ZASTOSOWANIA MATEMATYKI
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M. D. BENCHIBOUN (Lille)

# SOME RESULTS ON NON-LINEAR AND NON-LOGARITHMIC SEQUENCES AND THEIR ACCELERATION

*Abstract.* We first study the relation between the asymptotic behaviour of the ratio of the errors and the ratio of the differences for converging sequences. A classification of converging sequences is given.

Assuming that the ratio of the errors has $m$ limit points, we study the behaviour of the ratio of differences and three acceleration processes are deduced. In the particular case $m = 2$, these processes are studied and a characterization of some series is given.

**I. Introduction and notations.** The construction and the study of convergence acceleration methods for sequences and series requires a knowledge of the behaviour of the ratio of the errors $\varrho_n$ and the ratio of differences $R_n$ between two consecutive terms of a converging sequence. The majority of these methods fail if the sequences $(\varrho_n)$ or $(R_n)$ do not converge.

In this paper we suppose that the sequence $(\varrho_n)$ has $m$ limit points which may be zero, one or infinity. In Section 2 we study the behaviour of the sequence $(R_n)$ and we give some results, a classification of the converging sequences and a characterization of some series. A subset of these sequences was introduced in [5] where some results are given as well. These results are applied to continued fractions in [4].

In the third section we recover the $\Delta_m^2$ process given in [5], the $T_{+m}$ transformation introduced and studied in [6] and [13] and we propose a generalization $\theta_{2,m}$ of the process $\theta_2$ of Brezinski [2]. Some acceleration properties of these processes are given in the particular case $m = 2$.

Numerical examples illustrate the results of this paper in the fourth section.

Let $(S_n)$ be a real sequence converging to $S$. We set

$$e_n = S_n - S\,, \quad \varrho_n = (S_{n+1} - S)/(S_n - S)\,, \quad R_n = \Delta S_{n+1}/\Delta S_n$$

and we consider the two cases:

**1.** $(\varrho_n)$ *converges to* $\varrho$

a) If $\varrho \neq 1$, then the sequence $(R_n)$ converges to $\varrho$ and $(S_n)$ converges linearly; in this case we write $(S_n) \in LIN$.

b) If $\varrho = 1$, then $(S_n)$ converges logarithmically: $(S_n) \in LOG$, and for the sequence of ratios $(R_n)$ three cases can occur:

• $\lim_{n\to\infty} R_n = 1$; we write $(S_n) \in LOGSF$. This set of sequences was defined by Smith and Ford [12] and named by Kowalewski [8]. As examples, we consider the fixed point sequences $(S_n) \in \mathcal{A}_F^{(p)}(S)$ [8] and $(S_n) \in LOGF_p$ [11].

• $\lim_{n\to\infty} R_n = -1$; we write $(S_n) \in LOGL$ [9].

EXAMPLE 1. Let $(S_n)$ be the sequence defined by

$$S_{n+1} = S_n(1+\alpha_n), \quad S_1 = 1,$$

with $\alpha_{2n+1} = 1/(\alpha + \sqrt{2n+1})$, $\alpha_{2n} = -1/\sqrt{2n+1}$ $(n = 0, 1, \ldots)$. Then $\lim_{n\to\infty} S_n = 0$ and $(S_n) \in LOGL$. Note that $(S_n)$ represents the $n$th convergent of some continued fraction [9, p. 119].

• $(R_n)$ does not converge. We write $(S_n) \in LOGAP$.

EXAMPLE 2. Let $(S_n)$ be defined by $S_{n+1} = S_n(1+\alpha_n)$, $S_1 = 1$, with $\alpha_{2n} = a/(n+1)$, $\alpha_{2n+1} = b/(n+1)$, $a \neq b$; if $a, b < 0$, then $\lim_{n\to\infty} S_n = 0$, and

$$\lim_{n\to\infty} R_{2n} = b/a, \quad \lim_{n\to\infty} R_{2n+1} = a/b.$$

So $(S_n) \in LOGAP$.

DEFINITION 1. A sequence $(U_n)$ *has $m$ limit points* if there exist $m$ subsequences $(U_{\varphi_i(n)})$, $i = 1, \ldots, m$, such that $\lim_{n\to\infty} U_{\varphi_i(n)}$ exists.

**2.** $(\varrho_n)$ *does not converge*. Suppose that the sequence $(\varrho_n)$ has $m$ limit points $\varrho^{(0)}, \ldots, \varrho^{(m-1)}$ and

$$\lim_{n\to\infty} \varrho_{nm+i} = \varrho^{(i)} \quad \text{for } i = 0, 1, \ldots, m-1.$$

Set $\varrho^{(m)} = \varrho^{(0)}$. In this paper we assume that $\varrho^{(i)}$ may be zero, one or infinity.

Remark I.1. In the particular case where $\varrho^{(i)}$ $(i = 0, \ldots, m-1)$ is finite, different from 0, 1 and $|\prod_{i=0}^{m-1} \varrho^{(i)}| < 1$, the sequence $(S_n)$ is called *periodic-linear*. These sequences were introduced by Delahaye in [5] where some convergence results are given as well. These results are applied to continued fractions in [4].

*Relation between the asymptotic behaviour of the sequences* $(\varrho_n)$ *and* $(R_n)$. We have

$$R_n = \varrho_n(\varrho_{n+1} - 1)/(\varrho_n - 1) \tag{1.1}$$

and

$$\lim_{n\to\infty} R_{mn+i} = R^{(i)} = \varrho^{(i)}(\varrho^{(i+1)} - 1)/(\varrho^{(i)} - 1) \quad (i = 0, \dots, m-1),$$
$$R^{(m)} = R^{(0)}.$$

Note that $R^{(i)} \neq \varrho^{(i)}$ for $i = 0, \dots, m-1$; this is the reason why some processes do not accelerate the convergence of the sequences $(S_n)$, as is shown in the following examples.

EXAMPLE 3 [10]. We consider the series $S$ with partial sums

$$S_n = \sum_{i=0}^{n} (-1)^{[i/2]}/(i+1), \quad [x] = \text{greatest integer contained in } x,$$

and $S = \pi/2 + 0.5 \ln 2$. Then $m = 2$, and

$$R^{(0)} = -1, \quad R^{(1)} = +1, \quad \varrho^{(0)} = -\infty, \quad \varrho^{(1)} = 0,$$
$$\lim_{n\to\infty} \varrho_n \varrho_{n+1} = -1 = R^{(0)} R^{(1)}.$$

In [14] Aitken's $\Delta^2$ process applied to $(S_n)$ gives

$$\Delta_2^{(2n)} = S_{2n} + (-1)^n (2n+3)/(2n+2)(4n+5),$$
$$\Delta_2^{(2n+1)} = S_{2n+1} + (-1)^n (2n+4)/(2n+3).$$

So the $\Delta^2$ process is not regular for $(S_n)$, and the sequence $(\Delta_2^{(n)})$ contains three essentially distinct convergent subsequences.

We remark that $R_n$ is the acceleration factor of $\Delta^2$ (see Definition 2). So we can write

$$\Delta_2^{(n)} = (S_{n+1} - R_n S_n)/(1 - R_n) \tag{1.2}$$

and note that a subsequence of $(R_n)$ has limit 1.

