ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,3 (1992), pp. 371–385

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MAX-TYPE RANK TESTS IN THE TWO-SAMPLE PROBLEM

1. Introduction. In this paper we consider a method of construction of rank tests for the two-sample problem. Our aim is to extend the range of sensitivity of classical linear rank tests.

Behnen [5] has proved that in many testing problems asymptotic optimality of classical linear rank tests takes place if and only if the test statistic and alternative are generated by the same function, say b (cf. also [10] and [1]). Moreover, it is well known that under a larger class of alternatives the classical tests have rather small power. To overcome this disadvantage, Behnen and Neuhaus (B&N) [6], Neuhaus [12] and B&N [7] have recently adapted the classical rank statistics to larger classes of contiguous alternatives by estimating the function b.

In this paper we present a simpler alternative approach to the problem. Our method is naturally and easily applicable to some subclasses of alternatives, e.g. to a "stochastically larger" alternative or "more dispersed" alternative (see Sec. 6). Although our idea is quite general, we restrict attention to the two-sample problem. To extend the range of sensitivity of linear rank tests we propose to take max-type statistics, i.e. the maximum of some linear rank statistics. Before we present the content of the paper in more detail, note that some examples of taking the maximum or minimum of some test statistics in different testing problems can be found e.g. in [16] (the combination problem), [8] (testing goodness of fit), [11] (independence testing), [14] (change point problem). Observe also that though the results of [1], [3], [4], [11], [14] show some advantages of such a combination of test statistics, there is no general theory for this class of statistics.

1991 *Mathematics Subject Classification*: 62G10, 62G20, 62E20, 62E25.
Key words and phrases: rank test, two-sample problem, local alternatives, local asymptotic relative efficiency, asymptotic power, Monte Carlo studies, two samples differing in scale.

The content of the paper is as follows. In Section 2 we introduce the basic notation and the definition of a max-type statistic. The max-type statistics for one-sided (stochastically larger) and two-sided alternatives are discussed in Sections 3 and 4, respectively. The local asymptotic relative efficiency of a max-type test is calculated in Section 3.1. The asymptotic powers are investigated in Sections 3.2 and 4.1. The results are comparable with those obtained by Neuhaus [12]. To check the agreement of asymptotic results with their finite sample counterparts some simulations are reported in Sections 3.3 and 4.2. More numerical results are contained in [2]. All the results show that max-type statistics are sensitive for a larger class of alternatives than linear rank statistics. In Section 5 we discuss how to improve the famous Wilcoxon statistic via some max-type statistics. A construction of a max-type statistic for two samples differing in scale is shortly presented in Section 6.

2. Preliminaries

2.1. *A reparametrization of the problem.* Let $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ be independent rv's and suppose that the distribution function F (G) of X_i (Y_j) is continuous. Define $X_{m+j} = Y_j$, $j = 1, \ldots, n$, and let R_i be the rank of X_i in the pooled sample $(X_1, \ldots, X_N)$, $N = m + n$.

Now consider the testing problem

$$H_0 : F \equiv G$$

versus the *omnibus alternative*

$$K_0 : F \not\equiv G$$

and H_0 against the *stochastically larger alternative*

$$K_1 : F \leq G\,.$$

As in [12], we will replace the parameter (F, G) by an equivalent parameter (b, H), where

$$H = p_N F + (1 - p_N)G\,, \quad p_N = m/N\,,$$
$$b = (mn/N)^{1/2} \frac{d(F - G) \circ H^{-1}}{d\lambda}\,.$$

and λ is the Lebesgue measure on $(0, 1)$. By definition $H \in \mathcal{F}_c$, where $\mathcal{F}_c$ is the set of continuous df's on $\mathbb{R}$. Moreover, $b \in M_N$ where

$$M_N = \{f \in L_2^0(0,1) : -(Nm/n)^{1/2} \leq f \leq (Nn/m)^{1/2}\}\,,$$
$$L_2^0(0,1) = \left\{f \in L_2(0,1) : \int_0^1 f\, d\lambda = 0\right\}.$$

In terms of b the above testing problems take the equivalent form

$$H_0 : b \equiv 0 \quad \text{versus} \quad K_0 : b \not\equiv 0\,,$$

(1) $$H_0 : b \equiv 0 \quad \text{versus} \quad K_1 : \int_0^t b\, d\lambda \le 0 \text{ for all } t \in (0,1)\,,$$

respectively (for more details see [7], Section 1.3).

For further purposes it is convenient to present the alternatives (1) in a more general form. Consider the hypothesis

(2) $$H_0 : b \equiv 0 \quad \text{versus} \quad K : b \in L \setminus \{0\}\,,$$

where L is a cone in $L_2^0(0,1)$.

