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SOLVING INITIAL-BOUNDARY VALUE PROBLEMS FOR COUPLED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Abstract. In this paper an analytic solution of some initial-boundary value problems for coupled systems of second order partial differential equations is given. The method is based on some algebraic matrix transformations and a matrix separation of variables technique. By truncation of the infinite series solution, finite and computable approximate solutions and error bounds for them in terms of the data are given.

1. Introduction. Many physical systems cannot be described by a single partial differential equation but are, in fact, modelled by a system of coupled equations. So, the study of propagation of signals in a system of electrical cables led to the investigation of a system of linear partial differential equations. Some results related to these systems may be found in [3, 5, 6, 11]. Also, systems of linear partial differential equations appear in the study of temperature distribution in a composite heat conductor [4]. Numerical methods for solving such systems of coupled partial differential equations are given in [7, 8, 13]. Methods based on the transformation of the original system into a new system of uncoupled equations may be found in [3, 5, 6, 14].

The aim of this paper is to find an explicit analytic solution of the initial-boundary value problem

$$U_{xx}(x,t) - AU_{tt}(x,t) - BU(x,t) = 0\,, \quad 0 < x < p\,,\ t > 0\,, \tag{1.1}$$

$$U(0,t) = U(p,t) = 0\,, \quad t > 0\,, \tag{1.2}$$

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$$U(x,0) = f(x), \qquad 0 \le x \le p, \tag{1.3}$$

$$U_t(x,0) = 0, \tag{1.4}$$

where the unknown $U = (u_1, \ldots, u_m)^T$ and $f(x)$ take values in $\mathbb{R}^m$ and A, B are matrices in $\mathbb{R}^{m\times m}$ such that

(1.5) A is symmetric negative definite and $A^{-1}B$ is symmetric negative semi-definite

and f admits continuous derivatives up to order four in $[0,p]$ such that

$$f^{(i)}(0) = f^{(i)}(p) = 0, \quad 0 \le i \le 4. \tag{1.6}$$

The explicit solutions are of considerable interest both in explaining the physical phenomena and for checking results obtained by numerical methods. The paper is organized as follows. Section 2 provides a matrix separation of variables method to solve problem (1.1)–(1.4) by means of a series solution. In Section 3 we obtain computable finite approximations of the infinite series solution, and error bounds for the approximate solutions in terms of the data are given.

If C is a matrix in $\mathbb{R}^{m\times m}$, we denote by $\|C\|$ the operator norm, defined as the square root of the maximum eigenvalue of C^TC, where C^T is the transpose matrix of C [12, p. 41]. We denote by I the identity matrix in $\mathbb{R}^{m\times m}$, and the set of all eigenvalues of C is denoted by $\sigma(C)$.

2. A matrix separation of variables method. Suppose we are looking for solutions $U(x,t)$ of problem (1.1)–(1.2) of the form

$$U(x,t) = T(t)X(x) \tag{2.1}$$

where $T(t)$ lies in $\mathbb{R}^{m\times m}$ and $X(x)$ lies in $\mathbb{R}^m$. Take a real number λ such that there exists a non-trivial solution of the boundary value problem

$$X''(x) - \lambda X(x) = 0, \quad X(0) = X(p) = 0, \tag{2.2}$$

as well as a non-trivial solution $T(t)$ of the matrix differential equation

$$T''(t) + A^{-1}(B - \lambda I)T(t) = 0. \tag{2.3}$$

Then $U(x,t)$ defined by (2.1) satisfies

$$\begin{aligned} U_{xx}(x,t) - AU_{tt}(x,t) - BU(x,t) &= T(t)X''(x) - AT''(t)X(x) - BT(t)X(x) \\ &= T(t)\lambda X(x) + (B - \lambda I)T(t)X(x) - BT(t)X(x) = 0. \end{aligned}$$

Furthermore, from (2.2) it follows that

$$U(0,t) = T(t)X(0) = 0, \quad U(p,t) = T(t)X(p) = 0.$$

In consequence, for each value of λ such that there are non-trivial solutions $X(x)$ and $T(t)$ of (2.2) and (2.3), respectively, one gets a solution $U(x,t)$ of the homogeneous problem (1.1)–(1.2), given by (2.1).