EXAMPLE 4 [6]. Let $S$ be the series defined by

$$S = \sum_{n=1}^{\infty} a_n \quad \text{with} \quad a_n = 4 \sin(n\pi/2)/n.$$

We have

$$R_n = -\frac{n+1}{n+2} \tan \frac{n\pi}{2} \quad \text{and} \quad R_n R_{n+1} = -\frac{n+1}{n+3}.$$

In this case $m = 2$, and

$$R^{(0)} = 0, \quad R^{(1)} = -\infty, \quad \varrho^{(0)} = -1, \quad \varrho^{(1)} = 1,$$

$$\lim_{n\to\infty} R_n R_{n+1} = -1 = \varrho^{(0)}\varrho^{(1)}$$

and no subsequence of $(R_n)$ has limit 1; then the Aitken process is regular for $(S_n)$ but does not accelerate its convergence since $\varrho^{(i)} \neq R^{(i)}$ for $i = 0, 1$.

From (1.2), we remark that $\varrho^{(i)} = R^{(i)}$ $(i = 0, 1)$ is the necessary condition for the Aitken process to accelerate the convergence of $(S_n)$.

From (1.1) we deduce

$$\prod_{i=0}^{m-1} R^{(i)} = \prod_{i=0}^{m-1} \varrho^{(i)} \tag{1.3}$$

and if $k$ is the number of limit points of $(R_n)$ then $k \leq m$, so we consider the subsets

$$LAP(m) = \{(S_n) : (S_n) \text{ converges and } \lim_{n\to\infty} \varrho_{mn+i} = \varrho^{(i)}, \ \varrho^{(i)} \neq \varrho^{(j)} \text{ if } i \neq j\},$$

$$LAP(m,k) = \{(S_n) \in LAP(m) : (R_n) \text{ has } k \text{ limit points }\}.$$

We shall prove that

$$LAP(m,1) = \emptyset \quad \text{for } m \geq 3,$$
$$LAP(2,1) \neq \emptyset,$$
$$LAP(m,k) \neq \emptyset \quad \text{for } 2 \leq k \leq m.$$

Let $(S_n)$ be a sequence converging to $S$ and let $T : S_n \to T_n$ be a sequence transformation. We can write

$$T_n = (S_{n+1} - f_n S_n)/(1 - f_n),$$
$$(T_n - S)/(S_n - S) = (\varrho_n - f_n)/(1 - f_n).$$

DEFINITION 2. $f_n$ is called the *acceleration factor* of the sequence transformation $T$.

$f_n$ is introduced by Lembarki [9] and studied by Benchiboun [1]. Note that, in most cases, $f_n$ depends on some terms of the sequence $(R_n)$.

Furthermore, $T$ accelerates the convergence of $(S_n)$ if and only if $\lim_{n\to\infty}(1-\varrho_n)/(1-f_n) = 1$; therefore if $\lim_{n\in N'} \varrho_n = \varrho$ (where $N' \subset \mathbb{N}$) then a necessary condition for accelerating the convergence of $(S_n)$ is $\lim_{n\in N'} f_n = \varrho$. This condition is not satisfied by the majority of the processes if $(S_n)$ belongs to $LOGL$ or $LOGAP$ or $LAP(m,k)$.

In order to accelerate the convergence of the sequences in $LAP(m,k)$, we propose a generalization of the $\theta_2$ process of Brezinski [2]. From (1.3) we have the asymptotic approximation $e_{n+m}/e_n \sim \Delta S_{n+m}/\Delta S_n$, thus we recover the transformation $T_{+m}$ [6, 13] and the process $\Delta^2_m$ [5].

Finally, the particular case $m = 2$ is fully studied, so a characterization of some series of the set $LAP(2,1)$ and some numerical examples are given.

**II. Classification and characterization.** Let $(S_n)$ be a real sequence converging to $S$. We consider two cases, depending on whether $(\varrho_n)$ converges or not.

**A.** $(\varrho_n)$ *converges to* $\varrho$. We have $|\varrho| \le 1$ and,

- if $\varrho \neq 1$ then $(S_n)$ converges linearly, $(S_n) \in LIN$,
- if $\varrho = 1$ then $(S_n)$ converges logarithmically, $(S_n) \in LOG$.

First we present a theorem on the asymptotic comparison of the sequences $(\varrho_n)$ and $(R_n)$.

THEOREM II.1 [5]. *Let* $\lambda \in \mathbb{R}$, $|\lambda| \neq 1$. *Then*

$$\lim_{n\to\infty} \frac{S_{n+1}-S}{S_n - S} = \lambda \quad \textit{if and only if} \quad \lim_{n\to\infty} \frac{\Delta S_{n+1}}{\Delta S_n} = \lambda .$$

Moreover, from (1.1) we remark that, if $|\lambda| = 1$ and $\lambda \neq 1$ and if $\lim_{n\to\infty}(S_{n+1}-S)/(S_n-S) = \lambda$ then $\lim_{n\to\infty} \Delta S_{n+1}/\Delta S_n = \lambda$.

**1.** *Linear convergence*, $(S_n) \in LIN$. Let $(S_n) \in LIN$. Then $(\varrho_n)$ and $(R_n)$ converge to $\varrho \neq 1$. We set

$$\varrho_n = \varrho + \alpha_n \quad \text{where} \quad \lim_{n\to\infty} \alpha_n = 0 ,$$
$$R_n = \varrho + \beta_n \quad \text{where} \quad \lim_{n\to\infty} \beta_n = 0 .$$

We say that $(S_n) \in LIN_\varrho$. If $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = \alpha$ exists, then $|\alpha| \le 1$. In this case we write $(S_n) \in LIN_{\varrho,\alpha}$ and set $\alpha_{n+1}/\alpha_n = \alpha + \nu_n$, where $\lim_{n\to\infty} \nu_n = 0$.

THEOREM II.2. *Suppose that* $(S_n) \in LIN_{\varrho,\alpha}$.

(i) *If* $\varrho\alpha \neq 1$, *then* $\lim_{n\to\infty} \beta_{n+1}/\beta_n = \beta$ *exists*, $\beta = \alpha$ *and moreover* $\lim_{n\to\infty} \beta_n/\alpha_n = (1-\varrho\alpha)/(1-\varrho)$.

(ii) *If* $\alpha = 1$ *and* $\lim_{n\to\infty} \nu_{n+1}/\nu_n = \nu$ *exists*, *then* $|\nu| = 1$.

Proof. (i) Since

$$\beta_n = R_n - \varrho = \alpha_n \frac{\varrho\alpha_{n+1}/\alpha_n - 1 + \alpha_{n+1}}{\varrho - 1 + \alpha_n} ,$$

(i) is obvious.

(ii) Assume that $|\nu| \neq 1$. Since $\alpha_{n+1}/\alpha_n = 1 + \nu_n$, we have $\Delta\alpha_{n+1}/\Delta\alpha_n = \nu_{n+1}/\nu_n + \nu_{n+1}$, hence

$$\lim_{n\to\infty} \Delta\alpha_{n+1}/\Delta\alpha_n = \lim_{n\to\infty} \nu_{n+1}/\nu_n = \nu$$

and from Theorem II.1 it follows that $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n = \nu$ because $(\alpha_n)$ converges to 0 and $|\nu| \neq 1$, which yields a contradiction.

