(a) < 0\end{aligned}$$

for any a satisfying (11) and any p, $0\le p<1$. Moreover,

$$\varphi(P_1+) = \frac{1}{P(P_1)+\dfrac{p}{1-p}} > 1-p, \qquad \varphi(1) = 0.$$

Thus there always exists a unique solution a, $P_1<a<1$, of (11) if $0\le p<1$.

Moreover, from (5) and (9) we obtain

$$(12)\quad \left(\frac{2(1-P_1)^{E-1}(1-P_2)^{F-1}}{1-p}\int_a^1 \frac{P'(t)\,dt}{(P(t)-P_1)^E(P(t)-P_2)^F}+1\right)\alpha = 1\,.$$

From the above and (9) it follows that if $0 \le p < 1$ there always exists a unique solution C, α of (5) and (9) such that $0 < \alpha < 1$.

We now prove that $K_0(\xi_0^a, t) = K_0(s, \eta_0^a)$ for $a \le s \le 1$, $a \le t < 1$. We get

$$(13)\quad \begin{aligned} K_0(\xi_0^a, a) &= (-P(a) + p(1-P(a)))(1-\alpha) + (1-2P(a))\alpha \\ &\quad + (1-p)(1-P(a)) \int_a^1 P(s) f_1(s)\,ds \\ &= \int_a^1 P(s) f_1(s)\,ds - \alpha = K_0(\xi_0^a, 1-)\,, \end{aligned}$$

$$(14)\quad K_0(1-, \eta_0^a) = \int_a^1 (1-2P(t)) f_2(t)\,dt = P(a) = K_0(a, \eta_0^a)\,.$$

From (13) it follows that

$$(15)\quad \begin{aligned} &[1-(1-p)(1-P(a))] \int_a^1 P(s) f_1(s)\,ds \\ &\qquad = -P(a) + p(1-P(a)) + (2 - p(1-P(a)) - P(a))\alpha \end{aligned}$$

and

$$K_0(\xi_0^a, 1-) = \int_a^1 P(s) f_1(s)\,ds - \alpha = P(a) = K_0(a, \eta_0^a)$$

if

$$\alpha = \frac{(P(a)-P_1)(P(a)-P_2)}{2(1-P(a))}\,.$$

On the other hand,

$$\begin{aligned} \frac{\partial K_0(\xi_0^a, t)}{\partial t} \frac{1}{P'(t)}\bigg|_{t=a} &= \frac{C(1-p)}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}} \\ &\quad - \int_a^1 (1+p+(1-p)P(s)) f_1(s)\,ds - 2\alpha = 0 \end{aligned}$$

and by (15) we get

$$\frac{C(1-p)}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}} - (1+p)(1-\alpha)$$

$$-(1-p)\frac{p-(1+p)P(a)+[2-p-(1-p)P(a)]\alpha}{p+(1-p)P(a)}-2=0\,.$$

Using (9) we obtain

$$(17)\qquad \frac{(1-P_1)^{E-1}(1-P_2)^{F-1}}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}}2\alpha-(1+p)(1-\alpha) \\ -(1-p)\frac{p-(1+p)P(a)+[2-p-(1-p)P(a)]\alpha}{p+(1-p)P(a)}-2\alpha=0\,.$$

Moreover, from (6) and (7) we have

$$(18)\qquad \frac{\partial K_0(s,\eta_0^a)}{\partial t}\frac{1}{P'(s)}\bigg|_{s=1-}=-\frac{D(1-p)}{(1-P_1)^{F-1}(1-P_2)^{E-1}} \\ +(1-p)\int\limits_a^1(1-P(t))f_2(t)\,dt=0\,,$$

and by (14),

$$(19)\qquad \int\limits_a^1(1-P(t))f_2(t)\,dt=\frac{1+P(a)}{2}\,.$$

Then from (10), (18) and (19),

$$(20)\qquad \frac{(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}}{(1-P_1)^{F-1}(1-P_2)^{E-1}}=\frac{1}{2}(1-p)(1+P(a))\,.$$

But

$$(P-P_1)(P-P_2)=P^2+\frac{1+2p}{1-p}P-\frac{p}{1-p}$$

and

$$(1-P_1)(1-P_2)=\frac{2}{1-p}\,,\qquad E+F-2=1\,.$$

Dividing (17) by (20) and taking into account the above equations we obtain

$$\alpha=\frac{P^2(a)+\dfrac{1+2p}{1-p}P(a)-\dfrac{p}{1-p}}{2(1-P(a))}\,,$$

which is the same as (16). Thus $K_0(\xi_0^a,t)=K_0(s,\eta_0^a)$ for $a\le s\le 1$, $a\le t<1$.