An easy computation shows that taking $\lambda = \lambda_k = -(k\pi/p)^2$, for any vector c in $\mathbb{R}^m$, the vector-valued function

$$X_k(x) = \sin(k\pi x I/p)c$$

is a solution of (2.2). Now, consider the matrix differential equation (2.3) corresponding to the value λ_k,

$$T_k''(t) + A^{-1}(B + (k\pi/p)^2)T_k(t) = 0\,. \tag{2.4}$$

Following the ideas developed in [9], if we assume, for instance, that the matrix $C_k = -A^{-1}(B + (k\pi/p)^2 I))$ admits a square root R_k, then a set of solutions of the matrix differential equation (2.4) is given by

$$T_k(t) = \cos(tR_k)E_1 + \sin(tR_k)E_2$$

where E_1, E_2 are arbitrary matrices in $\mathbb{R}^{m\times m}$. Thus, for arbitrary vectors c_k, d_k in $\mathbb{R}^m$, the vector-valued functions

$$U_k(x,t) = \cos(tR_k)\sin(k\pi x I/p)c_k + \sin(tR_k)\sin(k\pi x I/p)d_k \tag{2.5}$$

satisfy equation (1.1) and the boundary value conditions of (1.2).

Now we prove a lemma which ensures the existence of square roots R_k for the matrix

$$C_k = -A^{-1}(B + (k\pi/p)^2 I) \tag{2.6}$$

for any integer $k \geq 1$.

LEMMA 1. *Let A and B be matrices in $\mathbb{R}^{m\times m}$ satisfying the hypothesis (1.5) and let k be a positive integer. If C_k is the matrix defined by (2.6) and $\sigma(C_k) = \{z_{ik}\,;\, 1 \leq i \leq m\}$, then there exist orthogonal matrices M_k in $\mathbb{R}^{m\times m}$ such that*

$$R_k = M_k^{-1}[\operatorname{diag}(z_{ik}^{1/2};\ 1 \leq i \leq m)]M_k \tag{2.7}$$

is a square root of C_k such that

$$\|R_k\| = O(k) \quad \text{as } k \to \infty\,. \tag{2.8}$$

Proof. From (1.5), for any positive integer the matrix $-(k\pi/p)^2 A$ is positive definite and the matrix $-A^{-1}B$ is positive semi-definite. Thus the matrix C_k defined by (2.6) is positive definite. Now, from Theorem 2 of [2, p. 59], there exists an orthogonal matrix M_k in $\mathbb{R}^{m\times m}$ such that

$$M_k C_k M_k^{-1} = [\operatorname{diag}(z_{ik};\ 1 \leq i \leq m)]$$

where $\sigma(C_k) = \{z_{ik};\ 1 \leq i \leq m\}$. Since z_{ik} is positive for each i, k with $k \geq 1$, $1 \leq i \leq m$, there exists a real positive square root of z_{ik} denoted by $z_{ik}^{1/2}$ and an easy computation shows that

$$R_k = M_k^{-1}[\operatorname{diag}(z_{ik}^{1/2};\ 1 \leq i \leq m)]M_k$$

is a symmetric square root of C_k. Moreover, as R_k is real and symmetric, its norm $\|R_k\|$ is the maximum of the numbers $|z_{ik}|^{1/2}$, for $1 \le i \le m$ [12, p. 41]. Also, as C_k is real and symmetric its norm $\|C_k\|$ is the maximum of $|z_{ik}|$, for $1 \le i \le m$. Since $\|M_k\| = \|M_k^{-1}\| = 1$, we have $\|R_k\| = \|C_k\|^{1/2}$ and by (2.6) the result is established.

Note that from (2.5) it follows that

$$(\partial U_k/\partial t)(x,t) = -R_k \sin(tR_k)\sin(k\pi x I/p)c_k + R_k\cos(tR_k)\sin(k\pi x I/p)d_k\,.$$

Assume that there exists a solution $U(x,t)$ of problem (1.1)–(1.4) of the form

$$U(x,t) = \sum_{k\ge1}\{\cos(tR_k)\sin(k\pi x I/p)c_k + \sin(tR_k)\sin(k\pi x I/p)d_k\} \tag{2.9}$$

for appropriate vectors c_k, d_k, to be chosen so that $U(x,t)$ defined by (2.9) satisfies the initial conditions (1.3) and (1.4). If we assume that we can compute the partial derivatives by termwise partial differentiation in the series (2.9), then we have

$$\begin{aligned} U_t(x,t) &= \sum_{k\ge1} R_k\{-\sin(tR_k)\sin(k\pi x I/p)c_k \\ &\qquad + \cos(tR_k)\sin(k\pi x I/p)d_k\}\,, \\ U_{xx}(x,t) &= \sum_{k\ge1}(k\pi/p)^2\{-\cos(tR_k)\sin(k\pi x I/p)c_k \\ &\qquad - \sin(tR_k)\sin(k\pi x I/p)d_k\}\,. \end{aligned} \tag{2.10}$$