**2.** *Logarithmic convergence*, $(S_n) \in LOG$. Let $(S_n) \in LOG$. Then $\lim_{n\to\infty}\varrho_n = 1$. We set $\varrho_n = 1+\alpha_n$, with $\lim_{n\to\infty}\alpha_n = 0$, hence

$$R_n = (1+\alpha_n)\frac{\alpha_{n+1}}{\alpha_n}.$$

Since $e_{n+1}/e_n = 1+\alpha_n$ we have $\Delta e_{n+1}/\Delta e_n = (1+\alpha_n)\alpha_{n+1}/\alpha_n$, by Theorem II.1 and similarly to assertion (ii) of Theorem II.2 we show that if $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n = \alpha$ exists then $|\alpha| = 1$, or equivalently, if $\lim_{n\to\infty}R_n = R$ exists then $|R| = 1$. So two cases can occur: either $R = \alpha = 1$, or $R = \alpha = -1$. Hence we find two subsets of $LOG$: $LOGSF$ and $LOGL$, which have been introduced by Smith and Ford [12] and Lembarki [9] respectively:

$$LOGSF = \{(S_n) \in LOG : \lim_{n\to\infty} R_n = 1\},$$
$$LOGL = \{(S_n) \in LOG : \lim_{n\to\infty} R_n = -1\}.$$

Notice that if $(S_n) \in LOGSF$, then the limits of $(\varrho_n)$ and $(R_n)$ are equal, but if $(S_n) \in LOGL$, then they have opposite values.

Let us now give examples of sequences in $LOGSF$ and $LOGL$.

EXAMPLE 5 [8]. Let $\mathcal{A}_F^p(S)$ be the set of sequences $(S_n)$ generated by $S_{n+1} = F(S_n)$ where $F : \mathbb{R} \to \mathbb{R}$, $F$ is analytic in a neighbourhood of $S$ which is the only fixed point of $F$, $F'(S) = 1$, $F^{(i)}(S) = 0$ for $2 \le i < p$ and $F^{(p)}(S) = c \ne 0$, where $c < 0$ for $p$ odd. In [7] it is proved that $\mathcal{A}_F^p(S) \subset LOGSF$.

For $LOGL$ we consider the sequence defined in Example 1.

In the preceding cases we have assumed that $\lim_{n\to\infty}R_n$ exists, but it is possible that the sequence $(R_n)$ does not converge. Thus we define the set

$$LOGAP = \{(S_n) \in LOG : (R_n) \text{ does not converge}\}.$$

In this paper we are interested in the cases where $(R_n)$ has $k$ limit points, so we set

$$LOGAP(k) = \{(S_n) \in LOGAP : \lim_{n\to\infty} R_{kn+i} = R^{(i)}\ (i = 0,\ldots,k-1),\ R^{(k)} = R^{(0)}\}.$$

The sequence $(S_n)$ defined in Example 2 belongs to $LOGAP(2)$.

**B.** $(\varrho_n)$ *does not converge*. In this case we suppose that $(\varrho_n)$ has limit points and we define

$$LAP(m) = \{(S_n) : (S_n) \text{ converges and } \lim_{n\to\infty} \varrho_{mn+i} = \varrho^{(i)}\ (i = 0,1,\ldots,m-1),\ \varrho^{(m)} = \varrho^{(0)}\}.$$

Note that if $(S_n) \in LAP(m)$, then $\varrho^{(i)}$ may be zero, one or infinity, and $\varrho^{(i)} \neq \varrho^{(j)}$ for $i \neq j$.

The link between $\varrho_n$ and $R_n$ is

$$R_n = \varrho_n(\varrho_{n+1} - 1)/(\varrho_n - 1)\,, \tag{2.1}$$

hence if $(S_n) \in LAP(m)$, then $(R_n)$ has limit points $R^{(i)}$ which satisfy

$$\begin{aligned} &\lim_{n\to\infty} R_{mn+i} = R^{(i)} = \varrho^{(i)}(\varrho^{(i+1)} - 1)/(\varrho^{(i)} - 1)\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots, m-1\,, \\ &R^{(m)} = R^{(0)}\,. \end{aligned} \tag{2.2}$$

Remark II.1. 1) If $(S_n) \in LAP(m)$ for $m \geq 2$, then $\varrho^{(i)} \neq \varrho^{(j)}$ $(i \neq j)$ but not necessarily $R^{(i)} \neq R^{(j)}$, so the number of limit points of $(R_n)$ does not exceed $m$.

2) If a subsequence of $(\varrho_n)$ has limit one, then $(R_n)$ has two subsequences having limits zero and infinity respectively.

Let us now define the subsets

$$LAP(m,k) = \{(S_n) \in LAP(m) : (R_n) \text{ has } k \text{ limit points}\}\,.$$

THEOREM II.3.

1) *$LAP(m,1) = \emptyset$ for $m \geq 3$.*

2) *$LAP(m,k) \neq \emptyset$ for $2 \leq k \leq m$.*

Proof. 1) Suppose $(S_n) \in LAP(m,1)$ with $m \geq 3$. Then $(\varrho_n)$ has $m$ limit points and $(R_n)$ converges. From (2.2) we can show that if there exists $i \in \{0, 1, \ldots, m-1\}$ such that $\varrho^{(i)} = 0$, or 1, or infinity, then $(R_n)$ does not converge, therefore these cases will not be considered. Moreover, the limit $R$ of $(R_n)$ satisfies $|R| = 1$, because if $|R| \neq 1$ then, by Theorem II.1, $(\varrho_n)$ converges.

So the following two cases can occur:

(i) $R = -1$. It follows from (2.1) that if we set $\lim_{n\to\infty} R_{mn+i} = -1$ and $\lim_{n\to\infty} R_{mn+i+1} = -1$ for $i \in \{0, 1, \ldots, m-2\}$ we obtain

$$\varrho^{(i)}\varrho^{(i+1)} = 1\,, \qquad \varrho^{(i+1)}\varrho^{(i+2)} = 1\,.$$

Then $\varrho^{(i)} = \varrho^{(i+2)}$, which gives a contradiction, because $(\varrho_n)$ has $m$ limit points and $m \geq 3$.

(ii) $R = 1$. In this case we have $\lim_{n\to\infty} R_{mn+i} = 1$ for $i = 0, 1, \ldots, m-1$;

hence

$$(2.3)\qquad \begin{cases} \varrho^{(0)}\varrho^{(1)} = 2\varrho^{(0)} - 1\,, \\ \varrho^{(1)}\varrho^{(2)} = 2\varrho^{(1)} - 1\,, \\ \vdots \\ \varrho^{(m-2)}\varrho^{(m-1)} = 2\varrho^{(m-2)} - 1\,, \\ \varrho^{(m-1)}\varrho^{(0)} = 2\varrho^{(m-1)} - 1\,. \end{cases}$$

We prove that the last equation of (2.3) is incompatible with the others. From the first $m-1$ equations it follows that

$$(2.4)\qquad \varrho^{(m-1)} = (m\varrho^{(0)} - m + 1)/((m-1)\varrho^{(0)} - m + 2)\,.$$

Substituting (2.4) into the last equation of (2.3), we obtain $\varrho^{(0)} = 1$ and the first equation gives $\varrho^{(1)} = 1$, which yields a contradiction.