Note that we get the stochastically larger alternative K_1 for

(3) $$L = C = \Big\{ b \in L_2^0(0,1) : \int_0^t b\, d\lambda \le 0 \text{ for all } t \in (0,1) \Big\}\,,$$

and the omnibus alternative K_0 for $L = L_2^0(0,1)$.

To simplify further considerations, we distinguish two specific properties of the cone L. The alternative described by L will be called *one-sided* if

$$b \in L \Rightarrow (-b) \notin L$$

and *two-sided* if

$$b \in L \Rightarrow (-b) \in L\,.$$

Note that in the two-sided case according to our definition the cone L is a subspace of $L_2^0(0,1)$.

2.2. *Linear and max-type rank statistics.* Now for any sequences $\{H_N\} \subset \mathcal{F}_c$, $\{b_N\} \subset M_N$ with $b_N \to b$ in $L_2^0(0,1)$ define the local alternatives $\{(F_N, G_N)\}$ by

(4) $$\frac{dF_N}{dH_N} = 1 + c_{N,1} b_N \circ H_N\,, \quad \frac{dG_N}{dH_N} = 1 + c_{N,N} b_N \circ H_N\,,$$

where

$$c_{N,i} = (mn/N)^{1/2} \cdot \begin{cases} m^{-1} & \text{for } 1 \le i \le m\,, \\ -n^{-1} & \text{for } m+1 \le i \le N\,. \end{cases}$$

The function b will be called the *asymptotic direction* of (4).

Behnen [5] has proved that the linear rank statistic

(5) $$S_N(h) = \sum_{i=1}^{N} c_{N,i} h_N(R_i)\,,$$

where

$$h_N(i) = N \int_{(i-1)/N}^{i/N} h(u)\, d\lambda, \quad 1 \le i \le N,\ h \in L_2^0(0,1),$$

is asymptotically optimal for testing H_0 against the alternatives of the form (4) with b such that $\int_0^t b\, d\lambda \le 0$ for all $t \in (0,1)$ iff there is a $c > 0$ such that $h = cb$ a.e., that is, for its "own" direction h. Moreover, the asymptotic power of (5) depends on b via $\langle h, b\rangle / \|h\|$ only, where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $L_2^0(0,1)$. The power is greater than the significance level only for h such that $\langle h, b\rangle > 0$. A similar assertion holds true for the local Bahadur optimality (cf. [10] and [1]).

Denote by $W_N(t)$, $t \in [0,1]$, the two-sample rank process

$$W_N(i/N) = \sum_{j=1}^{N} c_{N,j} I(R_j \le i), \quad i = 0, 1, \ldots, N, \tag{6}$$

and define $W_N(\cdot)$ to be linear in all intervals $[(i-1)/N, i/N]$. Under the alternatives (4), with $p_N \to p \in (0,1)$, we have

$$W_N \xrightarrow{\mathcal{D}} W_0 + \int_0^{\cdot} b\, d\lambda \quad \text{as } N \to \infty, \tag{7}$$

where W_0 is the Brownian bridge process on $[0,1]$ (cf. [12], formula (2.36)).

Note that if $h \in L_2^0(0,1)$ is of bounded variation then

$$S_N(h) = \int_0^1 h\, dW_N = -\int_0^1 W_N\, dh. \tag{8}$$

Moreover, it is a well known fact that for $L_2(0,1)$ functions $\{h_1, \ldots, h_k\}$ of bounded variation the rv's $\{\int h_i\, dW_0\}_{i=1,\ldots,k}$ are centered and jointly normal with covariances

$$E \int h_i\, dW_0 \cdot \int h_j\, dW_0 = \int h_i h_j\, d\lambda - \int h_i\, d\lambda \int h_j\, d\lambda.$$

Now let V_i, $i = 1, \ldots, k$, be continuous functionals on $C[0,1]$. The statistic

$$T_{N,k} = \max_{1 \le i \le k} V_i(W_N)$$

will be called a *max-type statistic.*

So, if we take the functionals $V_i(f) = -\int f\, dh_i$, where $h_1, \ldots, h_k$ are some $L_2^0(0,1)$ functions of bounded variation, we get a max-type statistic based on linear rank statistics. In Sections 3 and 4 we consider this kind of test statistics.

3. Max-type statistics for one-sided alternatives. As was noted by B&N [7], if we represent many relevant types of alternatives in the form (4), the function b_N can be approximated by a given finite set of special functions. For example, the first four functions of the orthonormal system (15) below give a suitable approximation of generalized shift models considered by B&N [7] (cf. our formulae (17)). Moreover, if we try to make a test sensitive in a whole infinite-dimensional cone, we obtain rather low power in all directions (cf. B&N [7], Section 3.2.B). Therefore for practical purposes we consider a finite system of score generating functions.