Taking $t=0$ in (2.10) and imposing the condition (1.4), it follows that

$$\sum_{k\ge1} R_k \sin(k\pi x I/p)d_k = 0\,.$$

Thus we can take $d_k = 0$ for $k \ge 1$ and (2.9) then takes the form

$$U(x,t) = \sum_{k\ge1}\cos(tR_k)\sin(k\pi x I/p)c_k\,. \tag{2.11}$$

Taking into account the condition (1.3), from (2.11) the coefficients c_k must satisfy

$$U(x,0) = f(x) = \sum_{k\ge1}\sin(k\pi x I/p)c_k\,. \tag{2.12}$$

If $c_k = (c_{k1},\dots,c_{km})^T$, $f = (f_1,\dots,f_m)^T$, the condition (2.12) is equivalent to the conditions

$$f_i(x) = \sum_{k\ge1}\sin(k\pi x)c_{ki}\,, \quad 1 \le i \le m\,,$$

From hypothesis (1.6) and scalar Fourier series theory, it follows that

$$c_{ki} = (2/p) \int_0^p \sin(k\pi x/p) f_i(x)\, dx\,, \quad 1 \le i \le m\,,\; k \ge 1\,. \tag{2.13}$$

Note that (2.13) may be written in vector form as

$$c_k = (2/p) \int_0^p \sin(k\pi x I/p) f(x)\, dx\,, \quad k \ge 1\,. \tag{2.14}$$

Taking into account (2.11) and (2.7) and that

$$\cos(tR_k) = M_k^{-1}[\mathrm{diag}(\cos(tz_{ik})\,;\; 1 \le i \le m)]M_k$$

we have

$$U(x,t) = \sum_{k\ge1} M_k^{-1}[\mathrm{diag}(\cos(tz_{ik})\,;\; 1 \le i \le m)]M_k \sin(k\pi x I/p) c_k \tag{2.15}$$

where c_k is defined by (2.14).

To prove that $U(x,t)$ defined by (2.14), (2.15), is a solution of problem (1.1)–(1.4), we have to justify the convergence of the series which defines $U(x,t)$, as well as that the partial derivatives with respect t and x may be computed by termwise partial differentiation in the series (2.15).

Note that from (1.6) the coefficients in the Fourier sine series expansion of the $f_i(x)$, for $1 \le i \le m$, satisfy

$$|c_{ki}| = O(k^{-4}) \quad \text{as } k \to \infty \tag{2.16}$$

(see [16, p. 71]). On the other hand, as M_k is an orthogonal matrix we have $\|M_k\| = \|M_k^{-1}\| = 1$. As $\|\sin(k\pi x I/p)\| \le 1$, from (2.5) and (2.16), it follows that

$$\|U_k(x,t)\| = O(k^{-4}) \quad \text{as } k \to \infty, \text{ uniformly for } (x,t) \in [0,p]\times]0,\infty[$$

Hence, from the Weierstrass majorant criterion [1], the vector series (2.15), (2.14) defines a continuous function $U(x,t)$ in $[0,p]\times]0,\infty[$. Also, note that the partial derivatives of the general term $U_k(x,t)$ of (2.15) take the form

$$\begin{aligned}
(U_k)_t(x,t) &= -R_k \sin(tR_k)\sin(k\pi x I/p) c_k\,,\\
(U_k)_x(x,t) &= (k\pi x/p)\cos(tR_k)\cos(k\pi x I/p) c_k\,,\\
(U_k)_{tt}(x,t) &= -R_k^2 \cos(tR_k)\sin(k\pi x I/p) c_k\,,\\
(U_k)_{xx}(x,t) &= -(k\pi x/p)^2 \cos(tR_k)\sin(k\pi x I/p) c_k\,.
\end{aligned} \tag{2.17}$$

From (2.8), (2.16) and (2.17), it follows that

$$\begin{aligned}
\|(U_k)_t(x,t)\| &= O(k^{-3})\,, \quad \|(U_k)_x(x,t)\| = O(k^{-3}) \quad \text{as } k \to \infty\,,\\
\|(U_k)_{tt}(x,t)\| &= O(k^{-2})\,, \quad \|(U_k)_{xx}(x,t)\| = O(k^{-2}) \quad \text{as } k \to \infty\,,
\end{aligned} \tag{2.18}$$

uniformly for $(x,t) \in [0,p]\times]0,\infty[$. From (2.18) and Theorem 9.14 of [1], it follows that $U(x,t)$ defined by (2.14), (2.15) is a continuous vector-valued function which admits second order partial derivatives that may be computed by termwise partial differentiation in (2.15). Thus the following result has been proved:

THEOREM 2. *Consider the coupled initial-boundary value system* (1.1)–(1.4), *where the matrices A and B satisfy* (1.5) *and* $f(x)$ *satisfies* (1.6). *Then a solution* $U(x,t)$ *of problem* (1.1)–(1.4), *is given by* (2.14), (2.15) *where* M_k *and* z_{ik} *are defined by Lemma* 1, *for* $1 \le i \le m$, $k \ge 1$.