2) For $2 \le k \le m$ we consider the sequence $(S_n)$ defined by

$$S_{mn+i} = a_i\lambda^n + b_i\nu^n\,, \quad i = 0, 1, \ldots, m-1\,, \quad S_{m(n+1)} = a_0\lambda^{n+1} + b_0\nu^{n+1}\,,$$

where $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m-1$. The sequence $(S_n)$ is a solution of the linear recurrence

$$S_{n+2m} = AS_{n+m} + BS_n\,,$$

where $A = \lambda + \nu$ and $B = -\lambda\nu$. We suppose that $|\nu| < |\lambda| < 1$. So $\lim_{n\to\infty} S_n = 0$,

$$\varrho^{(i)} = a_{i+1}/a_i\,, \quad i = 0, 1, \ldots, m-2\,, \quad \varrho^{(m-1)} = \lambda a_0/a_{m-1}$$

and

$$\begin{cases} R^{(i)} = (a_{i+2} - a_{i+1})/(a_{i+1} - a_i)\,, \quad i = 0, 1, \ldots, m-3\,, \\ R^{(m-2)} = (a_0\lambda - a_{m-1})/(a_{m-1} - a_{m-2})\,, \\ R^{(m-1)} = \lambda(a_1 - a_0)/(a_0\lambda - a_{m-1})\,. \end{cases}$$

We can choose $a_i$ $(i = 0, 1, \ldots, m-1)$ such that

$$\begin{cases} \varrho^{(i)} \ne \varrho^{(j)} \quad (i \ne j)\,, \\ R^{(0)} = R^{(1)} = \ldots = R^{(m-k)} = x\,, \\ R^{(i)} \ne x\,,\ R^{(i)} \ne R^{(j)}\ (i \ne j) \text{ for } i, j \in \{m-k+1, \ldots, m-1\}\,. \end{cases}$$

Thus $(S_n) \in LAP(m,k)$.

EXAMPLE. (i) If $a_i = ib + c$, $i = 0, 1, \ldots, m-1$, where $b = 1$, $c = 1/2 - m$ and $\lambda = 1/(1-2m)$, then $(S_n) \in LAP(m, 2)$.

(ii) If $a_i = ib + c$, $i = 0, 1, \ldots, m-1$, where $b = c = 1$, then $(S_n) \in LAP(m, 3)$.

(iii) For $4 \le k \le m$, if

$$\begin{cases} a_i = ib + c\,, & 0 \le i \le m-k+2, \\ a_i = i^2 b'\,, & m-k+3 \le i \le m-1, \end{cases}$$

where $b = b' = 1$, $c = -1/2$ and $\lambda > 0$, then $(S_n) \in LAP(m,k)$.

Remark II.2. If $m \ge 3$, then for all sequences $(S_n) \in LAP(m)$, the sequence $(R_n)$ does not converge.

In the particular case $m = 2$, we distinguish between two subcases $k = 1$ and $k = 2$.

THEOREM II.4.
1) $LAP(2,1) = \{(S_n) \in LAP(2) : \lim_{n\to\infty} R_n = -1\}$.
2) $(S_n) \in LAP(2,1)$ *if and only if* $\varrho^{(0)}\varrho^{(1)} = 1$.

Proof. 1) Let $(S_n) \in LAP(2,1)$. Then $(\varrho_n)$ has two limit points $\varrho^{(0)}$, $\varrho^{(1)}$ and $(R_n)$ converges to $R$. By (2.1) we obtain

$$\varrho^{(0)}(\varrho^{(1)}-1)/(\varrho^{(0)}-1) = \varrho^{(1)}(\varrho^{(0)}-1)/(\varrho^{(1)}-1) = R\,,$$

hence $\varrho^{(0)}\varrho^{(1)} = 1$ by the first equality and $R = -1$ by the second one.
2) follows from 1).

Remark II.3. 1) If $(S_n) \in LAP(2)$ then $(S_n) \in LAP(2,1)$ if $\varrho^{(0)}\varrho^{(1)} = 1$, and $(S_n) \in LAP(2,2)$ if $\varrho^{(0)}\varrho^{(1)} \ne 1$.

2) If $(S_n) \in LAP(2,1)$ and if the value of $\varrho^{(0)}$ is known, then $\varrho^{(1)} = 1/\varrho^{(0)}$.

3) From Theorems II.3 and II.4, we remark that if $(S_n) \in LAP(m)$ $(m \ge 2)$ and if $(R_n)$ converges then its limit is $-1$ and $m = 2$.

4) If $(S_n) \in LAP(2,1)$ then $\varrho^{(0)} \ne 1$ and $\varrho^{(1)} \ne 1$.

*Characterization of some series in* $LAP(2,1)$. We consider four polynomials $P_1, P_2, Q_1, Q_2$ defined by

$$P_1(X) = a_1 X^{p_1} + a_2 X^{p_1-1} + \ldots + a_{p_1+1}\,,$$
$$P_2(x) = c_1 X^{p_2} + c_2 X^{p_2-1} + \ldots + c_{p_2+1}\,,$$
$$Q_1(X) = b_1 X^{q_1} + b_2 X^{q_1-1} + \ldots + b_{q_1+1}\,,$$
$$Q_2(x) = d_1 X^{q_2} + b_2 X^{q_2-1} + \ldots + d_{q_2+1}\,,$$

where $a_1 b_1 c_1 d_1 \ne 0$. We have

THEOREM II.5. *Let* $S = \sum_{n=0}^{\infty} c_n$ *be a series, where* $c_{2n} = P_1(n)/Q_1(n)$, $c_{2n+1} = -P_2(n)/Q_2(n)$. *If* $p_1 < q_1$, $p_2 < q_2$ *and*

$$p_1 - q_1 = p_2 - q_2\,, \quad a_1 d_1 = b_1 c_1\,,$$

*then*

1) $\lim_{n\to\infty} c_{n+1}/c_n = -1\,,$

2) $$\lim_{n\to\infty} \varrho_{2n-1} = \varrho^{(1)} = \frac{a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 + (p_1 - q_1)a_1d_1}{a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1},$$

$$\lim_{n\to\infty} \varrho_{2n} = \varrho^{(0)} = \frac{1}{\varrho^{(1)}}.$$

Proof. 1) follows by a simple application of the assumption $p_1 - q_1 = p_2 - q_2$ and $a_1d_1 = b_1c_1$.

2) Let $A_n = c_{2n} + c_{2n+1}$. Then $A_n = (P_1(n)Q_2(n) - P_2(n)Q_1(n))/Q_1(n)Q_2(n)$ and as $a_1d_1 - b_1c_1 = 0$, we have $\deg(P_1Q_2 - P_2Q_1) \le p_1 + q_2 - 1$ and hence $\deg(P_1Q_2 - P_2Q_1) - \deg(Q_1Q_2) \le p_1 - q_1 - 1 \le -2$.