Let $g_1, \ldots, g_k$ be a finite orthonormal system in L of functions of bounded variation in L.

In the case of a one-sided alternative we propose the following one-sided max-type statistic:

$$T^1_{N,k} = \max_{1\le i\le k} c_i S_N(g_i)\,, \tag{9}$$

where the weights c_i are positive. Its asymptotic distribution is very simple. Under the alternatives (4)

$$T^1_{N,k} \xrightarrow{\mathcal{D}} \max_{1\le i\le k} c_i(Y_i + \langle g_i, b\rangle)\,, \tag{10}$$

where $Y_1, \ldots, Y_k$ are i.i.d. standard normal rv's and $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(0,1)$.

Note that Neuhaus' [12] test statistics for the stochastically larger alternative, which is a one-sided alternative, are more complicated and that he did not manage to calculate their asymptotic distributions.

Another kind of test statistics for the stochastically larger alternative was introduced in [7] using projection estimators of the function b_N (see (4)). However, the authors manage to evaluate the asymptotic distributions under H_0 only.

Because of the simple form of the asymptotic distribution of the statistic $T^1_{N,k}$, it is easy to examine some of its properties. For example, $T^1_{N,k}$ is asymptotically unbiased under (4) in all directions $b \in \{\sum_{i=1}^k \gamma_i g_i : \gamma_i \ge 0\}$.

3.1. *Local asymptotic relative efficiency.* Now, consider the case of equal weights c_i. $T^1_{N,k}$ with the weights $c_i = 1$, $i = 1, \ldots, k$, will be denoted by $\overline{T}^1_{N,k}$. We can compare $\overline{T}^1_{N,k}$ with the linear rank test based on $S_N(g_1)$ using the local asymptotic relative efficiency (LARE) defined in [9] (Section VII.2.3). We expand the asymptotic power of $S_N(g_1)$ ($\mathrm{AP}(S_N(g_1))$) and $\overline{T}^1_{N,k}$ ($\mathrm{AP}(T^1_{N,k})$) under the alternatives (4) in the direction $d\cdot b$, where $d > 0$, as follows:

$$\mathrm{AP}(S_N(g_1))(d) = \alpha + d\phi(u_\alpha)\langle b, g_1\rangle + o(d)\,,$$

$$\mathrm{AP}(\overline{T}^1_{N,k})(d) = \alpha + d(1-\alpha)^{(k-1)/k} \sum_{i=1}^{k} \langle b, g_i \rangle \phi(u^*_\alpha) + o(d)\,,$$

as $d \to 0$, where α is a common significance level, Φ and ϕ are the df and the density of the standard normal distribution, respectively, and $\Phi(u_\alpha) = 1-\alpha$, $\Phi(u^*_\alpha) = (1-\alpha)^{1/k}$. LARE is the square of the ratio of the asymptotic slopes, i.e.

$$\text{(11)} \qquad \mathrm{LARE}(\overline{T}^1_{N,k}, S_N(g_1)) = \left\{ (1-\alpha)^{(k-1)/k} \frac{\phi(u^*_\alpha)}{\phi(u_\alpha)} \sum_{i=1}^{k} \frac{\langle b, g_i \rangle}{\langle b, g_1 \rangle} \right\}^2 .$$

LARE enables a comparison of tests under contiguous alternatives in the direction of a small length. Now, we calculate the limit of LARE as $\alpha \to 0$:

PROPOSITION 1.

$$\lim_{\alpha \to 0} \mathrm{LARE}(\overline{T}^1_{N,k}, S_N(g_1)) = \left\{ \frac{1}{k} \sum_{i=1}^{k} \frac{\langle b, g_i \rangle}{\langle b, g_1 \rangle} \right\}^2 .$$

Proof. Note that

$$\frac{\phi(u^*_\alpha)}{\phi(u_\alpha)} \sim \frac{1}{k} \cdot \frac{u_\alpha}{u^*_\alpha} \quad \text{as } \alpha \to 0\,.$$

It is a known fact that $1 - \Phi(x) \sim \phi(x)/x$ as $x \to \infty$. Therefore

$$\frac{1}{k}\left[\frac{u_\alpha}{u^*_\alpha}\right]^2 \sim \frac{u_\alpha}{\phi(u_\alpha)} \cdot \frac{\phi(u^*_\alpha)}{u^*_\alpha} \sim \frac{1-(1-\alpha)^{1/k}}{\alpha} \to \frac{1}{k} \quad \text{as } \alpha \to 0\,. \ \blacksquare$$