LEMMA. *For a being the solution of equation* (11) *the strategy ξ_0^a is maximin and the strategy η_0^a is minimax in the game* $(1,1)^*$. *The value of the game is $v_{11}^0=P(a)$.*

Proof. We have proved that $K_0(\xi_0^a, t) = P(a)$ for $a \le t < 1$. Moreover,

$$K_0(\xi_0^a, 1) = \int_a^1 P(s) f_1(s)\, ds > \int_a^1 P(s) f_1(s)\, ds + (1 - 2P(1))\alpha$$
$$= \lim_{t\to 1-} K_0(\xi_0^a, t) = P(a).$$

If $t < a$ we have

$$K_0(\xi_0^a, t) = \int_a^1 (-P(t) + p(1 - P(t))$$
$$+ (1-p)(1 - P(t))P(s)) f_1(s)\, ds + (1 - 2P(t))\alpha$$
$$> \int_a^1 (-P(a) + p(1 - P(a)) + (1 - P(a))P(s)) f_1(s)\, ds$$
$$+ (1 - 2P(a))\alpha$$
$$= K_0(\xi_0^a, a) = P(a).$$

Thus $K_0(\xi_0^a, \eta) \ge P(a)$ for any strategy η of Player II.

On the other hand, $K_0(s, \eta_0^a) = P(a)$ for $a \le s \le 1$, and if $s < a$ then $K_0(s, \eta_0^a) = P(s) < P(a)$. Therefore $K_0(\xi, \eta_0^a) \le P(a)$ for any strategy ξ of Player I, which ends the proof of the lemma.

3. Solution of the duel $(1, 1)$. We now consider the duel $(1, 1)$ defined at the beginning of the paper. For given natural n such that $1/n \le 1 - \alpha$ let the constants a_k be defined as follows:

$$a_0 = a, \quad \int_{a_{k-1}}^{a_k} f_1(s)\, ds = \frac{1}{n}, \quad k = 1, \ldots, n_0, \quad a_{n_0+1} = 1,$$

where n_0 is defined from the inequalities $1 - \alpha - 1/n \le n_0/n < 1 - \alpha$.

Define the strategy ξ_ε of Player I in the game $(1, 1)$ as follows: Player I moves back and forth with maximal speed in the following manner: at first between 0 and a_1, then between 0 and $a_2, \ldots$, finally between 0 and a_{n_0+1}. At the kth step, $k = 1, \ldots, n_0 + 1$, he can fire his shot at random only if he is between the points a_{k-1} and a_k and goes forward, and he fires it with probability density $f_1(s)$. If he has fired at the kth step, he reaches the point a_k, escapes and never approaches Player II. If Player I has not fired between the points 0 and 1 and survives, he fires when he is at 1, as soon as possible.

The strategy η_0 of Player II is defined as follows: If Player I reaches the point t the first time and his velocity is $v_1(\tau)$, τ being the time, fire at random with density $v_1(\tau) f_2(t(\tau))$. Otherwise do not fire.

It is assumed that the function $v_1(\tau)$ is piecewise continuous.

THEOREM. *The strategy ξ_ε is ε-maximin and the strategy η_0 is minimax in the game $(1,1)$. The value of the game is $v_{11} = P(a)$.*

Proof. Assume that Player I applies the strategy ξ_ε. We say that Player II fires at (k, a'), $k = 1, \ldots, n_0 + 1$, if he fires when Player I is at the point a' and if this happens during the first player's approach to a_k or his escape from a_{k-1}.

Denote by $K(\xi_\varepsilon; k, a')$ the expected gain of Player I if he applies the strategy ξ_ε and Player II fires at (k, a'). We obtain

$$\begin{aligned} K(\xi_\varepsilon; k, a') \geq & \int_a^{a_{k-1}} P(s) f_1(s) + \int_{a_k}^{1} (-P(a') + p(1 - P(a')) \\ & + (1-p)(1 - P(a'))P(s)) f_1(s)\, ds + (1 - 2P(a'))\alpha - 1/n \\ \geq & \int_a^{a_{k-1}} P(s) f_1(s)\, ds + \int_{a_k}^{1} (-P(a_k) + p(1 - P(a_k)) \\ & + (1-p)(1 - P(a_k))P(s)) f_1(s)\, ds + (1 - 2P(a_k))\alpha - 1/n \\ \geq & \int_a^{a_k} P(s) f_1(s)\, ds + \int_{a_k}^{1} (-P(a_k) + p(1 - P(a_k)) \\ & + (1-p)(1 - P(a_k))P(s)) f_1(s)\, ds + (1 - 2P(a_k))\alpha - \varepsilon \\ = & P(a) - \varepsilon\,, \end{aligned}$$

where $\varepsilon = 2/n$, $k = 1, \ldots, n_0 + 1$.