It follows that the series $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} A_n$ converges and $A_n = \gamma_0 n^{p_1-q_1-1} + O(n^{p_1-q_1-1})$, where

$$\gamma_0 = \frac{a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1}{p_1 - q_1}.$$

Since $S - S_{2n-1} = \sum_{i=n}^{\infty} A_i$, if we apply the corollary of [14, p. 19] to the series $\sum_{n=0}^{\infty} A_n$, then we obtain

$$S_{2n-1} - S = -\gamma_0 n^{p_1-q_1} + O(n^{p_1-q_1}),$$

and similarly we show that

$$S_{2n} - S = -\gamma_0' n^{p_1-q_1} + O(n^{p_1-q_1}),$$

where

$$\gamma_0' = \frac{a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 + (p_1 - q_1)a_1d_1}{p_1 - q_1}.$$

Thus, $\varrho_{2n-1} = (S_{2n} - S)/(S_{2n-1} - S)$ converges to $\gamma_0'/\gamma_0 = \varrho^{(1)}$, and from 1) and Theorem II.4 we deduce that

$$\lim_{n\to\infty} \varrho_{2n} = \varrho^{(0)} = \frac{1}{\varrho^{(1)}},$$

and so $(S_n) \in LAP(2,1)$.

EXAMPLE 6 [7, p. 217]. We consider the series

$$\pi = x \tan\frac{\pi}{x}\left[1 - \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{2x-1} + \frac{1}{2x+1} - \dots\right],$$
$$x \neq 0,\ \pm 1,\ \pm 1/2,\ \pm 1/3,\ \dots$$

So $\pi = x\tan(\pi/x)\sum_{n=0}^{\infty} c_n$, where $c_{2n} = 1/(nx+1)$, $c_{2n+1} = -1/((n+1)x - 1)$. Applying Theorem II.5 to this series, we obtain

$$R = -1, \quad \varrho^{(0)} = \frac{2-x}{x}, \quad \varrho^{(1)} = \frac{x}{2-x},$$

and thus $(S_n) \in LAP(2,1)$.

Finally, we consider the case $m = k = 2$. Let $(S_n) \in LAP(2,2)$. We have

$$R^{(0)} = \varrho^{(0)}(\varrho^{(1)} - 1)/(\varrho^{(0)} - 1), \quad R^{(1)} = \varrho^{(1)}(\varrho^{(0)} - 1)/(\varrho^{(1)} - 1).$$

Remark II.4. 1) Note that $\varrho^{(0)}\varrho^{(1)} \neq 1$, because if $\varrho^{(0)}\varrho^{(1)} = 1$ then Theorem II.4 implies that $(S_n) \in LAP(2,1)$.

2) In all cases we have $\varrho^{(0)}\varrho^{(1)} = R^{(0)}R^{(1)} \neq 1$.

EXAMPLE 7. 1) Consider the series defined in Example 3 of Section I.

2) Consider the series defined by

$$\frac{2}{3}\ln 2 = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} + \ldots = \sum_{n=0}^{\infty} c_n,$$

where $c_0 = 1$, $c_{2n} = (-1)^n/(3n+1)$ and $c_{2n+1} = (-1)^{n+1}/(3n+2)$. Then

$$R^{(0)} = +1, \quad R^{(1)} = -1, \quad \varrho^{(0)} = 0, \quad \varrho^{(1)} = -\infty,$$

and $\lim_{n\to\infty} \varrho_n \varrho_{n+1} = R^{(0)}R^{(1)} = -1$.

We summarize the results obtained for the special case $m = 2$ in the table below.

| | $(R_n)$ converges to $R$ | $\lim_{n\to\infty} R_{2n} = R^{(0)}$; $\lim_{n\to\infty} R_{2n+1} = R^{(1)}$ |
|---|---|---|
| $(\varrho_n)$ converges to $\varrho$ | $LIN$ and $LOGSF : R = \varrho$<br>$LOGL : \varrho = -R = 1$ | $LOGAP\ (2) : \varrho = 1$ |
| $\lim_{n\to\infty} \varrho_{2n} = \varrho^{(0)}$<br>$\lim_{n\to\infty} \varrho_{2n+1} = \varrho^{(1)}$ | $LAP(2,1) : R = -1,$<br>iff $\varrho^{(0)}\varrho^{(1)} = 1$ | $LAP(2,2) : \varrho^{(0)}\varrho^{(1)} = R^{(0)}R^{(1)} \neq 1$ |

**III. Acceleration processes.** Let $(S_n) \in LAP(m)$. From (2.1), we have

$$\prod_{i=0}^{m-1} R^{(i)} = \prod_{i=0}^{m-1} \varrho^{(i)}. \tag{3.1}$$

Note that if $|\prod_{i=0}^{m-1} \varrho^{(i)}| \leq 1$ and $\prod_{i=0}^{m-1} \varrho^{(i)} \neq 1$ then the subsequences $(S_n, S_{n+m}, S_{n+2m}\ldots)$ converge linearly and can be accelerated by Aitken's $\Delta^2$ process. Thus we obtain

$$F_m^{(n)} = \frac{S_{n+2m}S_n - S_{n+m}^2}{S_{n+2m} + S_n - 2S_{n+m}}$$

and we recover the $\Delta_m^2$ process given in [5] by Delahaye. It can be written

(3.2)

$$\begin{cases} F_m^{(n)} = \dfrac{S_{n+m} - f_m^{(n)} S_n}{1 - f_m^{(n)}}, \quad \text{where} \\ f_m^{(n)} = \dfrac{S_{n+2m} - S_{n+m}}{S_{n+m} - S_n} = \dfrac{\Delta S_{n+m}}{\Delta S_n} \cdot \dfrac{1 + \dfrac{\Delta S_{n+m+1}}{\Delta S_{n+m}} + \ldots + \dfrac{\Delta S_{n+2m-1}}{\Delta S_{n+m}}}{1 + \dfrac{\Delta S_{n+1}}{\Delta S_n} + \ldots + \dfrac{\Delta S_{n+m-1}}{\Delta S_n}}. \end{cases}$$

From (3.1), we have the asymptotic approximation $e_{n+m}/e_n \sim \Delta S_{n+m}/\Delta S_n$ as $n \to \infty$, which gives

$$S \sim \frac{S_{n+m} - (\Delta S_{n+m}/\Delta S_n) S_n}{1 - \Delta S_{n+m}/\Delta S_n} = T_{+m}^{(n)}.$$

So that we recover the $T_{+m}$ transformation introduced and studied in [6] and [13]. It is a rank-two composite transformation of $(S_n)$ and $(S_{n+m})$, as defined by Brezinski [3].

Note that $T_{+m}^{(n)}$ can be written as

$$(3.3) \qquad T_{+m}^{(n)} = \frac{S_{n+m} - t_m^{(n)} S_n}{1 - t_m^{(n)}}, \quad \text{where} \quad t_m^{(n)} = R_n R_{n+1} \ldots R_{n+m-1}.$$

We recall that $R_n$ is the acceleration factor of Aitken's process. If we take $m = 1$, then we obtain $f_1^{(n)} = t_1^{(n)} = R_n$, so that the processes $T_{+1}$ and $F_1$ are identical with Aitken's process.

Let us now generalize the $\theta_2$-algorithm of Brezinski [2].