Some numerical results on LARE are contained in Table 1. These results confirm that the linear rank test is better in its own direction ($\psi = 0$) than the max-type test. However, it is worse in some intermediate directions ($\psi = \pi/4$ and the second part of Table 1) and very bad in almost orthogonal directions ($\psi = 0.9 \cdot \pi/2$). Note that in the first part of Table 1, i.e. for $b = g_1 \cos\psi + g_2 \sin\psi$, $k = 2$, we have

$$\lim_{\alpha \to 0} \mathrm{LARE}(\overline{T}^1_{N,2}, S_N(g_1)) = (1+\tan\psi)^2/4\,.$$

3.2. *Asymptotic power of some one-sided max-type statistics.* Consider $k = 2$ and investigate the asymptotic behaviour of $\overline{T}^1_{N,2}$ (the case of equal weights c_i) under the alternatives (4) in a direction b from $P_1 = \{\gamma g_1 + \beta g_2 : \gamma, \beta \geq 0, \gamma^2 + \beta^2 = \text{const}\}$. It is easy to calculate that the asymptotic power of $\overline{T}^1_{N,2}$ does not take local extremes for the main directions g_1, g_2 (as in the two-sided case, see Section 4.1) but for some $b = \gamma g_1 + \beta g_2$ with $\gamma, \beta > 0$. Numerical results depend on the level of the test and on the length $c = \|b\|$ of the direction b. Sometimes there are maxima near cg_1, cg_2 and a minimum for $b = 2^{-1/2}c(g_1 + g_2)$, and sometimes there is only a maximum for $b = 2^{-1/2}c(g_1 + g_2)$ (see Table 2).

TABLE 1

Some values of LARE($\overline{T}^1_{N,k}, S_N(g_1)$) for various levels α and the limit as $\alpha \to 0$, $b = g_1 \cos\psi + g_2 \sin\psi$, $k = 2$, $\psi = 0,\ \pi/4,\ 0.9\pi/2$ (first part) and for $b = \sum_{i=1}^{k} g_i/k^{1/2}$, $k = 2, 4, 10, 20$ (second part).

ψ	$\alpha = 0.1$	$\alpha = 0.001$	$\alpha = 0.00001$	$\alpha \to 0$
0	0.33	0.28	0.27	0.25
$\pi/4$	1.30	1.12	1.06	1.00
$0.9\pi/2$	17.31	14.90	14.29	13.40
k				
2	1.30	1.12	1.06	1.00
4	1.61	1.23	1.12	1.00
10	2.07	1.39	1.21	1.00
20	2.43	1.49	1.28	1.00

TABLE 2

Asymptotic powers (in %) of the one-sided test statistic $\overline{T}^1_{N,2}$ under the alternatives $b_N = d(g_1 \cos\psi + g_2 \sin\psi)$ for various d, α and $\psi = (r/10)\cdot(\pi/4)$, $r = 0, \ldots, 10$.

	$d = 0.5$				$d = 1$			
r \ α	0.1	0.05	0.01	0.001	0.1	0.05	0.01	0.001
0	17.35	9.64	2.39	0.31	30.14	19.09	6.24	1.15
1	17.69	9.84	2.44	0.32	30.69	19.43	6.32	1.16
2	17.99	10.02	2.49	0.32	31.14	19.69	6.36	1.15
3	18.26	10.17	2.52	0.33	31.50	19.87	6.36	1.13
4	18.48	10.30	2.55	0.33	31.77	19.99	6.33	1.11
5	18.67	10.41	2.57	0.33	31.98	20.06	6.28	1.08
6	18.82	10.49	2.59	0.33	32.13	20.09	6.23	1.05
7	18.94	10.56	2.60	0.33	32.23	20.10	6.17	1.02
8	19.02	10.60	2.61	0.33	32.29	20.10	6.12	1.00
9	19.07	10.63	2.61	0.33	32.33	20.10	6.09	0.98
10	19.09	10.64	2.62	0.33	32.34	20.10	6.08	0.98
	$d = 2$				$d = 3$			
r \ α	0.1	0.05	0.01	0.001	0.1	0.05	0.01	0.001
0	66.18	53.03	28.63	9.89	91.87	85.58	66.63	38.60
1	66.63	53.32	28.62	9.82	91.99	85.64	66.45	38.29
2	66.85	53.31	28.30	9.56	91.96	85.42	65.72	37.32
3	66.87	53.06	27.72	9.14	91.81	84.97	64.50	35.75
4	66.75	52.64	26.94	8.60	91.56	84.36	62.90	33.70
5	66.54	52.11	26.05	7.99	91.27	83.65	61.04	31.32
6	66.28	51.55	25.14	7.38	90.96	82.92	59.09	28.81
7	66.03	51.02	24.31	6.81	90.68	82.23	57.27	26.45
8	65.82	50.60	23.65	6.36	90.45	81.67	55.77	24.50
9	65.68	50.32	23.21	6.07	90.29	81.31	54.79	23.22
10	65.63	50.23	23.06	5.97	90.24	81.19	54.44	22.77