If Player II fires only when Player I reaches 1, the best for him is to fire as soon as possible. For such a strategy (call it η) we obtain

$$\begin{aligned} K(\xi_\varepsilon; \eta) \geq & \int_a^1 P(s) f_1(s)\, ds \\ & > \int_a^1 P(s) f_1(s)\, ds + (1 - 2P(1))\alpha = P(a)\,. \end{aligned}$$

From the above it follows that $K(\xi_\varepsilon; \eta) \geq P(a) - \varepsilon$ for any strategy η of Player II.

On the other hand, define

$\dot{a}$ = the farthest point reached by Player I before he fires,

a' = the point from which he fires,

$\dot{a}'$ = the farthest point reached by Player I after he fires,

$\widehat{a} = \max(\dot{a}, \dot{a}')$.

We have

$$a' \le \dot{a}, \quad a' \le \dot{a}'.$$

For such a strategy, say ξ, if $a < \dot{a}$, we have

$$\begin{aligned} K(\xi;\eta_0) = & \int_a^{\dot{a}} (-P(t) + p(1-P(t)) + (1-p)(1-P(t))P(a'))f_2(t)\,dt \\ & + \int_{\dot{a}}^{\hat{a}} (P(a') - (1-P(a'))P(t))f_2(t)\,dt \\ \le & \int_a^{\dot{a}} (-P(t) + p(1-P(t)) + (1-p)(1-P(t))P(\dot{a}))f_2(t)\,dt \\ & + \int_{\dot{a}}^{1} P(\dot{a})f_2(t)\,dt \\ = & P(a) \end{aligned}$$

by (6) and (12), and if $\dot{a} < a < \dot{a}'$ then

$$\begin{aligned} K(\xi;\eta_0) &= P(a') - (1-P(a')) \int_a^{\dot{a}'} P(t)f_2(t)\,dt \\ &\le P(a) - (1-P(a)) \int_a^{a'} P(t)f_2(t)\,dt \le P(a). \end{aligned}$$

Finally, if $\widehat{a} < a$ then

$$K(\xi;\eta_0) = P(a') \le P(a).$$

Thus $K(\xi;\eta_0) \le P(a)$ for any strategy ξ of Player I. The theorem is proved.

Duels under arbitrary motion, as far as the author knows, have never been considered before, except in the papers of the author (see [12]–[14]).

For other results in the theory of duels see [1], [2], [7], [10], [11].

References

[1] A. Cegielski, *Tactical problems involving uncertain actions*, J. Optim. Theory Appl. 49 (1986), 81–105.

[2] —, *Game of timing with uncertain number of shots*, Math. Japon. 31 (1986), 503–532.

[3] M. Fox and G. Kimeldorf, *Noisy duels*, SIAM J. Appl. Math. 19 (1969), 353–361.

[4] S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. 2, Addison-Wesley, Reading, Mass., 1959.

[5] G. Kimeldorf, *Duels: an overview*, in: *Mathematics of Conflict*, North-Holland, 1983, 55–71.

[6] K. Orłowski and T. Radzik, *Non-discrete silent duels with complete counteraction*, Optimization 16 (1985), 257–263.

[7] —, —, *Discrete silent duels with complete counteraction*, ibid., 419–429.

[8] R. Restrepo, *Tactical problems involving several actions*, in: *Contributions to the Theory of Games*, Vol. III, Ann. of Math. Stud. 39, Princeton Univ. Press, 1957, 313–335.

[9] A. Styszyński, *An n-silent-vs.-noisy duel with arbitrary accuracy functions*, Zastos. Mat. 14 (1974), 205–225.

[10] Y. Teraoka, *Noisy duels with uncertain existence of the shot*, Internat. J. Game Theory 5 (1976), 239–250.

[11] —, *A single bullet duel with uncertain information available to the duelists*, Bull. Math. Statist. 18 (1979), 69–80.

[12] S. Trybuła, *A noisy duel under arbitrary moving. I–VI*, Zastos. Mat. 20 (1990), 491–495, 497–516, 517–530; Zastos. Mat. 21 (1991), 43–61, 63–81, 83–98.

[13] —, *Solution of a silent duel under general assumptions*, Optimization 22 (1991), 449–459.

[14] —, *A mixed duel under arbitrary motion*, Zastos. Mat., to appear.

[15] N. N. Vorob'ev, *Foundations of the Theory of Games. Uncoalition Games*, Nauka, Moscow 1984 (in Russian).

[16] E. B. Yanovskaya, *Duel-type games with continuous firing*, Engrg. Cybernetics 1969 (1), 15–18.

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