Let $(S_n)$ be a sequence converging to $S$. The $\theta_2$-algorithm applied to $(S_n)$ is

$$\theta_2(n) = \frac{S_{n+2} - g_n S_{n+1}}{1 - g_n}, \quad \text{where} \quad g_n = R_{n+1} \frac{1 - R_n}{1 - R_{n+1}}.$$

$\theta_2$ is a composite transformation of $(S_{n+1})$ and $(\Delta_2^{(n)})$ [3]. Similarly to the transformation $T_{+m}$, the $\theta_2$-algorithm can be generalized as follows: we have

$$(\theta_2^{(n)} - S)/(S_{n+1} - S) = (\varrho_{n+1} - g_n)/(1 - g_n),$$

so that if $(S_n) \in LAP(m)$, then $\lim_{n\to\infty} g_{nm+i} \neq \lim_{n\to\infty} \varrho_{nm+i+1}$ and the $\theta_2$-algorithm does not accelerate the convergence of $(S_n)$.

We consider $g_m^{(n)} = g_n g_{n+1} \cdots g_{n+m-1}$ and

$$\theta_{2,m}^{(n)} = \frac{S_{n+m+1} - g_m^{(n)} S_{n+1}}{1 - g_m^{(n)}}.$$

Note that for $m = 1$ we have $\theta_{2,1}^{(n)} = \theta_2^{(n)}$.

*Convergence acceleration in the case* $m = 2$. We now give some results on convergence acceleration for the processes $T_{+2}$, $F_2$ and $\theta_{2,2}$ in the following cases: a) $(S_n) \in LAP(2,1)$ and b) $(S_n) \in LAP(2,2)$.

Let $(S_n) \in LAP(2)$. Then $(\varrho_n)$ has two limit points $\varrho^{(0)}$, $\varrho^{(1)}$; we set

$$\varrho_{2n} = \varrho^{(0)} + \alpha_{2n}\,, \qquad \varrho_{2n+1} = \varrho^{(1)} + \alpha_{2n+1}\,,$$

where $(\alpha_n)$ is a sequence converging to 0.

In the case $(S_n) \in LAP(2,1)$, Theorem II.4 gives $\lim_{n\to\infty} R_n = -1$. We set $R_n = -1 + \beta_n$.

THEOREM III.1. *Let* $(S_n) \in LAP(2,1)$. *If*

$$\lim_{n\to\infty} \alpha_{2n+1}/\alpha_{2n} = \alpha\,, \qquad \lim_{n\to\infty} \alpha_{2n+2}/\alpha_{2n+1} = \alpha'\,,$$

*with* $|\varrho^{(0)}\alpha| \neq |\varrho^{(1)}|$, *then*

1) $|\alpha\alpha'| = 1$,

2) $\lim_{n\to\infty} \beta_{2n+1}/\beta_{2n} = \beta$ *and* $\lim_{n\to\infty} \beta_{2n+2}/\beta_{2n+1} = \beta'$ *both exist*, $\beta$ *and* $\beta'$ *satisfy*

$$\beta = -\alpha\varrho^{(0)} \frac{\varrho^{(0)} + \varrho^{(1)}\alpha'}{\varrho^{(1)} + \varrho^{(0)}\alpha}\,, \quad \beta' = -\alpha'\varrho^{(1)} \frac{\varrho^{(1)} + \varrho^{(0)}\alpha}{\varrho^{(0)} + \varrho^{(1)}\alpha'}\,, \quad \beta\beta' = \alpha\alpha'\,,$$

3) $$\lim_{n\to\infty} \frac{\beta_{2n}}{\alpha_{2n}} = \frac{\varrho^{(0)}\alpha + \varrho^{(1)}}{\varrho^{(0)} - 1}\,, \ \lim_{n\to\infty} \frac{\beta_{2n+1}}{\alpha_{2n+1}} = \frac{\varrho^{(1)}\alpha' + \varrho^{(0)}}{\varrho^{(1)} - 1}.$$

Proof. 1) Suppose that $|\alpha\alpha'| \neq 1$. Remark that the subsequence $u_n = S_{2n}$ converges logarithmically, i.e. $\lim_{n\to\infty}(u_{n+1} - S)/(u_n - S) = 1$ because $\varrho^{(0)}\varrho^{(1)} = 1$. We have

$$\begin{aligned} \Delta u_{n+1}/\Delta u_n &= (S_{2n+4} - S_{2n+2})/(S_{2n+2} - S_{2n}) \\ &= \varrho_{2n}\varrho_{2n+1}(\varrho_{2n+2}\varrho_{2n+3} - 1)/(\varrho_{2n}\varrho_{2n+1} - 1)\,, \end{aligned}$$

and we can prove that

$$\lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \alpha\alpha' \quad \text{with } |\alpha\alpha'| \neq 1.$$

By Theorem II.1, we obtain $\lim_{n\to\infty}(u_{n+1} - S)/(u_n - S) = \alpha\alpha' \neq 1$, which yields a contradiction.

2), 3) We have $\beta_{2n} = R_{2n} + 1$, $\beta_{2n+1} = R_{2n+1} + 1$. So

$$\beta_{2n} = \frac{\varrho^{(1)}\alpha_{2n} + \varrho^{(0)}\alpha_{2n+1} + \alpha_{2n}\alpha_{2n+1}}{\varrho^{(0)} - 1 + \alpha_{2n}}\,,$$

$$\beta_{2n+1} = \frac{\varrho^{(0)}\alpha_{2n+1} + \varrho^{(1)}\alpha_{2n+2} + \alpha_{2n+1}\alpha_{2n+2}}{\varrho^{(1)} - 1 + \alpha_{2n+1}}\,.$$

Hence assertions 2) and 3) follow.

Remark III.1. From assertion 1) of Theorem III.1 two cases can occur: either $\alpha\alpha' = 1$ or $\alpha\alpha' = -1$. If $\alpha\alpha' = 1$ then $\beta = -\varrho^{(0)}$ and $\beta' = -\varrho^{(1)}$.

A simple application of Theorem III.1 in the case $\alpha\alpha' = 1$ gives

THEOREM III.2. *Let* $(S_n) \in LAP(2,1)$. *If* $\lim_{n\to\infty}\alpha_{2n+1}/\alpha_{2n} = \alpha$, $\lim_{n\to\infty}\alpha_{2n+2}/\alpha_{2n+1} = \alpha'$ *with* $|\varrho^{(0)}\alpha| \neq |\varrho^{(1)}|$ *and* $\alpha\alpha' = 1$, *then*

1) *the processes* $T_{+2}$ *and* $\theta_{2,2}$ *accelerate the convergence of* $(S_n)$,

2) *moreover, if* $\lim_{n\to\infty}(1/\beta_{n+2} - 1/\beta_n) = 0$, *then* $F_2$ *accelerates the convergence of* $(S_n)$.

Note that in the case $\alpha\alpha' = -1$ none of the three processes accelerate the convergence of $(S_n)$.