3.3. *Simulated powers of some one-sided max-type statistics.* Now consider a specific one-sided alternative, namely the stochastically larger alternative K_1. Note that the orthonormal system v_i, $i = 1, 2, \dots$ (see (15)), considered by Neuhaus [12] is not appropriate because the functions $\pm v_i$ for $i > 1$ do not belong to the corresponding cone C (see (3)). Therefore we propose the orthonormal system

$$w_i(u) = -2^{1/2}\sin(2\pi i u)\,, \quad i = 1, 2, \dots \tag{12}$$

Simulated powers of some statistics based on (12) are given in Table 3. The first column of Table 3 concerns $S_N(w_1)$, which is a one-sided linear rank statistic asymptotically optimal in the direction w_1. The next three statistics are one-sided max-type statistics (see (9)) with equal weights. Table 3 also gives asymptotic powers which are valid for an arbitrary orthonormal system.

Table 3 shows that the tests considered which are sensitive in more directions than $S_N(w_1)$, have smaller asymptotic power in the first direction, though the difference is not very large. However, in the next directions they have considerably greater power in contrast to the power 10% (the significance level) of the linear rank test. Note that the asymptotic power 77.7% of $\overline{T}^1_{N,10}$ is also valid for directions $5 \div 10$, for which the other tests have asymptotic power 10%. Table 3 also shows a good agreement of finite sample simulation results with its asymptotic counterpart.

4. Max-type statistics for two-sided alternatives. Consider the hypothesis H_0 versus the omnibus alternative K_0, and the max-type statistic

$$\max_{1\le i\le k} S_N(h_i)\,, \tag{13}$$

where $h_1, \dots, h_k$ are some $L_2^0(0,1)$ functions of bounded variation. First we formulate a result on the asymptotic admissibility of this test. Note that asymptotic admissibility of a test means that its asymptotic power under the local alternatives (4) cannot be improved for some direction b without diminishing the power for some other direction.

PROPOSITION 2. *The test based on* (13) *is asymptotically admissible for all directions* $b \in L_2^0(0,1)$.

Proof. Note that the set

$$\Big\{f \in B : \max_{1\le i\le k}\Big(-\int f\,dh_i\Big) \le c\Big\}$$

is closed and convex in B, where $B = \{f \in C[0,1] : f(0) = f(1) = 0\}$. As in [12], the rest of the proof follows the lines of Example 82.23 of [15].

4.1. *Asymptotic power of some two-sided max-type statistics.* Similarly to Section 3, let $g_1, \dots, g_k$ be a finite orthonormal system of $L_2^0(0,1)$ functions

TABLE 3

Powers (in %) of some one-sided tests at level $\alpha = 0.1$ under the alternatives (4) with $b_N = 3w_i$ for $i = 1, 2, 3$ (see (12)), $m = n$, $N = m + n = 10$, 40, 80 (obtained by 3000 Monte Carlo runs) and $N = \infty$ (the asymptotic power).

N	alt. i	$S_N(w_1)$	$\overline{T}^1_{N,2}$	$\overline{T}^1_{N,4}$	$\overline{T}^1_{N,10}$
	1	96.1	92.7	87.5	82.1
20	2	21.6	68.3	56.8	50.4
	3	14.1	27.0	39.2	31.1
	1	95.3	92.1	86.5	78.6
40	2	15.4	77.8	71.4	59.5
	3	11.0	21.9	55.2	43.7
	1	95.3	90.5	85.8	76.0
80	2	13.6	84.7	78.7	66.5
	3	11.0	16.0	66.2	53.1
	1	95.7	91.9	86.6	77.7
∞	2	10.0	91.9	86.6	77.7
	3	10.0	10.0	86.6	77.7

of bounded variation. Since $\max(S_N(g_i), S_N(-g_i)) = |S_N(g_i)|$, for a two-sided alternative we propose the following test statistic which is a special case of (13):

$$T^2_{N,k} = \max_{1\le i\le k} c_i |S_N(g_i)|\,, \qquad c_i > 0\,,\ i = 1, \ldots, k\,.$$

Just as in (10),

$$T^2_{N,k} \xrightarrow{\mathcal{D}} \max_{1\le i\le k} c_i |Y_i + \langle g_i, b\rangle|\,. \tag{14}$$

As in the one-sided case, $T^2_{N,k}$ is asymptotically unbiased under (4) in all directions $b \in \{\sum_{i=1}^k \gamma_i g_i : \gamma_i \in \mathbb{R}\}$.