Remark 1. If we do not impose the condition (1.6), then if one extends $f(x)$ to an odd function on the interval $[-p,p]$ and next periodically to the whole real line, then the extended function is not necessarily continuous and we only can assure that

$$\|c_k\| \le \alpha/k + \beta/k^2 + \varrho/k^3 + \gamma/k^4$$

for some positive constants α, β, ϱ and γ (see [16, p. 71]).

3. Finite approximate solutions and error bounds. The series solution $U(x,t)$ of problem (1.1)–(1.4), provided by Theorem 2, has two numerical drawbacks. First of all, the infinite series is not computable, and secondly, the general term of (2.15) requires the computation of the orthogonal matrices M_k as well as the eigenvalues z_{ik} of the matrix C_k defined by (2.6). Although the eigenvalues of any real matrix Q can be found by using its Hessenberg form, a sequence of orthogonal transformations to get the real Schur form of Wintner–Murnaghan and the QR-algorithm of Francis (see contribution II/14 of [15]), it is interesting to compute finite approximations to the exact solution $U(x,t)$, thereby avoiding the explicit computation of the eigenvalues and expressing the approximate solutions in terms of data.

Note that if $Q^{1/2}$ is a square root of a matrix $Q \in \mathbb{R}^{m\times m}$, then by using the series expansion of $\cos(tQ^{1/2})$, it follows that

$$\cos(tQ^{1/2}) = \sum_{j\ge 0}(-1)^j (tQ^{1/2})^{2j}/(2j)! = \sum_{j\ge 0}(-t^2Q)^j/(2j)!$$

and from Theorem 11.2.4 of [8, p. 390], and Lemma 1, we have

$$\left\| \cos(tR_k) - \sum_{j=0}^{q}(-t^2C_k)^j/(2j)! \right\| \le m/(q+1)! \tag{3.1}$$

for any positive real number t.

Let ε be an admissible error. From (2.16) there exists a positive constant L and a positive k_0 such that

$$\|c_k\| \le L/k^4 \quad \text{for } k \ge 1 \tag{3.2}$$

and

$$\sum_{k\geq k_0} \|c_k\| < \varepsilon/2\,. \tag{3.3}$$

Since $\sum_{n\geq 1} n^{-4} = \pi^4/90$, let q_0 be a positive integer such that

$$1/(q_0+1)! \leq 45\varepsilon/(mL\pi^4)\,. \tag{3.4}$$

From (3.1)–(3.3), if we set

$$U(x,t,k_0,q_0) = \sum_{k=1}^{k_0}\sum_{j=0}^{q_0}(-t^2C_k)^j \sin(k\pi xI/p)c_k/(2j)! \tag{3.5}$$

it follows that

$$\begin{aligned}&\|U(x,t) - U(x,t,k_0,q_0)\| \\ &\leq \sum_{k>k_0}\|c_k\| \sum_{k=1}^{k_0}\Big\|\Big\{\cos(tR_k) - \sum_{j=0}^{q_0}(-t^2C_k)^j/(2j)!\Big\}\sin(k\pi xI/p)c_k\Big\| \\ &\leq \varepsilon/2 + mL\Big(\sum_{n\geq 1} n^{-4}\Big)/(q_0+1)! < \varepsilon\end{aligned}$$

for any $(x,t) \in [0,p]\times]0,\infty[$.

Note that from (2.6), the finite series (3.5) may be written in terms of the data, in the following form:

$$U(x,t,k_0,q_0) = \sum_{k=1}^{k_0}\sum_{j=0}^{q_0}(B + (k\pi/p)^2 I)^j A^{-j} \sin(k\pi xI/p)c_k t^{2j}/(2j)!\,.$$

Thus the following result has been established:

THEOREM 3. *Let $\varepsilon > 0$, and let k_0, q_0 be positive integers satisfying (3.3) and (3.4), respectively. Then $U(x,t,k_0,q_0)$ is an approximate solution of the problem (1.1)–(1.4), such that if $U(x,t)$ is the exact solution of the problem given by Theorem 2, then $\|U(x,t) - U(x,t,k_0,q_0)\| \leq \varepsilon$, uniformly for $(x,t) \in [0,p]\times]0,\infty[$.*

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