Let now $(S_n) \in LAP(2,2)$. Then $(\varrho_n)$ and $(R_n)$ have two limit points $\varrho^{(0)}$, $\varrho^{(1)}$ and $R^{(0)}$, $R^{(1)}$, respectively. We set

$$R_{2n} = R^{(0)} + \beta_{2n}, \qquad R_{2n+1} = R^{(1)} + \beta_{2n+1}.$$

By Remark II.4 if $R^{(0)} \neq -1$, $R^{(1)} \neq -1$, then from (3.1)–(3.3) we deduce that the acceleration factors satisfy

$$\lim_{n\to\infty} t_2^{(n)} = \lim_{n\to\infty} f_2^{(n)} = R^{(0)}R^{(1)} = \varrho^{(0)}\varrho^{(1)} \neq 1,$$

and if $R^{(0)} \neq 1$, $R^{(1)} \neq 1$, then

$$\lim_{n\to\infty} g_2^{(n)} = R^{(0)}R^{(1)} = \varrho^{(0)}\varrho^{(1)} \neq 1.$$

In the cases $R^{(0)} = 1$ or $R^{(1)} = 1$ we give sufficient conditions for $\lim_{n\to\infty} g_2^{(n)} = \varrho^{(0)}\varrho^{(1)}$.

THEOREM III.3. *If* $(S_n) \in LAP(2,2)$, *then*

1) *the processes* $T_{+2}$ *and* $F_2$ *accelerate the convergence of* $(S_n)$ *if* $R^{(0)} \neq -1$, $R^{(1)} \neq -1$,

2) *the process* $\theta_{2,2}$ *accelerates the convergence of the sequences* $(S_n)$ *that satisfy one of the following assumptions*:

(i) $R^{(0)} \neq 1$ *and* $R^{(1)} \neq 1$,
(ii) $R^{(0)} = 1$, $R^{(1)} = 0$ *and* $\exists M$ *such that* $|\beta_{2n}/\beta_{2n+2}| < M$, $\forall n$,
(iii) $R^{(0)} = 1$ *and* $(\beta_{2n}/\beta_{2n+2})$ *converges to* 1.

Note that in assertion 2), if $R^{(1)} = 1$ then in (ii) and (iii), $\beta_{2n}/\beta_{2n+2}$ can be replaced by $\beta_{2n+1}/\beta_{2n+3}$ and $R^{(1)}$ by $R^{(0)}$.

The three processes accelerate the convergence of other sequences than those which belong to $LAP(2,1)$ and $LAP(2,2)$.

Let $(S_n) \in LOGSF$. Then $\lim_{n\to\infty}\varrho_n = 1$, $\lim_{n\to\infty} R_n = 1$. Setting $\varrho_n = 1 + \alpha_n$ and $R_n = 1 + \beta_n$, we have

THEOREM III.4. *If* $(S_n) \in LOGSF$ *and if* $\lim_{n\to\infty}\beta_n/\alpha_n = K \neq 0$, *then*

1) $T_{+2}$ *and* $F_2$ *accelerate the convergence of the sequences for which* $K = 1$,

2) *moreover, if* $\lim_{n\to\infty}(1/\beta_{n+1} - 1/\beta_n) = b$ *then* $\theta_{2,2}$ *accelerates the convergence of the sequences for which* $b = 1/k - 1$.

Let us give some sequences satisfying the conditions of assertion 2).

Let $(S_n) \in \mathcal{A}_F^p(S)$ (defined in Example 4 of Section II). Then

$$S_{n+1} = F(S_n) = S + (S_n - S) + \frac{(S_n - S)^p}{p!} F^{(p)}(S + \theta(S_n - S)),$$

where $\theta \in ]0, 1[$. So

$$\varrho_n = 1 + \frac{(S_n - S)^{p-1}}{p!} F^{(p)}(S + \theta(S_n - S)) = 1 + \alpha_n,$$
$$R_n = 1 + \frac{(S_n - S)^{p-1}}{(p-1)!} F^{(p)}(S + \theta(S_n - S)) = 1 + \beta_n,$$

hence $(\beta_n/\alpha_n)$ converges to $p$ and $(1/\beta_{n+1} - 1/\beta_n)$ converges to $1/p-1 = b$; therefore $b = 1/k - 1$, and thus $\theta_{2,2}$ accelerates the convergence of $(S_n) \in \mathcal{A}_F^p(S)$.

Let now $(S_n) \in LOGAP(2)$. Then $(R_n)$ has two limit points $R^{(0)}$, $R^{(1)}$. If we apply the processes $T_{+2}$, $F_2$ and $\theta_{2,2}$ to a sequence $(S_n)$ for which $R^{(0)}R^{(1)} = 1$ and $\lim_{n\to\infty}\beta_n/\alpha_n = K \neq 0$, we obtain three sequences $(T_{+2}^{(n)})$, $(F_2^{(n)})$ and $(\theta_{2,2}^{(n)})$ which belong to $LAP(2,1)$. So in order to accelerate the convergence of $(S_n)$, we once more apply one of the three processes to one of the three sequences $(T_{+2}^{(n)})$, $(F_2^{(n)})$ and $(\theta_{2,2}^{(n)})$.

Note that the three processes also accelerate linear convergence (in the case $\varrho \neq -1$) and the convergence of some sequences in $LOGL$.

**IV. Numerical results.** In the following figures the number of exact digits is represented as a function of the number of terms used. We use the notations:

$$T(n) \equiv T_{+2}(n), \quad F(n) \equiv F_2(n), \quad G(n) \equiv \theta_{2,2}(n).$$

In the case $(S_n) \in LAP(2,1)$, we consider two examples:

(i) Let $S$ be the series defined in Example 5 of Section II,

$$S = \frac{\pi}{y\tan(\pi/y)} = \left[1 - \frac{1}{y-1} + \frac{1}{y+1} - \frac{1}{2y-1} + \frac{1}{2y+1} - \dots\right],$$

where $y \neq 0, \pm1, \pm1/2, \dots$ Then $\alpha\alpha' = \beta\beta' = 1$, $|\varrho^{(0)}\alpha| \neq |\varrho^{(1)}|$, and $\lim_{n\to\infty}(1/\beta_{n+2} - 1/\beta_n) \neq 0$. So, by Theorem III.2, $T_{+2}$ and $\theta_{2,2}$ accelerate the convergence of $S$.

For $y = 0.98$ the results are given in Figure 1.

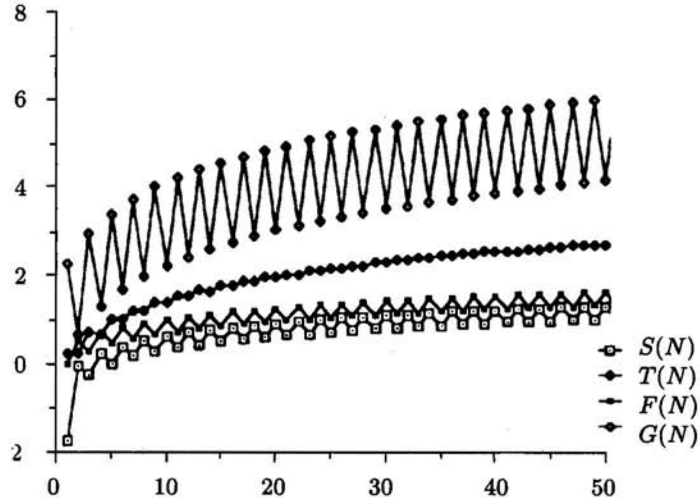


Fig. 1

(ii) Let $(S_n)$ be the sequence defined by $S_1 = 1$,

$$S_{2n} = \frac{2n+1}{n(n+1)}, \quad S_{2n+1} = \frac{1}{n+1}, \quad n = 1, 2, \ldots$$