We would like to compare the above statistic with the statistics defined by Neuhaus [12] for the omnibus alternative K_0. For this purpose, we have chosen the statistic $T^2_{N,k}$ with $k = 4$ and

$$c_1 = 0.95\,, \quad c_2 = 0.81\,, \quad c_3 = 0.67\,, \quad c_4 = 0.5\,,$$

and the statistic Par-2 defined by Neuhaus [12], which can be written in the form

$$\begin{aligned}\text{Par-2} = 0.9 \cdot S_N(g_1)^2 + 0.66 \cdot S_N(g_2)^2 + 0.38 \cdot S_N(g_3)^2 \\ + 0.16 \cdot S_N(g_4)^2 + 0.05 \cdot S_N(g_5)^2 + 0.01 \cdot S_N(g_6)^2\,.\end{aligned}$$

As can be seen in Table 4 the asymptotic power of $T^2_{N,4}$ is greater than the power of Par-2 in the directions g_i, $i = 1, \dots, 4$. However, in some intermediate directions it must be smaller because both tests are asymptotically admissible.

TABLE 4

Asymptotic power (in %) of the tests Par-2 and $T^2_{N,4}$ at level $\alpha = 0.1$ for H_0 versus (4) with $b = 3g_i$, $i = 1, \dots, 4$.

i	1	2	3	4
Par-2	86.7	80.2	62.1	29.2
$T^2_{N,4}$	87.9	81.1	67.8	37.1

Now consider $k = 2$ and as in Section 3.2 investigate the asymptotic behaviour of $\overline{T}^2_{N,2}$ (the case of equal weights c_i) under the alternatives (4) in directions b from $P_2 = \{\gamma g_1 + \beta g_2 : \gamma^2 + \beta^2 = \text{const.}\}$. It can be proved by a straightforward calculation that the asymptotic power of $\overline{T}^2_{N,2}$ takes a local maximum for the main directions $\pm g_1$, $\pm g_2$. Our numerical results presented in Table 5 show that minimum powers are observed for the intermediate directions $2^{-1/2}(\pm g_1 \pm g_2)$.

TABLE 5

Asymptotic powers (in %) of the two sided test statistic $\overline{T}^2_{N,2}$ under the alternatives $b_N = d(g_1 \cos\psi + g_2 \sin\psi)$ for various d, α and $\psi = (r/10)\cdot(\pi/4)$, $r = 0, \dots, 10$.

	$d = 2$			$d = 3$		
r \ α	0.1	0.05	0.01	0.1	0.05	0.01
0	54.51	42.16	21.40	86.09	78.31	57.89
1	54.41	42.03	21.27	85.99	78.13	57.59
2	54.14	41.67	20.88	85.68	77.61	56.68
3	53.71	41.10	20.27	85.19	76.78	55.23
4	53.16	40.37	19.49	84.55	75.71	53.33
5	52.55	39.56	18.62	83.83	74.48	51.12
6	51.94	38.75	17.74	83.10	73.21	48.81
7	51.39	38.00	16.93	82.42	72.02	46.64
8	50.94	37.41	16.29	81.86	71.06	44.85
9	50.66	37.02	15.87	81.51	70.43	43.67
10	50.56	36.89	15.72	81.38	70.21	43.25

4.2. *Simulated powers of some two-sided max-type statistics.* We present (Table 6) some power values of some tests for the omnibus alternative K_0. The simulations have been done for the orthonormal system

$$v_i(u) = 2^{1/2}\cos(\pi i u), \quad i = 1, 2, \dots, \tag{15}$$

considered by Neuhaus [12]. In Table 6, $|S_N(g_1)|$ is a two-sided linear rank statistic. $\overline{T}^2_{N,2}$, $\overline{T}^2_{N,4}$, $T^2_{N,10}$ are two-sided max-type statistics (see (14)) with equal weights, i.e. $c_i = 1$, $i = 1, \dots, k$. As in Table 3, the lower power of $\overline{T}^2_{N,4}$ and $\overline{T}^2_{N,10}$ in the direction v_1 is compensated in other orthogonal directions.

TABLE 6

Powers (in %) of some two-sided tests at level $\alpha = 0.1$ for H_0 versus K_0 (see (1)) under the alternatives (4) with $b_N = 3v_i$ for $i = 1, 2, 3, 4$ (see (15)), $m = n$, $N = m + n = 20$, 40, 80 (obtained by 3000 Monte Carlo runs) and $N = \infty$ (the asymptotic power).

N	alt. i	$\lvert S_N(v_1)\rvert$	$\overline{T}^2_{N,2}$	$\overline{T}^2_{N,4}$	$\overline{T}^2_{N,10}$
20	1	96.1	91.9	86.6	79.6
	2	18.6	86.8	78.6	69.2
	3	11.1	21.6	66.7	56.0
	4	11.2	14.7	55.0	46.6
40	1	94.2	89.6	84.0	73.4
	2	14.0	85.4	78.8	67.5
	3	12.4	18.2	72.6	60.8
	4	10.3	13.6	63.3	52.0
80	1	91.5	87.1	80.9	70.8
	2	11.9	85.0	78.6	67.7
	3	10.6	14.5	76.0	64.5
	4	10.4	13.2	69.2	58.7
∞	1	91.2	86.1	79.7	70.0
	2	10.0	86.1	79.7	70.0
	3	10.0	10.0	79.7	70.0
	4	10.0	10.0	79.7	70.0

5. On some improvements of the Wilcoxon statistic. Now, consider the two-sample Wilcoxon statistic which is a linear rank statistic determined by the function $e_1(u) = 3^{1/2}(2u - 1)$.

If we want to improve the two-sided Wilcoxon statistic (the absolute value of the one-sided Wilcoxon statistic) in some directions other than e_1, we can use, instead of (15), the orthonormal system of Legendre polynomials whose first elements are

$$e_1(u) = 3^{1/2}(2u - 1), \quad e_2(u) = 5^{1/2}(6u^2 - 6u + 1).$$

This kind of orthonormal system has been used by Neyman [13] to construct some smooth tests of fit.

If we want to improve the one-sided Wilcoxon test for testing H_0 against the stochastically larger alternative using methods considered in this paper, we need an orthonormal system in C which contains the above function e_1. Unfortunately, such a system does not exist, also for some other functions. This can be seen from the following simple but interesting fact (analogous to a part of Lemma 7.3.1 of B&N [7]).

PROPOSITION 3. *Consider a strictly increasing function h in $L_2(0,1)$. Then for all functions $f \not\equiv 0$ a.e. from the cone C we have*

$$\int_0^1 h(u)f(u)\,du > 0\,.$$

Proof. Note that if $f \in C$ then for every $t \in (0,1)$, $F(t) = \int_0^t f(u)\,du \leq 0$. Moreover,

$$\int_0^1 h(u)f(u)\,du = -\int_0^1 F(u)\,dh(u)$$

is nonnegative and can be 0 only for $F(u) \equiv 0$. This contradicts $f \not\equiv 0$. ■

The above proposition also shows that the linear rank statistic determined by a strictly increasing function h (e.g. Wilcoxon or van der Waerden tests) has an asymptotic power greater than the significance level for the alternatives (4) with an arbitrary direction $b \in C$.

Because of the above difficulties in constructing the appropriate orthonormal system we propose the following solution. Define two functions:

$$f_1 = 0.8 \cdot e_1 + 0.6 \cdot e_2\,, \qquad f_2 = 0.6 \cdot e_1 - 0.8 \cdot e_2\,,$$

from the cone C. Consider

$$\overline{T}'^1_{N,2} = \max_{1\leq i\leq 2} S_N(f_i)\,, \qquad T^1_{N,3} = \max\{S_N(f_1), S_N(f_2), S_N(e_1)\}\,.$$

These statistics are not much more complicated than Wilcoxon's but they are sensitive in a greater number of directions.

In Table 7 we present some simulation results for $\overline{T}'^1_{N,2}$, $T^1_{N,3}$ and $S_N(e_1)$ — the Wilcoxon test statistic. Note that the calculations of powers under the alternatives (4) would give results similar to those in Table 3. To make it more interesting we consider another kind of alternative, viz. the generalized shift alternative (see [12]):

$$F \leq G\,, \quad \text{where } F = G(x - D(x)) \tag{16}$$

for some shift functions $D \geq 0$. Such an alternative is more realistic than the traditional constant shift (for more details see B&N [7]). We have chosen (as in [12])

$$(17)\qquad \begin{aligned} &\text{upper shift}: && D(x) = G(4x)/2\,,\\ &\text{central shift}: && D(x) = 2G(2x)\cdot(1 - G(2x))\,,\\ &\text{lower shift}: && D(x) = (1 - G(4x))/2\,,\\ &\text{pure shift}: && D(x) = 1/2\,, \end{aligned}$$

with the standard logistic and Cauchy df G. Additionally, we consider the double-exponential df G. It is seen that under the above alternatives the new max-type tests are frequently better than the Wilcoxon test (except for the pure shift), especially for the upper and lower shifts. Moreover, new tests have powers similar to that of the one-sided test considered by Neuhaus [12]. Note also that in the exact logistic shift model (pure shift), the Wilcoxon test is an asymptotically optimal and locally most powerful rank test.

In Table 8 we give critical values of some tests considered in this paper.

TABLE 7

Simulated powers (in %) of some one-sided tests considered in Section 5 at level $\alpha = 0.1$ under generalized shift alternatives (16) with G the standard double-exponential (exp), logistic (log) and Cauchy df. The shifts are defined according to (17). The powers have been obtained by 3000 Monte Carlo runs in the cases $m = n$ and $N = m + n = 20,\ 80$.