We have $S = \lim S_n = 0$, $\varrho^{(0)} = 2$, $\varrho^{(1)} = 1/2$ and $R = -1$. $(S_n)$ represents the $n$th convergent of a continued fraction [9, p. 112]. The results are given in Figure 2. In this example, the sequence $G(n)$ is defined by $G_{2n} = 0$, $G_{2n+1} = 4/(n+3)(2n+3)$ for $n = 1, 2, \ldots$

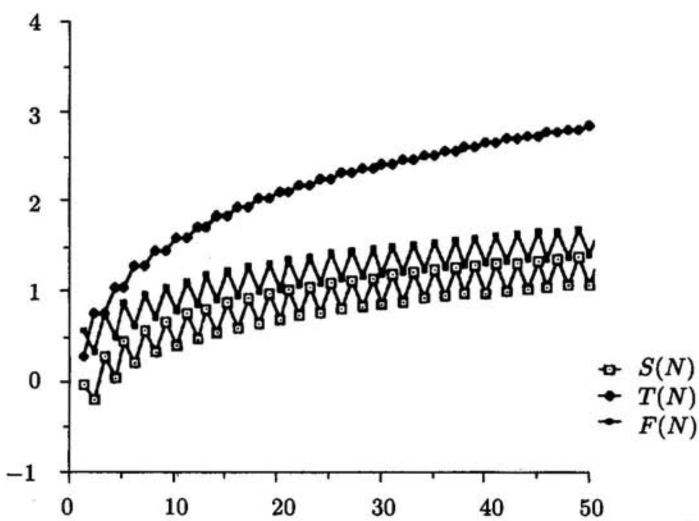


Fig. 2

In the case $(S_n) \in LAP(2,2)$, we consider the series defined in Example 7 of Section II.

(i) For $S = \sum_{i=0}^{\infty} (-1)^{[i/2]}/(i+1)$, we have Figure 3.

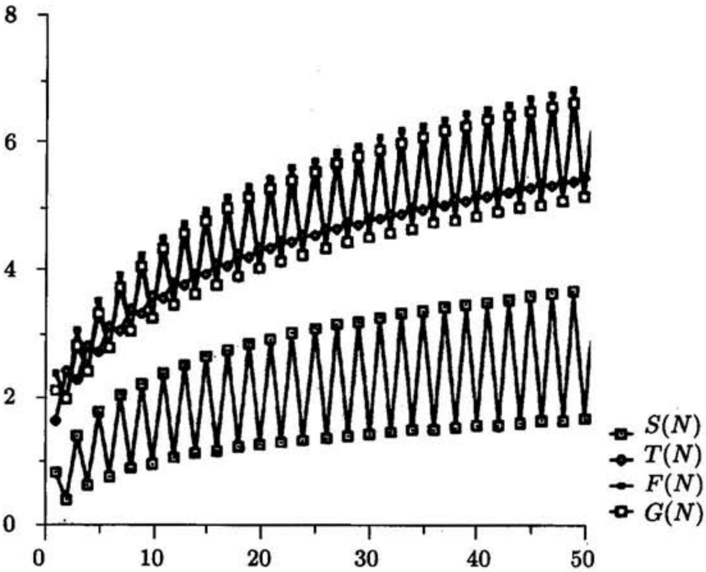


Fig. 3

(ii) For $\frac{2}{3}\ln 2 = 1 - 1/2 - 1/4 + 1/5 + 1/7 - 1/8 - 1/10 + \ldots$, see Figure 4.

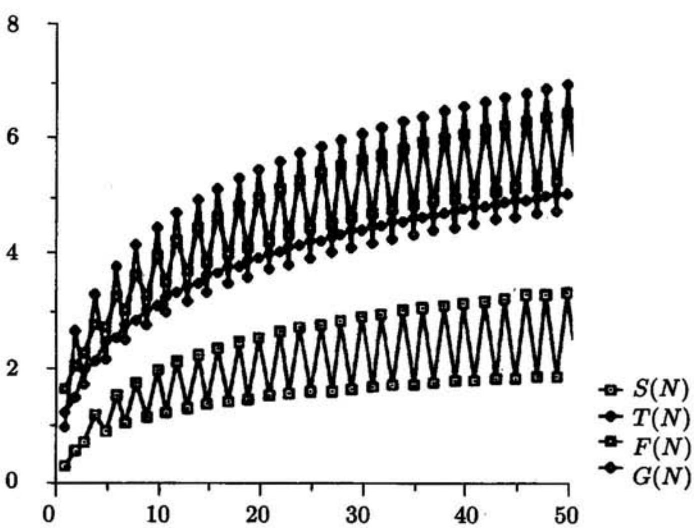


Fig. 4

For these examples one can show that assertion (iii) of Theorem III.3 is satisfied, so $T_{+2}$, $F_2$ and $\theta_{2,2}$ accelerate the convergence.

(iii) Let $(S_n)$ be the sequence defined by

$$S_1 = 1\,, \qquad S_{2n} = \alpha\lambda^n + \beta\nu^n\,, \qquad S_{2n+1} = \gamma\lambda^n + \delta\nu^n\,.$$

$S_n$ is a solution of a linear recurrence.

If $\lambda = 1/2$, $\nu = 1/3$ and $\alpha = 1$, $\beta = 0.2$, $\gamma = 0.3$; $\delta = 1$ then $\lim_{n\to\infty} S_n = S = 0$ and $(S_n) \in LAP(2,2)$. Assertion (i) of Theorem III.3 is satisfied, so $T_{+2}$, $F_2$ and $\theta_{2,2}$ accelerate the convergence of $(S_n)$. The results are given in Figure 5.

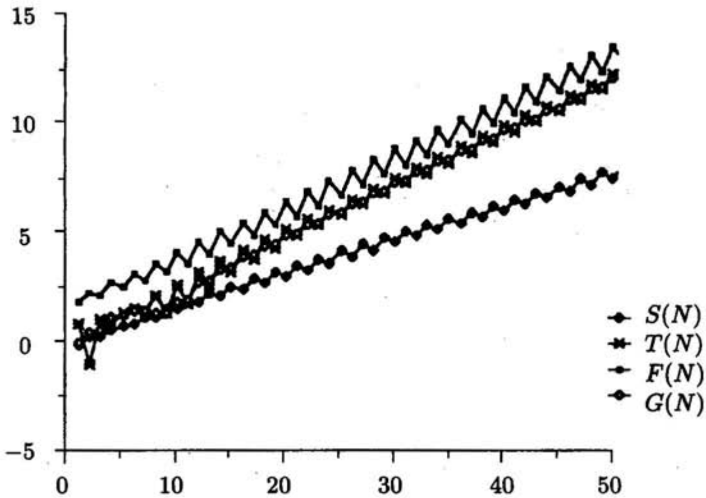


Fig. 5

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M. D. BENCHIBOUN
UNIVERSITÉ DES SCIENCES ET TECHNIQUES DE LILLE FLANDRES-ARTOIS
U.F.R. I.E.E.A.
LABORATOIRE D'ANALYSE NUMÉRIQUE ET D'OPTIMISATION
F-59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

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