		$N = 20$			$N = 80$		
Test		$\overline{T}'^{1}_{N,2}$	$T^{1}_{N,3}$	$S_N(w_1)$	$\overline{T}'^{1}_{N,2}$	$T^{1}_{N,3}$	$S_N(w_1)$
G	shift						
exp	upper	23.1	23.4	21.0	48.4	49.3	43.1
	central	24.5	25.4	22.8	41.9	43.8	46.8
	lower	21.2	22.5	18.5	40.1	41.6	32.8
	pure	31.3	33.9	33.7	63.3	68.0	72.0
log	upper	18.9	18.6	17.1	31.7	32.3	28.4
	central	18.2	19.0	17.1	26.4	28.9	29.3
	lower	18.0	18.8	15.1	29.9	30.8	23.6
	pure	22.4	24.1	23.3	41.9	45.7	50.0
Cauchy	upper	17.9	18.8	16.5	27.3	27.3	26.5
	central	20.4	20.8	19.3	28.8	31.8	34.8
	lower	17.7	18.3	15.9	25.3	26.6	23.0
	pure	22.6	24.0	22.7	39.8	42.8	46.1

6. Two samples differing in scale. In Sections 3.3 and 5 we consider, as an example of one-sided alternative, the stochastically larger alternative. Now we present another example to show how to construct max-type statistics for a different kind of alternatives.

TABLE 8

Critical values of some one-sided and two-sided tests considered in Sections 3.3, 4.2 and 5 at various levels α for $N = 20, 40, 80$ (obtained by 8000 Monte Carlo runs) and for $N = \infty$ (asymptotic values).

α	N	$\overline{T}^2_{N,2}$	$\overline{T}^2_{N,4}$	$\overline{T}^2_{N,10}$	$\overline{T}'^1_{N,2}$	$\overline{T}^1_{N,4}$	$\overline{T}^1_{N,10}$
0.1	20	2.02	2.28	2.49	1.66	1.94	2.17
	40	1.98	2.25	2.56	1.65	1.96	2.26
	80	1.95	2.24	2.56	1.63	1.94	2.31
	∞	1.96	2.23	2.56	1.63	1.95	2.31
0.05	20	2.32	2.51	2.73	1.95	2.22	2.42
	40	2.26	2.50	2.76	1.95	2.25	2.46
	80	2.23	2.50	2.80	1.96	2.25	2.53
	∞	2.24	2.49	2.80	1.96	2.24	2.57
0.01	20	2.78	2.94	3.12	2.48	2.73	2.81
	40	2.81	2.97	3.21	2.55	2.77	2.97
	80	2.86	3.09	3.26	2.59	2.83	3.06
	∞	2.81	3.02	3.29	2.58	2.81	3.08

Under the notation of Section 2.1 we deal with testing of the hypothesis $H_0 : F \equiv G$ against the alternative of dispersion about $\mu \in [0,1]$:

$$K_2^\mu : \quad F \not\equiv G \text{ and } F(x) \ge G(x) \text{ if } G(x) < \mu, \quad F(x) \le G(x) \text{ if } G(x) \ge \mu,$$

considered by B&N [7].

The reparametrization as in Section 2.1 yields the testing problem in the form (2) with the cone

$$L = S^\mu = \Big\{ b \in L_2^0(0,1) : \int_0^t b\,d\lambda \ge 0 \text{ for all } t \in (0,\mu) \text{ and } \int_0^t b\,d\lambda \le 0 \text{ for all } t \in [\mu, 1) \Big\}.$$

For more details and general information as well as motivation see B&N [7], Chapter 4.1. To construct one-sided max-type statistics for this case, we should have an orthonormal system of functions in S^μ. Given any $\mu \in [0,1]$ we propose the orthonormal system

(18) $$z_i^\mu(x) = -2^{1/2} \sin\{4\pi i |x-\mu| / [1 + (1-2\mu)\,\mathrm{sgn}(x-\mu)]\},$$

$i = 1, 2, \ldots$, in S^μ.

Now consider two special cases of S^μ.

1) $\mu = 0$. Then $S^0 = C$ (cf. (3)). In other words, S^0 is the stochastically larger alternative. Moreover, the orthonormal system (18) with $\mu = 0$ coincides with w_i (see Section 3.3, formula (12)).

2) $\mu = 1/2$. This is the case of dispersion about the median and the orthonormal system (18) takes a simpler form

$$z_i^{1/2}(x) = -2^{1/2}\sin(2\pi i|2x-1|).$$

Acknowledgments. I would like to thank T. Ledwina and E. Schulte-Geers for helpful discussions.

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Received on 30.11.1989